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Multiple Scattering of Elastic Waves by Two Arbitrary Cylinders

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A perturbation method for the treatment of multiple scattering of acoustic waves by two arbitrary cylinders is extended to the case of elastic waves of plane strain. Interaction terms of monopole and dipole type are given explicitly.

INTRODUCTION

This paper deals with the multiple scattering of plane elastic waves of plane strain by two homogeneous parallel cylinders of arbitrary shape and composition embedded in an arbitrary, homogeneous elastic medium. The method described here applies to a wide class of boundary conditions and includes free boundaries, rigid boundaries, and cases where both the stresses and displacements are continuous across the boundaries.

The scattering of elastic waves by a single circular cylinder has been treated by numerous authors, including: Sezawa,[1] Kuskov,[2] Kato,[3] Tyutekin,[4] White,[5] Miles,[6] and Golubev.[7] Sezawa[1] has also considered elliptic cylinders. However, as far as the author can determine, no one has treated multiple scattering of elastic waves by two arbitrary cylinders. In the present paper, a method developed by Zitron and Karp[8] for the scalar or acoustic case is extended to the elastic case where one is confronted with both a scalar and a vector wavefunction which are coupled at the boundary between two media. In the elastic case, the expressions obtained are somewhat analogous to those obtained in the scalar case, but are much more complicated, and the number of terms of each type in the elastic case will be 2^{n-1} times the number of terms of corresponding type in the acoustic case where n is the number of times the wave is scattered. Terms are said to be of the same type if they represent the same polarity and the same number of scatterings. For example, all monopole terms that have been scattered twice are of the same type.

I. STATEMENT OF THE PROBLEM

The displacement $\bar{u}$ of an elastic medium may be decomposed into two elementary types of displacements,[9] a compressional or longitudinal displacement

$$\bar{u}_p=\text{grad}\phi, \qquad (1)$$

and a shear or transverse displacement

$$\bar{u}_s=\text{curl}\bar{\psi}, \qquad (2)$$

where $\phi(x,y,z)$ and $\bar{\psi}(x,y,z)$ are scalar and vector potentials, respectively, which are sufficient to determine $\bar{u}_p$ and $\bar{u}_s$. Thus

$$\bar{u}=\bar{u}_p+\bar{u}_s=\text{grad}\phi+\text{curl}\bar{\psi}. \qquad (3)$$

For time dependence $e^{-i\omega t}$, ϕ and $\bar{\psi}$ satisfy Helmholz equations:

$$\nabla^2\phi+\beta_p{}^2\phi=0, \qquad (4)$$

and

$$\nabla^2\bar{\psi}+\beta_s{}^2\bar{\psi}=0, \qquad (5)$$

where $\beta_p=\omega(\rho/\mu)^{\frac{1}{2}}$, $\beta_s=\omega[\rho/(\lambda+2\mu)]^{\frac{1}{2}}$, ρ is the density of the medium, and μ and λ are the Lamé constants of the medium. β_p and β_s are the respective propagation constants of the compressional and shear waves.

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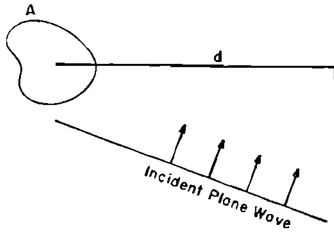

FIG. 1. Plane wave incident upon two parallel cylinders.

In the case of plane strain, the displacements in the direction normal to the plane of strain, which is designated as the z direction, vanish, as do the derivatives of the above potentials with respect to z, since the displacements and stresses are the same for all cross sections normal to the z axis. Equation 3 then becomes, in polar coordinates,

$$u_r=\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial\psi_z}{\partial\theta}, \tag{6}$$

$$u_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}-\frac{\partial\psi_z}{\partial r}. \tag{7}$$

It is clear from Eqs. 6 and 7 that the plane strains are determined completely by ϕ and ψ_z and are independent of the components of $\bar{\psi}$ that lie in the plane of strain. It is possible, therefore, to treat the problem entirely in terms of ϕ and ψ_z.

