Multiple scattering by two cylinders

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Theoretical calculations and experimental measurements of the backscattering of a plane acoustic wave by two parallel, rigid cylinders are presented. The effects of multiple scattering between cylinders are included in the calculations. Two methods of solution of the equations are discussed. Good agreement is found between theory and experiment. Multiple-scattering calculations show the influence of the shadow of one cylinder upon the other. Variations in level of the interference peaks are correctly predicted.

Subject Classification: 30,30, 30.40; 20.30, 20.15.

INTRODUCTION

The scattering of an incident wave by two cylinders is of particular interest since the multiple-reflection terms can be explicitly evaluated. In this paper, we examine both theoretically and experimentally the scattered field produced by a plane acoustic wave normal to the axis of two parallel, rigid cylinders. The general solution of this problem was first published by Twersky[1,2] in 1952. He expressed the total radiation field as an incident field plus various orders of scattered fields. The necessary expansion coefficients were to be generated by an iterative procedure. In a later work,[3] Twersky developed a particularly elegant formulation of the coefficient equations which relates the multiple-scattering coefficients to their single-scattering analogs. The case of two arbitrary circular cylinders was considered in some detail. The series solutions were truncated at two terms, monopole and dipole, the resulting four linear equations were written in matrix form, and were explicitly solved. A completely general matrix formulation of the multiple-scattering equations was given by Burke, Censor, and Twersky[4] in 1965. Solution of the two-body problem in terms of an iterative expansion of the inverse matrix was also developed in this paper.

Olaofe,[5] in 1970, applied Twersky's iterative method to the numerical calculations of the electromagnetic scattering cross section of two dielectric cylinders. He also noted that the linear equations involved could be solved by direct matrix inversion techniques. More re-recently, Yousri and Fahy[6] employed the iterative approach to calculate the radiation from two driven, parallel cylinders. In addition, Radinski and Meyers[7] computed the radiation from one large, driven cylinder surrounded by several smaller, rigid cylinders using the matrix inversion method. These latter two papers give some comparisons of the theoretical calculations with experimental data.

Although experimental measurements of the scattering by two spheres are given by several authors,[8–10] we know of no data available on the two-cylinder problem. Faran[11] and Shenderov[12] have considered the scattering of a cylindrical wave by a rigid cylinder. The latter author has achieved reasonably good agreement between theory and experiment. This is, however, a simpler problem since only single scattering is involved.

We have performed the multiple-scattering calculations using both direct matrix inversion and the iterative procedure. For the cases we have considered, the results are comparable although the inversion technique appears more efficient. In addition, we have compared the theoretical predictions with experiments on two solid steel cylinders. The agreement achieved is, we believe, sufficient to confirm the numerical results.

I. MATHEMATICAL FORMULATION

In order to make clear the specific form of the multiple-scattering equations which we have used, we shall briefly review the theoretical analysis. The The geometrical situation, as shown in Fig. 1, consists of two identical, rigid cylinders of radius a with centers a distance $2b$ apart. An incident plane wave,

$$\Phi^{\text{inc}} = e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1}$$

impinges at an angle α with respect to the perpendicular bisector of the line of centers. Two cylindrical coordinate systems are defined with respect to the center of the cylinders. The incident wave can be expanded in the two coordinate systems as follows:

$$\Phi^{\text{inc}}(r_1, \theta_1) = e^{-ikb\sin\alpha} \sum_n i^{-n} J_n(kr_1) e^{in(\theta_1-\alpha)} \tag{2a}$$

and

$$\Phi^{\text{inc}}(r_2, \theta_2) = e^{ikb\sin\alpha} \sum_n i^{-n} J_n(kr_2) e^{in(\theta_2-\alpha)}, \tag{2b}$$

where $-\infty < n < \infty$. The scattered fields from the respective cylinders are given by

$$\Phi_1^s = \sum_n A_n H_n(kr_1) e^{in\theta_1} \tag{3a}$$

and

$$\Phi_2^s = \sum_n B_n H_n(kr_2) e^{in\theta_2} . \tag{3b}$$

In the above equations, J_n and H_n are ordinary Bessel functions and Hankel functions of the first kind, respectively. The latter represent outgoing cylindrical waves at infinity for $e^{-i\omega t}$ time dependence. By means of the Graf Addition Theorem,[13] the outgoing waves from one cylinder can be expressed in the coordinate system of the other as follows:

$$H_n(kr_1) e^{in\theta_1} = \sum_m i^{n-m} H_{m-n}(2kb) J_m(kr_2) e^{im\theta_2} \tag{4a}$$

