Addition theorems for spherical wave solutions of the vector Helmholtz equation

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Addition theorems for spherical wave solutions of the vector Helmholtz equation are discussed. The theorems allow one to expand a vector spherical wave about a given origin into spherical waves about a shifted origin. A simplified derivation of the results obtained earlier by Cruzan [O. R. Cruzan, Q. Appl. Math. **20**, 33 (1962)] is presented.

I. INTRODUCTION

In many physical problems it is necessary to expand a multipole wave centered about a given origin into mulipole waves centered about a shifted origin. In this article we consider the expansion of spherical wave solutions of the vector Helmholtz equation. These are the well-known **L**, **M**, and **N** waves of electromagnetic theory.[1] The expansions are referred to as addition theorems since the expansion coefficients themselves satisfy the scalar wave equation.

Addition theorems for vector spherical waves have been found earlier by Stein[2] and by Cruzan.[3] They started from the addition theorem for scalar spherical waves,[4] which is recalled in Sec. II of this article. The addition theorems for the **L**, **M** and **N** waves are then derived by a tedious calculation in spherical coordinates. We sketch the procedure in Sec. III.

An independent derivation along the same lines for the special case when the origin is shifted in the z direction was given by Langbein,[5] who was led to a different form which is not related in an obvious manner to the results of Stein[2] and Cruzan.[3] Langbein's[5] expansion was generalized to arbitrary directions of the shift vector by Gérardy and Ausloos.[6] In the resulting form of the expansion the coefficients do not obviously satisfy the scalar wave equation. This makes the expansion of Ref. 6 less elegant and less satisfying from a theoretical point of view.

In Sec. IV of this article we show that addition theorems in the desired form may be derived straightforwardly and quickly from an extension of the scalar wave addition theorem to tensor multipole fields. The basic theorem was found by Danos and Maximon,[7] who derived an addition theorem for tensor multipole fields by coupling unit tensors to both sides of the scalar equation and using known quantum mechanical angular momentum algebra. We refer to their article for an interesting review of the history of the problem.

In Sec. V we compare in some detail with Langbein's[5] results. Throughout this article we adhere to the notation used by Edmonds.[8]

II. SCALAR WAVE ADDITION THEOREMS

The scalar wave equation

$$\nabla^2\psi + k^2\psi = 0 \tag{2.1}$$

has spherical wave solutions

$$\Psi_{lm}(\mathbf{r}) = j_l(kr)Y_{lm}(\theta,\varphi), \tag{2.2}$$

where $j_l(kr)$ is a regular spherical Bessel function and $Y_{lm}(\theta,\varphi)$ is a spherical harmonic. We wish to expand the solution $\Psi_{lm}(\mathbf{r})$ into spherical waves centered about a shifted origin. The expansion yields the simplest wave addition theorem. We consider the three vectors $\mathbf{r}, \boldsymbol{\rho}$, and $\mathbf{r}'$ related by

$$\mathbf{r} = \boldsymbol{\rho} + \mathbf{r}', \tag{2.3}$$

and expand the corresponding plane-wave identity

$$e^{i\mathbf{k}\cdot\mathbf{r}} = e^{i\mathbf{k}\cdot\boldsymbol{\rho}}e^{i\mathbf{k}\cdot\mathbf{r}'}, \tag{2.4}$$

into spherical waves using[9]

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi\sum_{lm} i^l\Psi_{lm}(\mathbf{r})Y^*_{lm}(\theta_k,\varphi_k). \tag{2.5}$$