An elastic wave scattering problem is formulated in the following way. Let V_j^{inc} represent either an incident ϕ wave $(j=p)$ or an incident ψ_z wave $(j=s)$. Let V_{jk}^{scat} represent either a ϕ type of response to V_j^{inc} $(k=p)$ or a ψ_z type of response to V_j^{inc} $(k=s)$. It is well known[10] that, in general, both types of responses will occur at the boundaries between two media. Then, the total field V_j^{tot} arising from V_j^{inc} is

$$V_j^{\text{tot}}=V_j^{\text{inc}}+\sum_{k=p,s}V_{jk}^{\text{scat}}. \tag{8}$$

Each of the fields in Eq. 8 will satisfy the appropriate Helmholz equation, Eq. 4 or Eq. 5, depending on whether it is a scalar or vector potential. The scattered fields V_{jk}^{scat} will satisfy a radiation condition

$$\lim_{r\to\infty}r^{\frac{1}{2}}\left(\frac{\partial V_{jk}^{\text{scat}}}{\partial r}-i\beta_k V_{jk}^{\text{scat}}\right)=0, \tag{9}$$

and will satisfy conditions at the boundaries imposed upon the stresses and displacements. This is discussed in greater detail by Roseau.[11] For example, the displacements will vanish at the boundary of a rigid scatterer, the stresses will vanish at a free boundary, and continuity conditions will apply to both displacements and stresses at the boundary between two different media.[10] The stresses are related to the displacements by the stress–strain relations[9]

$$\tau_{i_1i_2}=2\mu\epsilon_{i_1i_2}+\lambda\delta_{i_1i_2}\theta, \tag{10}$$

where the τ's are the stresses, the ϵ's are the strains, $\delta_{i_1i_2}$ is the Kronecker delta, and θ is the sum of the principal strains. Thus, the stresses may be expressed in terms of normal and tangential second derivatives of the potentials by combining Eq. 3 and 10, whereas the displacements may be expressed in terms of the normal and tangential first derivatives of the potentials by Eq. 3. For example, in polar coordinates,[6,7]

$$\tau_{rr}=-\lambda\beta_p^2\phi+2\mu\frac{\partial^2\phi}{\partial r^2}+2\mu\left(\frac{1}{r}\frac{\partial^2\psi_z}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\psi_z}{\partial\theta}\right), \tag{11}$$

$$\tau_{r\theta}=2\mu\left(\frac{1}{r}\frac{\partial^2\phi}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\phi}{\partial\theta}\right)-2\mu\left(\frac{\beta_s^2}{2}\psi_z+\frac{\partial^2\psi_z}{\partial r^2}\right). \tag{12}$$

A potential satisfying Eq. 9 and either Eq. 4 or Eq. 5 is called a cylindrically radiated field. At large distances from a cylindrical region containing the sources or scatterers that give rise to the radiated field, the field has the asymptotic representation known as the far field

$$V_{jk}(r,\theta,\alpha)\sim H(\beta_k r)g_{jk}(\theta,\alpha),\qquad H(\beta_k r)=(2/\pi kr)^{\frac{1}{2}}e^{i(\beta_k r-\frac{1}{4}\pi)}, \tag{13}$$

where r and θ are the cylindrical coordinates of the field point, and α is the angle of incidence. The first coefficient $g_{jk}(\theta,\alpha)$ is called the complex scattering amplitude of the far field.

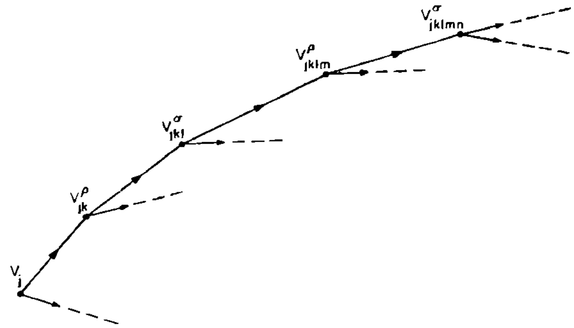

FIG. 2. This graph is a general representation for a sequential multiple scattering process. The first node represents the incident wave, the second represents a singly scattered wave, the third a doubly scattered wave, etc. The subscripts j, k, l, m, n may each be set equal to p or s. When read from left to right, the sequence of subscripts represents the sequence of the types of waves which ultimately gave rise to the wave under consideration. The superscripts ρ and σ assume the equal a or b, but $\rho\neq\sigma$, thus identifying the cylinders that alternately scatter a wave.

