ERRATUM

Addendum and Erratum: "Comparison of the *T*-matrix and Helmholtz integral equation methods for wave scattering calculations" [J. Acoust. Soc. Am. 77, 369–374 (1985)]

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Certain errors appeared in our paper [J. Acoust. Soc. Am. 77, 369–374 (1985)]. In this letter we attempt to rectify and clarify the problem.

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The author is grateful to Dr. P. C. Waterman for pointing out certain errors in the paper in question.[1] Evidently, in transcribing equations from notes to the manuscript, expressions from two different *T*-matrix formalisms were inappropriately juxtaposed. However, the author believes that the *T*-matrix calculations reported in the paper are based on mathematically correct expressions. An attempt to rectify the situation is made here.

As Dr. Waterman observed, all the l's and m's inside the curly brackets in Eq. (8) should be primed. In addition, the factor $(4\pi)^{-1}$ should be deleted from Eq. (8), and the comma should be removed from Eq. (11). Most important, Eqs. (9b) and (9c) are inappropriate. They should be replaced by

$$J_{lm,l'm'} = i^{l'-l+1}k\oint_s dS'\, Y_{l'}^{m'}(\hat{r}')j_{l'}(kr') \times \mathbf{n}'\cdot\nabla' Y_l^m(\hat{r}')^* j_l(kr'), \tag{1}$$

$$H_{lm,l'm'} = i^{l'-l+1}k\oint_s dS'\, Y_{l'}^{m'}(\hat{r}')h_{l'}^{(1)}(kr') \times \mathbf{n}'\cdot\nabla' Y_l^m(\hat{r}')^* j_l(kr'). \tag{2}$$

The formalism developed in Eqs. (8)–(11) of the paper is not very closely related to the *T*-matrix formalism of Waterman.[2] It is, in fact, not the formalism used to do the *T*-matrix calculations reported in the paper, and the presentation of that formalism was entirely inadvertent.

The author's intention was to present the following formalism. For the wave field $\psi(\mathbf{r}')$ under the integral in Eq.(1) of the paper use the following expansion in spherical harmonics:

$$\psi(\mathbf{r}') = 4\pi\sum_{l,m} i^l Y_l^m(\hat{r}')\alpha_{lm}, \quad \mathbf{r}'\in S. \tag{3}$$

Then substitution of Eq. (3) and Eqs. (5), (6), and (7) of the paper into Eq. (1a) of the paper gives

$$t_{lm} = \sum_{l',m'} \bar{J}_{lm,l'm'}\alpha_{l'm'}, \tag{4a}$$

$$\bar{J}_{lm,l'm'} = i^{l'+1-l}k\oint_s dS'\, Y_{l'}^{m'}(\hat{r}') \times \mathbf{n}'\cdot\nabla' Y_l^m(\hat{r}')^* j_l(k,r'), \tag{4b}$$

where Eq. (4b) is identical to Eq. (9b) in the paper. Next, Eq. (3) and Eqs. (5) and (7) of the paper are substituted into Eq. (1c) of the paper with the result

$$-Y_l^m(\hat{k}_0)^* = \sum_{l',m'} \bar{H}_{lm,l'm'}\alpha_{l'm'}, \tag{5a}$$

$$\bar{H}_{lm,l'm'} = i^{l'+1-l}k\oint_s dS'\, Y_{l'}^{m'}(\hat{r}') \times \mathbf{n}'\cdot\nabla' Y_l^m(\hat{r}')^* h_l^{(1)}(kr'), \tag{5b}$$

where Eq. (5b) is identical to Eq. (9c) of the paper. Finally, eliminating α between Eqs. (4a) and (5a) gives

$$t = -\bar{J}\bar{H}^{-1}Y^* \tag{6}$$

in place of Eq. (11) of the paper.