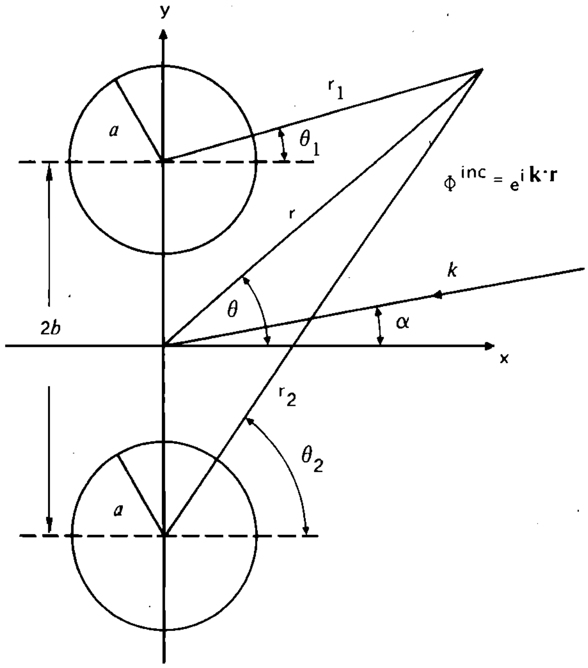

FIG. 1. Two-cylinder coordinate systems.

and

$$H_n(kr_2)e^{in\theta_2} = \sum_m i^{m-n} H_{m-n}(2kb) J_m(kr_1) e^{im\theta_1} . \tag{4b}$$

The total field,

$$\Phi = \Phi^{inc} + \Phi_1^s + \Phi_2^s , \tag{5}$$

can now be written completely in either coordinate system, and the appropriate boundary conditions at each cylinder can be applied. In the case of two perfectly rigid cylinders, we have

$$\left.\frac{\partial\Phi}{\partial r_1}\right|_{r_1=a} = \left.\frac{\partial\Phi}{\partial r_2}\right|_{r_2=a} = 0. \tag{6}$$

These conditions along with the orthogonality of the angular functions lead to the following symmetrized coefficient relations:

$$C_l^{\pm} = S_l i^{-l} e^{\pm il\alpha} e^{\pm ikb\sin\alpha} + \sum_n M_{ln} C_n^{\mp} , \tag{7}$$

in which

$$S_l = \frac{-J_l'(ka)}{H_l'(ka)} = -\frac{J_{l+1}(ka) - (l/ka)J_l(ka)}{H_{l+1}(ka) - (l/ka)H_l(ka)} \tag{8}$$

and

$$M_{ln} = S_l i^{l-n} H_{l+n}(2kb) . \tag{9}$$

The symmetrized coefficients are related to the previously defined expansion coefficients by

$$C_l^- = A_l \tag{10a}$$

and

$$C_l^+ = (-1)^l B_{-l} . \tag{10b}$$

The equations represented by Eq. 7 are now in the form given by Twersky in Eq. 82 of Ref. 3. The first term on the right-hand side of Eq. 7 corresponds to the lth single-cylinder scattering coefficient. The iterative solution is generated by taking these values as first estimates of $C_l^{\pm}$ and inserting them into the summation term for successively higher-order approximations. The process can be continued to an arbitrary order of scattering.

By considering the coefficients $C_l^{\pm}$ as components of column vectors $\mathbf{C}^{\pm}$, we may rewrite the coefficient equations as

$$\mathbf{C}^{\pm} = e^{\pm ikb\sin\alpha}\,\mathbf{K}^{\pm}(\alpha) + M\,\mathbf{C}^{\mp} , \tag{11}$$

where $\mathbf{K}^{\pm}(\alpha)$ have components

$$K_l^{\pm} = S_l i^{-l} e^{\pm il\alpha} \tag{12}$$

and M is an infinite dimensional matrix with elements M_{ln}. The solution of the coupled linear equations can be formally written as

$$\mathbf{C}^{\pm} = (I - M^2)^{-1}\left[e^{\pm ikb\sin\alpha}\,\mathbf{K}^{\pm}(\alpha) + e^{\mp ikb\sin\alpha} M\,\mathbf{K}^{\mp}(\alpha)\right], \tag{13}$$

where I is the identity matrix. In this form, we have a special case of Eq. 55 of Ref. 4.