II. MULTIPLE SCATTERING OF ELASTIC WAVES

In the case of scattering of elastic waves by two or more scatterers, the problem is to find the far field of the

[10] H. Kolsky, *Stress Waves in Solids* (The Clarendon Press, Oxford, England, 1953), p. 31.

[11] M. Roseau, J. Math. Pure Appl. 39, 2, 173–196 (1960).

TABLE I. A perturbation table giving the various terms in each type of multiply scattered field, their orders of magnitude, the number of times they have been scattered, their locations in the "Feynman diagrams," and their composition.

Field corresponding to a single path	Corresponding node of Feynman diagram	Type of term	Order of magnitude	Number of times the field is scattered	Total field of each type (summation over all scattering paths)
$V_{jk}^{(0)}$	1	Noninteraction	d^0	1	$V_{jk}^{(0)}=V_{jk}^{a(0)}+V_{jk}^{b(0)}$
$V_{jkl}^{(1)}$	2	Monopole	$d^{-\frac{1}{2}}$	2	$V_{jl}^{(1)}=\sum_{k}(V_{jkl}^{a(1)}+V_{jkl}^{b(1)})$
$V_{jklm}^{(2)}$	3	Monopole	d^{-1}	3	$V_{jm}^{(2)}=\sum_{k,l}(V_{jklm}^{a(2)}+V_{jklm}^{b(2)})$
$V_{jklmn}^{(3)}$	4	Monopole	$d^{-\frac{3}{2}}$	4	$V_{jn}^{(3)}=\sum_{k,l,m}(V_{jklmn}^{a(3)}+V_{jklmn}^{b(3)})$
$V_{jkl}^{(3\delta)}$	2	Dipole	$d^{-\frac{3}{2}}$	2	$V_{jl}^{(3\delta)}=\sum_{k}(V_{jkl}^{a(3\delta)}+V_{jkl}^{b(3\delta)})$

combination of scatterers in terms of the far field amplitudes which the individual scatterers would radiate if they were isolated from each other. The far field of the combination cannot be obtained by a pure superposition of the far fields of the individual scatterers because an interaction occurs. Zitron and Karp[8] have developed a perturbation method to take a certain degree of this interaction into account in the case of acoustic and electromagnetic waves where only one type of wave is excited in response to an incident plane wave. In the present investigation, the method is extended to the case of elastic waves of plane strain where two waves, a p wave and an s wave, are excited in response to either an incident p wave or an s wave. The method is valid for widely spaced scatterers where the spacing d is large compared to the wavelength and the diameters of the scatterers.

III. STATEMENT OF THE PROBLEM

In the case considered here (see Fig. 1), a plane wave of unit amplitude,

$$V_j^{\text{inc}}=e^{i\beta_j(x\cos\alpha+y\sin\alpha)}, \tag{14}$$

is incident on two widely spaced parallel cylindrical scatterers of arbitrary shape and composition denoted by the letters A and B. It is assumed that the unperturbed scattering amplitudes $g_{jk}^a(\theta,\alpha)$ and $g_{jk}^b(\theta,\alpha)$ of cylinders A and B, respectively, are given. The problem is to obtain an asymptotic solution in inverse half-integral and integral powers of d whose coefficients are in terms of $g_{jk}^a(\theta,\alpha)$ and $g_{jk}^b(\theta,\alpha)$ up to order of magnitude $d^{-\frac{3}{2}}$ as was done in the acoustic case. In the elastic case, it is more difficult to keep track of all of the various types of waves produced in the successive stages of the perturbation process than in the acoustic case. Fortunately, this task is facilitated by the use of a "Feynman diagram," which is a type of linear graph known as a tree Fig. 2. In this diagram, each path along a sequence of branches from left to right traces the history of a perturbed wave. The first node represents the incident wave, the second node represents a singly scattered wave, the third node a doubly scattered wave, etc. The subscripts from left to right, each of which may take on the value of s or p, represent the sequence of types of waves which ultimately gave rise to the wave under consideration. The superscripts ρ and σ that take on the values a and b (or b and a) identify the cylinder that was the last in the sequence to scatter the wave under consideration.