Multiplying (2.4) by $Y_{lm}(\theta_k,\varphi_k)$ and integrating over the directions of $\mathbf{k}$ one obtains

$$\Psi_{lm}(\mathbf{r}) = \sum_{l'm'} A^{lm}_{l'm'}(\boldsymbol{\rho})\Psi_{l'm'}(\mathbf{r}'), \tag{2.6}$$

where

$$A^{lm}_{l',m'}(\boldsymbol{\rho}) = \sum_{\lambda\mu} c(lm|l'm'|\lambda\mu)\Psi_{\lambda\mu}(\boldsymbol{\rho}), \tag{2.7}$$

with coefficients

$$c(lm|l'm'|\lambda\mu) = i^{l'+\lambda-l}(-1)^m[4\pi(2l+1)(2l'+1)(2\lambda+1)]^{1/2} \times\begin{pmatrix} l & l' & \lambda\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & \lambda\\ -m & m' & \mu\end{pmatrix}. \tag{2.8}$$

The coefficients arise as integrals of products of three spherical harmonics[10]:

$$c(lm|l'm'|\lambda\mu) = 4\pi i^{l'+\lambda-l}\int Y_{lm}Y^*_{l'm'}Y^*_{\lambda\mu}\,d\Omega. \tag{2.9}$$

They are related to the coefficients in the expansion of a product of two associated Legendre functions:

$$P^{m'}_{l'}P^{\mu}_{\lambda} = \sum_l a(l'm'|\lambda\mu|l)P^{m'+\mu}_l \tag{2.10}$$

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by[3]

$$c(lm|l'm'|\lambda\mu) = i^{l'+\lambda-l}\left[4\pi\frac{(2l'+1)(2\lambda+1)}{2l+1}\right]^{1/2} \times\left[\frac{(l+m)!(l'-m')!(\lambda-\mu)!}{(l-m)!(l'+m')!(\lambda+\mu)!}\right]^{1/2} a(l'm'|\lambda\mu|l)\,, \tag{2.11}$$

where $m = m' + \mu$.

The addition theorem (2.6) may be generalized to singular solutions of the wave equation, which are the product of a singular spherical Bessel function and a spherical harmonic. In the following we generalize (2.2) to

$$\Psi_{lm}(\mathbf{r}) = f_l(kr)Y_{lm}(\theta,\varphi)\,, \tag{2.12}$$

where $f_l(\zeta)$ is any of the spherical Bessel functions $j_l(\zeta)$, $y_l(\zeta)$, $h_l^{(1)}(\zeta)$, or $h_l^{(2)}(\zeta)$. The regular solution (2.2) will be distinguished by a superscript: $\Psi_{lm}^{+}(\mathbf{r})$. The generalization of (2.6) then reads[7]

$$\Psi_{lm}(\mathbf{r}) = \sum_{\substack{l'm'\\ \lambda\mu}} c(lm|l'm'|\lambda\mu)\,\Psi_{l'm'}(\mathbf{r}_>)\Psi_{\lambda\mu}^{+}(\mathbf{r}_<)\,, \tag{2.13}$$

where $\mathbf{r}_<$ is the smaller and $\mathbf{r}_>$ is the larger of $\boldsymbol{\rho}$ and $\mathbf{r}'$. It is understood that $\Psi_{lm}(\mathbf{r})$ and $\Psi_{l'm'}(\mathbf{r}_>)$ contain spherical Bessel functions of the same type. Clearly, the expansion (2.13) may be written in two ways. We may either write

$$\Psi_{lm}(\mathbf{r}) = \sum_{l'm'} A_{l'm'}^{lm+}(\mathbf{r}_<)\Psi_{l'm'}(\mathbf{r}_>)\,, \tag{2.14}$$

or

$$\Psi_{lm}(\mathbf{r}) = \sum_{l'm'} A_{l'm'}^{lm}(\mathbf{r}_>)\Psi_{l'm'}^{+}(\mathbf{r}_<)\,, \tag{2.15}$$

where we have used the symmetry of the coefficients $c(lm|l'm'|\lambda\mu)$ in the pairs $(l'm')$ and $(\lambda\mu)$ which follows from (2.9).