Since these equations are of infinite dimensionality, some truncation must be performed in order to generate a numerical solution. It is well known that for reasonably large values of ka a number of Bessel function terms $N \approx 2ka$ should be computed to give a valid solution to the single-cylinder scattering problem. We have assumed that the same rule applies to the two-cylinder case. This is clearly true for $kb \gg ka$ since multiple scattering will be unimportant. For the cases we have investigated experimentally ($5.4 < ka < 22$ and $10.9 < kb < 55$) taking $N \approx 2ka$ appears to be sufficient. However, for $ka \approx 1$ or less, a number of terms $N = 2ka + Q$, where Q is of the order of 5 to 10, are needed for accuracy.

In order to generate numerical results, we have written a computer program which constructs the required matrices and performs the inversion. Bessel and Hankel functions of argument ka are computed up to order N by a standard subroutine using backward recursion for the J_n and forward recursion for the Y_n. Hankel functions of argument $2kb$ are computed up to order $2N$ for use in generating the M matrix elements.

It should be noted that the matrix $I - M^2$ is a function of ka and kb only and does not depend on the angle of incidence. Thus, it need be inverted only once for each frequency. This is accomplished by a subroutine which employs the Crout factorization technique.[14] The inverse matrix

$$D = (I - M^2)^{-1} \tag{14}$$

and the matrix product DM are then multiplied by different "forcing functions," $\mathbf{K}^{\pm}(\alpha)$, to give $\mathbf{C}^{\pm}$ as functions of α. For convenience in notation, we define the following vectors:

$$\mathbf{X}^{\pm}(\alpha) = DM\,\mathbf{K}^{\pm}(\alpha) \tag{15a}$$

and

$$\mathbf{W}^{\pm}(\alpha) = D\,\mathbf{K}^{\pm}(\alpha) . \tag{15b}$$

We can now express the original expansion coefficients of the two scattered fields as

$$A_n = e^{ikb\sin\alpha} X_n^+(\alpha) + e^{-ikb\sin\alpha} W_n^-(\alpha) \tag{16a}$$

and

$$B_n = (-1)^n [e^{-ikb\sin\alpha} X_{-n}^-(\alpha) + e^{ikb\sin\alpha} W_{-n}^+(\alpha)]. \tag{16b}$$

The total scattered field is given by

$$\Phi^s = \Phi_1^s + \Phi_2^s = \sum_{n=-N}^{N} [A_n H_n(kr_1) e^{in\theta_1} + B_n H_n(kr_2) e^{in\theta_2}]. \tag{17}$$

At distances such that $r \gg 2b$, we may make the farfield approximations

$$r_1 \approx r - b\sin\theta,$$
$$r_2 \approx r + b\sin\theta, \tag{18}$$
$$\theta_1 \approx \theta_2 \approx \theta.$$

In addition, if $kr \gg 1$, we may use the asymptotic values of the Hankel functions

$$H_n(kr) \approx \left(\frac{2}{\pi kr}\right)^{1/2} e^{ikr} e^{-i\pi/4} i^{-n}. \tag{19}$$

Under these assumptions, the total scattered field is

$$\Phi^s \approx \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\pi/4)} \sum_{n=-N}^{N} i^{-n} (A_n e^{-ikb\sin\theta} + B_n e^{ikb\sin\theta}) e^{in\theta}. \tag{20}$$

Using our expansions of the A_n and B_n, we can write

$$\Phi^s \approx \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\pi/4)} \sum_{n=-N}^{N} i^{-n} [e^{ikb(\sin\alpha-\sin\theta)} X_n^+(\alpha) + (-1)^n e^{-ikb(\sin\alpha-\sin\theta)} X_{-n}^-(\alpha) + (-1)^n e^{ikb(\sin\alpha+\sin\theta)} W_{-n}^+(\alpha) + e^{-ikb(\sin\alpha+\sin\theta)} W_n^-(\alpha)] e^{in\theta}. \tag{21}$$

For monostatic scattering, $\alpha = \theta$, and

$$\Phi^s(\alpha) = \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\pi/4)} \sum_{n=-N}^{N} i^{-n} \phi_n(\alpha) e^{in\alpha}, \tag{22}$$

where

$$\phi_n(\alpha) = X_n^+(\alpha) + (-1)^n X_{-n}^-(\alpha) + (-1)^n e^{2ikb\sin\alpha} W_{-n}^+(\alpha) + e^{-2ikb\sin\alpha} W_n^-(\alpha). \tag{23}$$