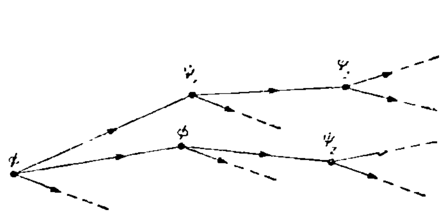

FIG. 3. The tree describing the shear waves of order $d^{-\frac{3}{2}}$ excited by an incident compressional wave is shown above. The upper path represents the scattering sequences $[V_p, V_{ps}^a, V_{pss}^b]$ and $[V_p, V_{ps}^b, V_{pss}^a]$ while the lower path represents the scattering sequences $[V_p, V_{pp}^a, V_{pps}^b]$ and $[V_p, V_{pp}^b, V_{pps}^a]$.

If the complete tree were to be drawn, and all nodes of a fixed number three from the left were selected, their aggregate would represent all fields which have been scattered three times. For example, the singly scattered s waves excited by an incident p wave would be

$$V_{ps}^a+V_{ps}^b. \tag{15}$$

It is now feasible to make a classification of the various terms in the perturbed solution up to order of magnitude $d^{-\frac{3}{2}}$. In the multiply scattered fields, the singly scattered fields (at the second node of the Feynman diagram) that are rescattered consist of a term of order $d^{-\frac{1}{2}}$, which is referred to as a monopole term and a term of order $d^{-\frac{3}{2}}$, which is referred to as a dipole term. It is clear that a quadruply scattered dipole field (corresponding to the fifth node of the Feynman diagram) has the same order of magnitude ($d^{-\frac{3}{2}}$) as the doubly scattered dipole term. The physical significance of these terms is that the dipole term involves a directivity pattern, i.e., a variable curvature of the wavefront.

The classification of fields is given in Table I. The superscripts in parentheses give the power of $d^{-\frac{1}{2}}$ which is the order of magnitude of the monopole terms. The superscript 3δ gives the order of magnitude of the dipole term and distinguishes it from the monopole terms. The scattered field up to order $d^{-\frac{3}{2}}$ may be obtained by adding up the terms in Table I. The result is

$$V_{jk}^{\text{scat}}=V_{jk}^{(0)}+V_{jl}^{(1)}+V_{jm}^{(2)}+V_{jn}^{(3)}+V_{jl}^{(3\delta)}, \quad (16)$$

where

$$j=\begin{Bmatrix}p\\s\end{Bmatrix} \quad \text{and} \quad k=l=m=n=\begin{Bmatrix}p\\s\end{Bmatrix}.$$

The first term on the right represents single scattering and is independent of d the second, third, and fourth terms are the monopole terms of order $d^{-\frac{1}{2}}$, d^{-1}, and $d^{-\frac{3}{2}}$, respectively, and the fifth term is the dipole term of order $d^{-\frac{3}{2}}$.

These terms may be obtained explicitly with the aid of Eq. 31 of Zitron and Karp[8]:

$$V_{jk}^{(0)}=H(\beta_k r)[e^{i\gamma_{jk}^-}g_{jk}^a(\theta,\alpha)+e^{-i\gamma_{jk}^-}g_{jk}^b(\theta,\alpha)], \quad (17)$$

$$V_{jl}^{(1)}=H(\beta_l r)\sum_k H(\beta_k d)[e^{i\gamma_{jl}^+}g_{kl}^a(\theta,\pi)g_{jk}^b(\pi,\alpha)+e^{-i\gamma_{jl}^+}g_{kl}^b(\theta,0)g_{jk}^a(0,\alpha)], \quad (18)$$

$$V_{jm}^{(2)}=H(\beta_m r)\sum_{k,l} H(\beta_k d)H(\beta_l d)[e^{i\gamma_{jm}^-}g_{lm}^a(\theta,\pi)g_{kl}^b(\pi,0)g_{jk}^a(0,\alpha)+e^{-i\gamma_{jm}^-}g_{lm}^b(\theta,0)g_{kl}^a(0,\pi)g_{jk}^b(\pi,\alpha)], \quad (19)$$

$$V_{jn}^{(3)}=H(\beta_n r)\sum_{k,l,m} H(\beta_k d)H(\beta_l d)H(\beta_m d)[e^{i\gamma_{jm}^+}g_{mn}^a(\theta,\pi)g_{lm}^b(\pi,0)g_{kl}^a(0,\pi)g_{jk}^b(\pi,\alpha)$$
$$+e^{-i\gamma_{jm}^+}g_{mn}^b(\theta,0)g_{lm}^a(0,\pi)g_{kl}^b(\pi,0)g_{jk}^a(0,\alpha)], \quad (20)$$