Equation (6) is the one that was used for the *T*-matrix calculations reported in the paper. It is very similar to the *T*-matrix formalism of Waterman. Yet the two are not quite identical. To get Waterman's formalism Eq. (3) must be replaced by

$$\psi(\mathbf{r}') = 4\pi\sum_{l,m} i^l Y_l^m(\hat{r}')j_l(r')\beta_{lm}, \quad \mathbf{r}'\in S. \tag{7}$$

Then Eq. (6) gets replaced by

$$t = -\hat{J}\hat{H}^{-1}Y^*, \tag{8}$$

where $\hat{J}$ and $\hat{H}$ differ from $\bar{J}$ and $\bar{H}$ by virtue of having an additional factor $j_l(kr')$ in the integrands.

$$\hat{J}_{lm,l'm} = i^{l'+1-l}k\oint_s dS'\, Y_{l'}^{m'}(\hat{r}')j_{l'}(kr') \times \mathbf{n}'\cdot\nabla' Y_l^m(\hat{r}')^* j_l(kr'), \tag{9}$$

$$\hat{H}_{lm,l'm'} = i^{l'+1-l}k\oint_s dS'\, Y_{l'}^{m'}(\hat{r}')j_{l'}(kr') \times \mathbf{n}'\cdot\nabla' Y_l^m(\hat{r}')^* h_l^{(1)}(kr'). \tag{10}$$

It appears that both Eqs. (6) and (8) are valid. Indeed, the coefficients α and β must be related by

$$\alpha_{lm} = i^{-l}\oint_s dS'\, \mathbf{n}'\cdot\mathbf{r}'r'^{-3}Y_l^m(\hat{r}')^* \times \sum_{l',m'} i^{l'}Y_{l'}^{m'}(\hat{r}')j_{l'}(kr')\beta_{l'm'}. \tag{11}$$

A series of calculations has been carried out to compare the responses to numerical evaluation of the two T-matrix formalisms displayed in Eqs. (6) and (8) with that of the HIEM. The results are displayed in Tables I through VIII. The T-matrix method of Eq. (6) has been labeled as TMM and the T-matrix method of Eq. (8), the Waterman formalism, as TMW.

It is seen that for a spherical target the TMM and TMW

TABLE I. Scattering amplitude $|T(\theta)|$ for scattering angle θ for a plane acoustic wave of wavenumber $k = 1.0$ incident on a rigid prolate spheroid of major semi-axis $c = 1.0$ and minor semi-axis $a = 1.0$. The direction of incidence is perpendicular to the axis of symmetry, and the scattering plane is that of the symmetry axis and the direction of incidence. HIEM identifies results calculated by the Helmholtz integral equation method, TMM identifies results calculated by the T-matrix method of Eq. (6), and TMW identifies results calculated by the T-matrix method of Eq. (8). N is the number of surface patches used to discretize the Helmholtz integral equation, C.No. is the condition number of the matrix that was inverted, and NPW is the number of partial waves used in the T-matrix calculation. The number of Gaussian quadrature points NGP used in the T-matrix calculations was 40.

	HIEM				TMM				TMW		
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
12	1.08	1.94E-1	4.76E-1	2	1.11	1.37E-1	5.27E-1	2	1.17	1.37E-1	5.27E-1
34	1.04	1.99E-1	4.80E-1	3	2.75	1.90E-1	4.66E-1	3	1.20	1.90E-1	4.66E-1
64	1.03	1.97E-1	4.75E-1	4	13.4	1.92E-1	4.69E-1	4	1.19	1.92E-1	4.69E-1
108	1.02	1.95E-1	4.73E-1	5	93.1	1.92E-1	4.69E-1	5	1.18	1.92E-1	4.69E-1
158	1.01	1.94E-1	4.70E-1	6	831	1.92E-1	4.69E-1	6	1.17	1.92E-1	4.69E-1
exact		1.92E-1	4.69E-1								

TABLE II. Same as Table I except that $a = 0.5$.

	HIEM				TMM				TMW		
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
10	1.12	7.22E-2	2.10E-1	2	5.37	4.95E-2	1.84E-1	2	1.04	6.40E-2	1.96E-1
24	1.06	7.08E-2	2.02E-1	3	7.61	5.06E-2	1.83E-1	3	1.06	6.46E-2	1.95E-1
46	1.04	7.00E-2	2.01E-1	4	61.4	6.44E-2	1.96E-1	4	1.08	6.85E-2	1.99E-1
78	1.03	6.95E-2	2.00E-1	5	750	6.49E-2	1.96E-1	5	1.09	6.86E-2	1.99E-1
114	1.02	6.94E-2	2.00E-1	6	1.2E4	6.75E-2	1.98E-1	6	1.18	6.86E-2	1.99E-1
156	1.02	6.93E-2	2.00E-1	7	2.5E5	6.76E-2	1.98E-1	7	6.8E3	6.86E-2	1.99E-1
				8	6.1E6	6.83E-2	1.99E-1	8	2.0E6	6.89E-2	1.99E-1
				9	overflow			9	1.9E11	6.87E-2	2.00E-1
								10	overflow		

TABLE III. Same as Table I except that $a = 0.2$ and NGP = 48.