III. VECTOR WAVE ADDITION THEOREMS

The vector wave equation

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \tag{3.1}$$

has spherical wave solutions that may be derived from scalar potentials which are solutions of the scalar wave equation. Thus one finds the three vector spherical waves[1]

$$\begin{aligned}\mathbf{L}_{JM}(\mathbf{r}) &= k^{-1}\nabla\Psi_{JM}(\mathbf{r})\,,\\ \mathbf{M}_{JM}(\mathbf{r}) &= \nabla\times(\mathbf{r}\Psi_{JM}(\mathbf{r}))\,,\\ \mathbf{N}_{JM}(\mathbf{r}) &= k^{-1}\nabla\times[\nabla\times(\mathbf{r}\Psi_{JM}(\mathbf{r}))]\,.\end{aligned} \tag{3.2}$$

The L wave is longitudinal and the M and N waves are transverse. The latter are related by

$$\mathbf{M}_{JM} = k^{-1}\nabla\times\mathbf{N}_{JM}, \quad \mathbf{N}_{JM} = k^{-1}\nabla\times\mathbf{M}_{JM}\,. \tag{3.3}$$

More explicitly, the solutions may be written as

$$\begin{aligned}\mathbf{L}_{JM}(\mathbf{r}) &= [1/(2J+1)]f_{J-1}(kr)\mathbf{A}_{JM}(\theta,\varphi)\\ &\quad + [1/(2J+1)]f_{J+1}(kr)\mathbf{B}_{JM}(\theta,\varphi)\,,\\ \mathbf{M}_{JM}(\mathbf{r}) &= f_J(kr)\mathbf{C}_{JM}(\theta,\varphi)\,,\\ \mathbf{N}_{JM}(\mathbf{r}) &= [(J+1)/(2J+1)]f_{J-1}(kr)\mathbf{A}_{JM}(\theta,\varphi)\\ &\quad - [J/(2J+1)]f_{J+1}(kr)\mathbf{B}_{JM}(\theta,\varphi)\,,\end{aligned} \tag{3.4}$$

where the vector spherical harmonics are given by

$$\begin{aligned}\mathbf{A}_{JM} &= JY_{JM}\mathbf{e}_r + \frac{\partial Y_{JM}}{\partial\theta}\mathbf{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_{JM}}{\partial\varphi}\mathbf{e}_\varphi\,,\\ \mathbf{B}_{JM} &= -(J+1)Y_{JM}\mathbf{e}_r + \frac{\partial Y_{JM}}{\partial\theta}\mathbf{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_{JM}}{\partial\varphi}\mathbf{e}_\varphi\,,\\ \mathbf{C}_{JM} &= \frac{1}{\sin\theta}\frac{\partial Y_{JM}}{\partial\varphi}\mathbf{e}_\theta - \frac{\partial Y_{JM}}{\partial\theta}\mathbf{e}_\varphi\,.\end{aligned} \tag{3.5}$$

The latter are related to the normalized vector spherical harmonics defined by Edmonds[11] by

$$\begin{aligned}\mathbf{A}_{JM} &= \sqrt{J(2J+1)}\,\mathbf{Y}_{J,J-1,M}\,,\\ \mathbf{B}_{JM} &= \sqrt{(J+1)(2J+1)}\,\mathbf{Y}_{J,J+1,M}\,,\\ \mathbf{C}_{JM} &= -i\sqrt{J(J+1)}\,\mathbf{Y}_{JJM}\,.\end{aligned} \tag{3.6}$$

The angular momentum operator for the vector waves is $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L} = -i\mathbf{r}\times\nabla$ and $\mathbf{S}$ is the spin operator for spin 1. The vector spherical waves (3.4) are eigenfunctions of J^2 with eigenvalue $J(J+1)$ and eigenfunctions of J_z with eigenvalue M.