For a single cylinder, the backscattered field, Φ^s, takes on the special form

$$\Phi_0^s = \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\pi/4)} \sum_{n=-N}^{N} (-1)^n S_n. \tag{24}$$

The ratio of backscattering cross section of the two-cylinder system to that of a single cylinder is given by

$$\frac{\sigma}{\sigma_0} = \frac{|\Phi_1^s + \Phi_2^s|^2}{|\Phi_0^s|^2}. \tag{25}$$

In logarithmic form, we define the "two-cylinder gain" to be

$$\mathrm{TCG} = 10\log\left(\frac{\sigma}{\sigma_0}\right) = 10\log\left(\frac{\left|\sum_{n=-N}^{N} i^{-n}\phi_n(\alpha)e^{in\alpha}\right|^2}{\left|\sum_{n=-N}^{N}(-1)^n S_n\right|^2}\right). \tag{26}$$

Simple theory, which neglects multiple scattering, predicts the two-cylinder gain to be

$$(\mathrm{TCG})_0 = 20\log|2\cos(2kb\sin\alpha)|. \tag{27}$$

Therefore we expect that $\mathrm{TCG} - (\mathrm{TCG})_0$ will be a measure of the significance of multiple-scattering effects.

For comparison with the matrix inversion approach, we wrote a computer program to calculate the TCG using the iterative method. Five iterations were found to be sufficient for convergence at $ka = 10.64$, $kb = 15.95$ and at $ka = 17.02$, $kb = 25.5$. Comparing the TCG of the interference peaks calculated by both methods, we found that the disagreement was generally less than 0.02 dB. The greatest difference was 0.2 dB. This difference is well within experimental error.

Although we have specifically dealt with the problem of two rigid cylinders, the case of two ideally soft cylinders can be handled by the same formalism. It is only necessary to change the coefficients S_l to

$$S_l = -J_l(ka)/H_l(ka) \tag{28}$$

in computing Φ_1^s, Φ_2^s, and Φ_0^s.

II. EXPERIMENTAL PROCEDURE

Two solid steel cylinders, 2 in. in diameter and 24 in. in length, were mounted at center-to-center separations of 4.05, 8, and 20 in. using PVC pipe crossbars. The pipes were perforated to assure flooding when submerged. Acoustic scattering measurements were made at the Transducer Evaluation Center (TRANSDEC) of the Naval Undersea Center, San Diego. The fluid medium there is fresh water, which was at a temperature of 10.8 °C during the tests. The experimental geometry was as shown in Fig. 2. The idealized two-dimensional situation on which the calculations are based can, of course, only be approximated in an experimental measurement. The receiver-to-target separation of 6.5 m was chosen to be large compared to the cylinder separation, yet still well within the cylindrical scattering region. For the test cylinder at 50 kHz, the farfield transition range L^2/λ is approximately 12 m. Backscattering measurements versus aspect angle were made by rotating the cylinder assembly. Data were taken at frequencies close to 50, 75, and 100 kHz. The transmit pulse length was 1.0 msec. At 11.5-msec delay after transmit, the receiver

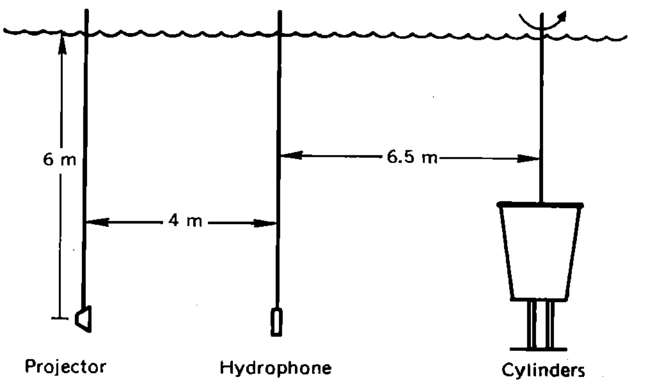

FIG. 2. Experimental geometry for scattering measurements.