	HIEM				TMM				TMW		
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
16	1.39	1.30E-2	3.66E-2	2	4.39	2.34E-3	2.70E-2	2	1.00	1.26E-2	3.62E-2
26	1.21	1.27E-2	3.76E-2	3	43.1	1.97E-3	2.73E-2	3	2.01	1.17E-2	3.71E-2
40	1.12	1.29E-2	3.81E-2	4	853	6.52E-3	3.19E-2	4	1.01	1.27E-2	3.80E-2
60	1.08	1.28E-2	3.81E-2	5	2.7E4	6.54E-3	3.19E-2	5	2.0E3	1.29E-2	3.78E-2
82	1.05	1.28E-2	3.82E-2	6	1.1E6	8.59E-3	3.39E-2	6	1.6E5	5.48E-2	3.04E-2
106	1.04	1.28E-2	3.82E-2	7	5.2E7	8.46E-3	3.40E-2	7	1.6E11	1.34E-2	3.69E-2
				8	4.1E9	overflow		8	1.4E13	2.21E-2	1.58E-2
								9	overflow		

TABLE IV. Same as Table I except that $a = 0.1$ and NGP = 80.

	HIEM				TMM				TMW		
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
34	1.54	3.08E-3	8.85E-3	2	8.80	6.52E-4	5.70E-3	2	1.00	3.28E-3	9.33E-3
44	1.37	3.16E-3	9.22E-3	3	171	7.56E-4	5.81E-3	3	1.00	3.02E-3	9.59E-3
60	1.25	3.35E-3	9.88E-3	4	6.7E3	2.62E-4	6.82E-3	4	1.09	3.34E-3	9.90E-3
76	1.18	3.34E-3	9.89E-3	5	4.2E5	2.76E-4	6.81E-3	5	1.5E7	5.42E3	7.83E-3
92	1.14	3.34E-3	9.92E-3	6	3.7E7	8.32E-4	7.36E-3	6	2.6E8	3.44E-2	3.19E-2
114	1.11	3.32E-3	9.89E-3	7	3.2E9	overflow		7	3.8E13	6.58E2	2.90E-3
								8	overflow		

TABLE V. Same as Table I except that $k = 6.0$ and NGP = 48.

	HIEM			TMM				TMW			
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
12	5.97	1.20E0	1.27E1	2	12.6	1.73E-1	1.63E-1	2	1.92	1.73E-1	1.63E-1
34	1.68	4.32E0	1.99E0	3	1.00	9.04E-1	9.18E-1	3	2.41	9.04E-1	9.18E-1
64	1.46	2.54E0	8.49E-1	4	1.00	1.68E0	8.63E-1	4	2.08	1.68E-0	8.63E-1
108	1.28	2.34E0	3.99E-1	5	1.01	1.80E0	6.00E-1	5	2.13	1.80E0	6.00E-1
158	1.19	2.32E0	5.08E-1	6	1.01	1.84E0	1.08E-1	6	1.93	1.84E0	1.08E-1
222	1.14	2.30E0	5.16E-1	7	317	2.11E0	7.60E-1	7	188	2.11E0	7.60E-1
exact		2.27E0	5.31E-1	8	1.1E3	2.23E0	4.77E-1	8	546	2.24E0	4.77E-1
				9	6.0E7	2.26E0	5.40E-1	9	4.0E7	2.26E0	5.40E-1
				10	overflow			10	2.8E8	2.27E0	5.30E-1
								11	overflow		

TABLE VI. Same as Table I except that $k = 6.0$, $a = 0.5$, and NGP = 56.

	HEIM			TMM				TMW			
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
10	2.97	1.67	5.99E-1	3	1.05	5.32E-1	4.81E-1	3	1.34	4.67E-1	4.88E-1
24	1.31	1.09	9.56E-1	4	1.17	7.30E-1	3.96E-1	4	1.70	7.25E-1	4.03E-1
46	1.21	1.17	3.57E-1	5	1.22	8.54E-1	4.71E-1	5	1.67	8.66E-1	4.16E-1
78	1.13	1.13	4.65E-1	6	2.28	9.76E-1	7.09E-1	6	2.52	1.01	6.30E-1
114	1.09	1.12	4.62E-1	7	6.07	1.05	5.17E-1	7	3.56	1.09	5.01E-1
156	1.07	1.12	4.62E-1	8	5.6E3	1.06	5.57E-1	8	4.0E4	1.09	5.18E-1
				9	2.8E5	1.07	5.82E-1	9	6.0E5	1.08	6.36E-1
				10	overflow			10	overflow		

TABLE VII. Same as Table I except that $k = 6.0$, $a = 0.2$, and NGP = 72.