$$V_{jl}^{(3\delta)}=H(\beta_l r)\sum_k \frac{H(\beta_k d)}{2ikd}[e^{i\gamma_{jl}^+}(D_l^2[g_{kl}^a(\theta,l)g_{jk}^b(l,\alpha)]_{l=\pi}+\tfrac{1}{4}g_{kl}^a(\theta,\pi)g_{jk}^b(\pi,\alpha))$$
$$+e^{-i\gamma_{jl}^+}(D_l^2[g_{kl}^b(\theta,l)g_{jk}^a(l,\alpha)]_{l=0}+\tfrac{1}{4}g_{kl}^b(\theta,0)g_{jk}^a(0,\alpha))], \quad (21)$$

where

$$\gamma_{ji}^+=\tfrac{1}{2}d(\beta_i\cos\theta+\beta_j\cos\alpha), \quad i=k,\,l,\,m,\,n,$$
$$\gamma_{ji}^-=\tfrac{1}{2}d(\beta_i\cos\theta-\beta_j\cos\alpha),$$
$$D_l=d/dl,$$

and the summations over k, l, m include all possible combinations of p and s. The factors γ_{ji}^+ and γ_{ji}^- result from the fact that multiple scattering formulas are normalized in terms of a common origin located midway between the two scatterers.[8]

IV. CASE OF TWO CIRCULAR CYLINDERS

For an example of how the above results would be applied in a particular problem, consider the following case. A plane compressional wave is incident upon a combination of two widely spaced parallel circular cylinders, A and B. Let us assume that the object is to find the s waves of order $d^{-\frac{1}{2}}$ resulting from this incident wave.

The complete multiple scattering tree pertaining to the history of the s waves of order $d^{-\frac{1}{2}}$ resulting from an incident p wave is shown in Fig. 3 in a more conventional notation. Let the response of A to an incident plane p wave be

$$H(\beta_k r)\sum_{\nu_1=-\infty}^{\infty} a_{pk}^{\nu_1}e^{i\nu_1(\theta-\alpha)}, \quad (22)$$

and the response of B to an incident plane p wave be

$$H(\beta_k r)\sum_{\nu_2=-\infty}^{\infty} b_{pk}^{\nu_2}e^{i\nu_2(\theta-\alpha)}. \quad (23)$$

It should be noted that the superscripts on the a's and b's are indices of summation and not exponents. A direct substitution of Eqs. 22 and 23 into Eq. 18 yields

$$V_{ps}^{(1)}=H(\beta_s r)\left\{\begin{aligned}&H(\beta_p d)\Big(e^{i\gamma_{ps}^+}\sum_{\nu_1}(-1)^{\nu_1}a_{ps}^{\nu_1}e^{i\nu_1(\theta-\alpha)}\sum_{\nu_2}(-1)^{\nu_2}b_{pp}^{\nu_2}e^{-i\nu_2\alpha}+e^{-i\gamma_{ps}^+}\sum_{\nu_2}b_{ps}^{\nu_2}e^{i\nu_2\theta}\sum_{\nu_1}a_{pp}^{\nu_1}e^{-i\nu_1\alpha}\Big)\\&+H(\beta_s d)\Big(e^{i\gamma_{ps}^+}\sum_{\nu_1}(-1)^{\nu_1}a_{ps}^{\nu_1}e^{i\nu_1\theta}\sum_{\nu_2}(-1)^{\nu_2}b_{pp}^{\nu_2}e^{-i\nu_2\alpha}+e^{-i\gamma_{ps}^+}\sum_{\nu_2}b_{ss}^{\nu_2}e^{i\nu_2\theta}\sum_{\nu_1}a_{ps}^{\nu_1}e^{-i\nu_1\alpha}\Big)\end{aligned}\right\}.$$

V. CONCLUDING REMARKS

This method might also be applied to diffraction between two rigid half-planes or two cracks in an elastic medium which Resende[12] treated by using Keller's[13] geometrical theory of diffraction. Since that treatment was based on a high-frequency method, it gives only monopole interaction terms. This method gives higher-order terms and thus is valid for even smaller spacing.

The reviewers have kindly pointed out the possibility of using expansion theorems.[14–18] The author has obtained the results presented here without the aid of these more general expansions. However, they would undoubtedly be useful to anyone who wanted to compute the general term of the series or specific terms of order higher than the dipole terms.

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