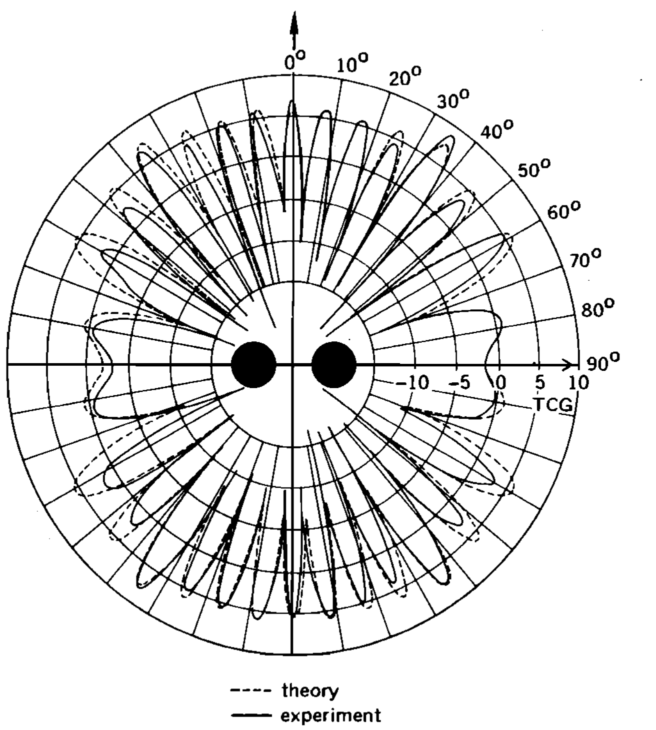

FIG. 3. Two-cylinder gain for $ka = 5.41$, $kb = 10.97$. Frequency = 49.6 kHz, $b/a = 2.03$.

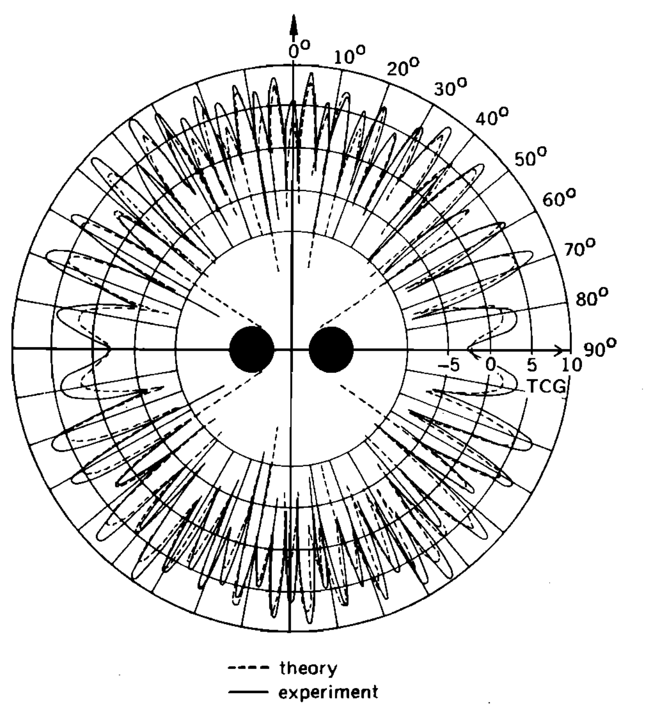

FIG. 5. Two-cylinder gain for $ka = 10.82$, $kb = 21.94$. Frequency = 99.1 kHz, $b/a = 2.03$.

gate was opened for 0.5 msec. The received signal was bandpass filtered with 3-dB-down points of the filters set at ± 10 kHz around the transmit frequency. The recorded data represent the peak level achieved by the pressure field during the gate period. The delay and gate were chosen to permit measurements which exclude initial and final transients. We believe the data represent good approximations to steady-state values. In order to provide a reference level, the scattering strength of one of the cylinders was measured under the same conditions. Fluctuations in this level of from 1 to 3 dB as a function of angle were observed, with the greatest variations occurring at 75 kHz. This phenomenon, which limits the accuracy of the two-cylinder measurements, is presumably due to scattering from the mounting apparatus.