	HIEM			TMM				TMW			
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
8	1.79	4.07E-1	3.33E-1	3	1.67	1.80E-1	2.85E-1	3	1.01	1.72E-1	2.56E-1
16	1.55	3.53E-1	5.75E-1	4	3.92	2.33E-1	3.61E-1	4	1.03	2.67E0	3.72E-1
26	1.31	3.66E-1	5.35E-1	5	17.9	2.92E-1	5.67E-1	5	1.03	6.75E-1	9.22E-1
42	1.19	3.70E-1	5.00E-1	6	117	5.07E-1	3.24E-1	6	13.4	8.74E-1	6.86E-1
60	1.13	3.71E-1	5.00E-1	7	996	1.02E0	8.33E-1	7	464	5.72E-1	5.18E-1
82	1.09	3.71E-1	5.03E-1	8	1.0E4	1.06E0	7.85E-1	8	2.9E7	6.76E-1	8.66E-1
106	1.07	3.71E-1	5.04E-1	9	1.2E5	overflow		9	6.0E8	overflow	
134	1.06	3.71E-1	5.05E-1								

TABLE VIII. Same as Table I except that $k = 6.0$, $a = 0.1$, and NGP = 96.

	HIEM			TMM				TMW			
N	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$	NPW	C.No.	$\|T(0°)\|$	$\|T(180°)\|$
6	4.38	3.53E-7	3.53E-7	2	1.65	3.06E-2	9.93E-2	2	1.00	6.87E-2	1.17E-1
12	3.69	2.03E-1	2.65E-1	3	4.83	2.96E-2	1.36E-1	3	1.00	4.26E-2	1.51E-1
18	2.60	1.45E-1	2.49E-1	4	29.2	4.74E-2	1.78E-1	4	1.00	1.13E-1	2.25E-1
24	2.07	1.36E-1	2.63E-1	5	299	4.11E-2	1.93E-1	5	1.09	4.35E-1	6.74E-1
38	1.67	1.37E-1	3.07E-1	6	4.2E3	2.75E-1	1.16E-1	6	563	9.87E-1	2.69E-1
46	1.45	1.40E-1	3.16E-1	7	6.7E4	5.21E-1	1.93E-1	7	5.6E4	8.24E-1	1.72E-1
60	1.31	1.36E-1	3.01E-1	8	1.7E6	1.57E-1	5.44E-1	8	1.1E6	3.00E0	3.47E0
76	1.21	1.35E-1	2.99E-1	9	overflow			9	overflow		
92	1.16	1.35E-1	2.99E-1								
114	1.13	1.34E-1	2.97E-1								
132	1.11	1.35E-1	2.97E-1								

are pretty much equivalent and converge more rapidly than the HIEM. For $a = 0.5$ the HIEM converges better than the TMM and the TMW converges better than the HIEM for $k = 1.0$ and a little worse than the HIEM for $k = 6.0$. For $a = 0.2$ and $a = 0.1$ the HIEM converges very well and the TMM converges not at all. For $a = 0.2$ and $a = 0.1$ the TMW fails to converge for $k = 6.0$ while for $k = 1.0$ it gives a good result only if fewer than five partial waves are used.

It is concluded that the TMW is only slightly better than the TMM. These two methods work well only for spherical or approximately spherical targets. The HIEM, on the other hand, shows admirable numerical stability in all the cases tested here.

Finally, it is noted that Waterman is correct in his remark concerning Fig. 4. The scale factor should be 1/0.0092 instead of 1/0.038.

In conclusion, the author wishes to express his gratitude to the Center for Automation and Intelligent Systems Research of Case Western Reserve University for the use of their VAX-11/782 computer in carrying out these calculations.

[1]W. Tobocman, J. Acoust. Soc. Am. 77, 369–374 (1985).
[2]P. C. Waterman, J. Acoust. Soc. Am. 45, 1417 (1969).