By applying the operator $k^{-1}\nabla_>$ to (2.14) and noting that $\mathbf{r} = \mathbf{r}_< + \mathbf{r}_>$, one finds the addition theorem

$$\mathbf{L}_{JM}(\mathbf{r}) = \sum_{J'M'} A_{J'M'}^{JM+}(\mathbf{r}_<)\mathbf{L}_{J'M'}(\mathbf{r}_>)\,. \tag{3.7}$$

In the same manner, by applying the operator $k^{-1}\nabla_<$ to (2.15), one finds

$$\mathbf{L}_{JM}(\mathbf{r}) = \sum_{J'M'} A_{J'M'}^{JM}(\mathbf{r}_>)\mathbf{L}_{J'M'}^{+}(\mathbf{r}_<)\,. \tag{3.8}$$

A similar method may be used for the derivation of addition theorems for the M and N spherical waves. One starts from the definition of $\mathbf{M}_{JM}(\mathbf{r})$ in (3.2) and applies either the operator $\nabla_>\times(\mathbf{r}$ to (2.14) or the operator $\nabla_<\times(\mathbf{r}$ to (2.15). These procedures each yield an addition theorem for $\mathbf{M}_{JM}(\mathbf{r})$. The addition theorems for $\mathbf{N}_{JM}(\mathbf{r})$ then follow from (3.3). The first procedure leads to

$$\begin{aligned}\mathbf{M}_{JM}(\mathbf{r}) &= \sum_{J'M'}[F_{J'M'}^{JM+}(\mathbf{r}_<)\mathbf{M}_{J'M'}(\mathbf{r}_>) + G_{J'M'}^{JM+}(\mathbf{r}_<)\mathbf{N}_{J'M'}(\mathbf{r}_>)]\,,\\ \mathbf{N}_{JM}(\mathbf{r}) &= \sum_{J'M'}[G_{J'M'}^{JM+}(\mathbf{r}_<)\mathbf{M}_{J'M'}(\mathbf{r}_>) + F_{J'M'}^{JM+}(\mathbf{r}_<)\mathbf{N}_{J'M'}(\mathbf{r}_>)]\,.\end{aligned} \tag{3.9}$$

The second procedure yields

$$\begin{aligned}\mathbf{M}_{JM}(\mathbf{r}) &= \sum_{J'M'}[F_{J'M'}^{JM}(\mathbf{r}_>)\mathbf{M}_{J'M'}^{+}(\mathbf{r}_<) + G_{J'M'}^{JM}(\mathbf{r}_>)\mathbf{N}_{J'M'}^{+}(\mathbf{r}_<)]\,,\\ \mathbf{N}_{JM}(\mathbf{r}) &= \sum_{J'M'}[G_{J'M'}^{JM}(\mathbf{r}_>)\mathbf{M}_{J'M'}^{+}(\mathbf{r}_<) + F_{J'M'}^{JM}(\mathbf{r}_>)\mathbf{N}_{J'M'}^{+}(\mathbf{r}_<)]\,.\end{aligned} \tag{3.10}$$

Explicit expressions for the coefficient functions $F_{J'M'}^{JM}(\boldsymbol{\rho})$

and $G^{JM}_{J'M'}(\boldsymbol{\rho})$ may be found by a tedious calculation in spherical coordinates. The calculation was first performed by Stein[2] and by Cruzan.[3] An independent derivation for the case where $\boldsymbol{\rho}$ points in the z direction was given by Langbein.[5] Cruzan's[3] result for the coefficient function $F^{JM}_{J'M'}(\boldsymbol{\rho})$ may be written as

$$F^{JM}_{J'M'}(\boldsymbol{\rho}) = \sum_{\lambda\mu} f(JM|J'M'|\lambda\mu)\Psi_{\lambda\mu}(\boldsymbol{\rho}) \tag{3.11}$$

and his coefficients $f(JM|J'M'|\lambda\mu)$ may be cast in the form

$$f(JM|J'M'|\lambda\mu) = \frac{J(J+1)+J'(J'+1)-\lambda(\lambda+1)}{2J'(J'+1)} \times c(JM|J'M'|\lambda\mu). \tag{3.12}$$