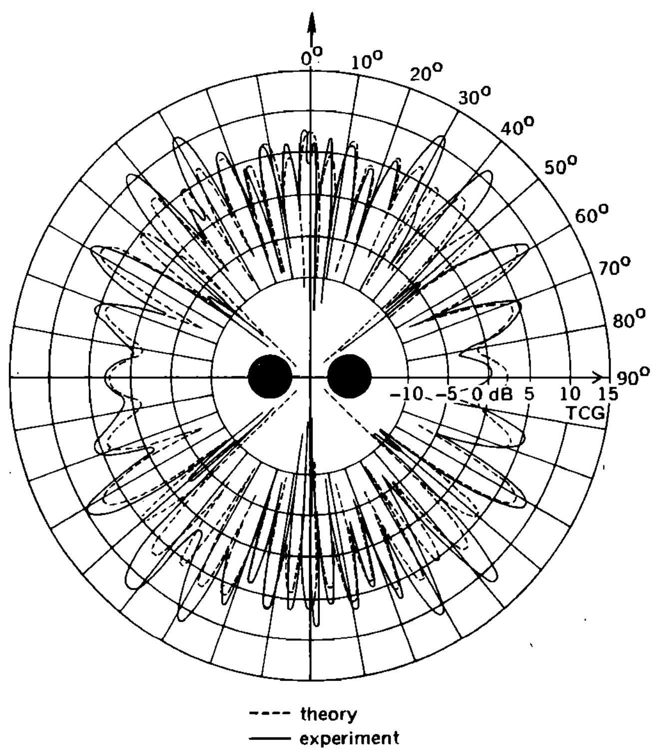

FIG. 4. Two-cylinder gain for $ka = 8.12$, $kb = 16.46$. Frequency = 74.3 kHz, $b/a = 2.03$.

III. RESULTS AND DISCUSSION

The experimental measurements and theoretical calculations for a b/a ratio of 2 (i.e., 4-in. separation) are compared in Figs. 3, 4, and 5.[15] In general, agreement is within 1 or 2 dB near broadside and endfire, with somewhat larger discrepancies in the intermediate-aspect regions. The values of ka and kb used in the computations correspond to a sound velocity of 1464 m/sec which is appropriate for fresh water at 10.8 °C. It was found that changes in these parameters of only 1% cause significant variations in the scattering patterns.

Two principal consequences of multiple scattering are evident in both the theoretical and experimental data. First, the influence of the shadow of one cylinder upon the other is apparent near "endfire" incidence. Simple theory, of course, does not predict the existence of a shadow. This can be seen clearly in Fig. 6, which compares the single- and multiple-scattering calculations for the 100-kHz case. For a b/a ratio of 2, the shadowing begins at an angle of incidence of 60°, and half of

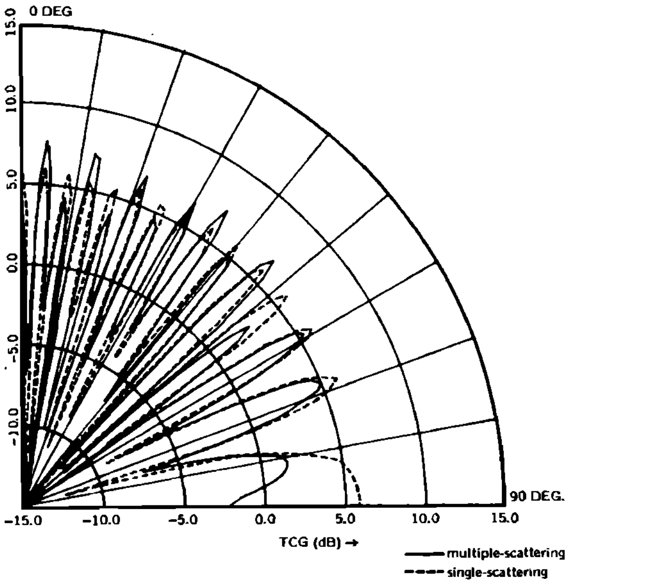

FIG. 6. Comparison of single- and multiple-scattering calculations for $ka=10.83$, $kb=21.94$, $b/a=2.03$.

the second cylinder is obscured at 76°. Beyond this point, the two-cylinder gain (TCG) drops toward zero but still shows variations of about ±2 dB versus angle and frequency. This indicates that some scattering from the second cylinder occurs even in the geometrical shadow region. Outside the shadow, the maximum TCG levels vary considerably from the uniform +6 dB predicted by simple theory. Peak locations calculated by the simple model are, however, quite accurate except within the shadow.

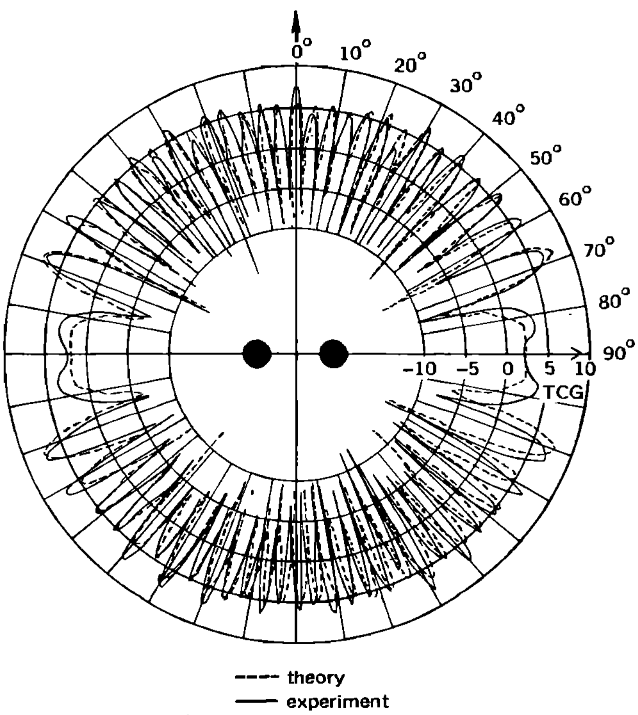

FIG. 7. Two-cylinder gain for $ka=5.49$, $kb=21.94$. Frequency = 50.04 kHz, $b/a=4.0$.

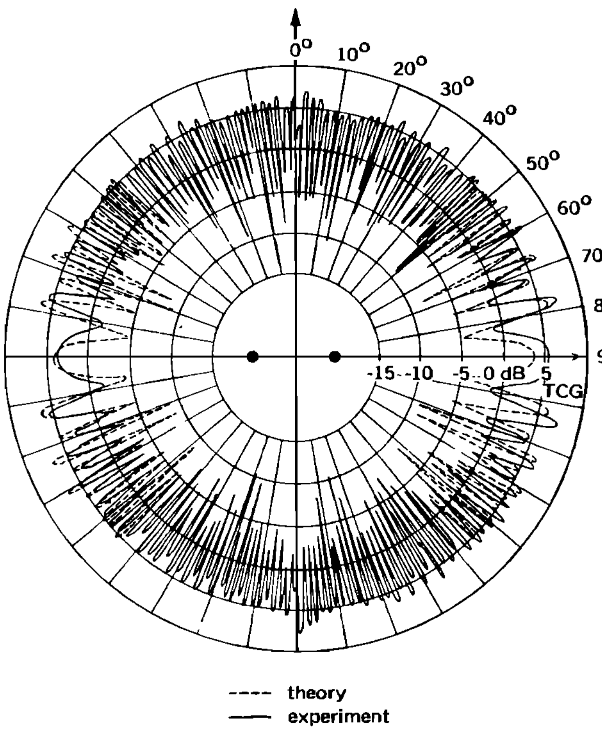

FIG. 8. Two-cylinder gain for $ka=5.46$, $kb=54.60$. Frequency = 50.05 kHz, $b/a=10$.

The effects of increasing b/a with ka fixed are shown in Figs. 7 and 8. The angular size of the shadow region is seen to decrease as expected, but there remain significant variations in the peak levels outside of the shadow. Thus, multiple scattering is not negligible even for a b/a ratio of 10.

IV. CONCLUSIONS

We have shown that it is possible to calculate the multiple-scattering interaction of two rigid cylinders in close proximity and achieve good agreement with experimental measurements. Both an iterative method and a direct matrix inversion technique can be successfully used, although the latter approach is more efficient for the cases we have considered. Either way, truncation of the infinite series at $N \approx 2ka$ is apparently justifiable for the ka and kb values we have considered. Inclusion of multiple scattering effects is necessary to predict correctly the levels of the interference peaks both in and out of the geometrical shadow region. The simple theory does, however, produce correct peak locations. The relative simplicity and modest cost of the two-cylinder calculations suggest that much larger problems may be easily tractable. Thus, it should be possible to include multiple-interaction effects in the analysis of scattering and radiation from sizable arrays of cylinders.

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[15]In Figs. 3, 4, 5, 7, and 8, the orientation and b/a ratios of the cylinders are indicated by the two solid circles.