Similarly, the coefficient function $G^{JM}_{J'M'}(\boldsymbol{\rho})$ may be written in the form

$$G^{JM}_{J'M'}(\boldsymbol{\rho}) = \sum_{\lambda\mu} g(JM|J'M'|\lambda\mu)\Psi_{\lambda\mu}(\boldsymbol{\rho}), \tag{3.13}$$

with coefficients

$$g(JM|J'M'|\lambda\mu) = \frac{[(J+J'-\lambda)(J+\lambda-J')(\lambda+J'+J+1)(\lambda+J'-J+1)]^{1/2}}{2J'(J'+1)} d(JM|J'M'|\lambda\mu), \tag{3.14}$$

where, in analogy to (2.8), the coefficients $d(JM|J'M'|\lambda\mu)$ are given by

$$d(JM|J'M'|\lambda\mu) = i^{J'+\lambda-J}(-1)^{M+1} \times[4\pi(2J+1)(2J'+1)(2\lambda+1)]^{1/2} \times\begin{pmatrix} J-1 & J' & \lambda \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} J & J' & \lambda \\ -M & M' & \mu\end{pmatrix}. \tag{3.15}$$

Using Edmonds'[8] relation (3.7.16) one sees that (3.14) differs in sign from Cruzan's[3] result. The sign error in Cruzan's result was also noted by other authors.[12,13]

The derivation of the results (3.9) and (3.10) following the method outlined above is lengthy and tedious. In Sec. IV we show how these results may be derived more quickly and more elegantly.

IV. IMPROVED DERIVATION

An improved derivation of the addition theorems for vector spherical waves may be based on the generalization of the scalar addition theorem (2.13) to tensor multipole fields as presented by Danos and Maximon.[7] Tensor wave fields are defined by coupling unit tensors to the scalar fields, for example,

$$\Psi^{[J]}_{lSM}(\mathbf{r}) = f_l(kr)\sum_{m's'}(lm'Ss'|lSJM)Y_{lm'}(\theta,\varphi)\hat{\mathbf{e}}^{[S]}_{s'} = f_l(kr)[\mathbf{Y}^{[l]}(\hat{\mathbf{r}})\times\hat{\mathbf{e}}^{[S]}]^{[J]}_M \tag{4.1}$$

in the notation of Fano and Racah.[14] By coupling unit tensors to both sides of the scalar addition theorem (2.13), one finds

$$\Psi^{[J]}_{lSM}(\mathbf{r}) = \sum_{\substack{J'l'M'\\ \lambda\mu}} c^{[S]}(JlM|J'l'M'|\lambda\mu) \times\Psi^{[J']}_{l'SM'}(\mathbf{r}_>)\Psi^+_{\lambda\mu}(\mathbf{r}_<), \tag{4.2}$$

with coefficients

$$c^{[S]}(JlM|J'l'M'|\lambda\mu) = i^{l'+\lambda-l}(-1)^{S-M} \times[4\pi(2J+1)(2l+1)(2J'+1) \times(2l'+1)(2\lambda+1)]^{1/2} \times\begin{pmatrix} l & l' & \lambda \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} J & J' & \lambda \\ -M & M' & \mu\end{pmatrix}\begin{Bmatrix} \lambda & J' & J \\ S & l & l'\end{Bmatrix} \tag{4.3}$$

(we have corrected the prefactor given by Danos and Maximon[7]). An alternative generalization of (2.13) is

$$\Psi^{[J]}_{lSM}(\mathbf{r}) = \sum_{\substack{J'l'M'\\ \lambda\mu}} c^{[S]}(JlM|J'l'M'|\lambda\mu) \times\Psi^{[J']+}_{l'SM'}(\mathbf{r}_<)\Psi_{\lambda\mu}(\mathbf{r}_>). \tag{4.4}$$

For $S=1$ the definition (4.1) becomes

$$\Psi^{[J]}_{l1M}(\mathbf{r}) \equiv \Psi_{JlM}(\mathbf{r}) = f_l(kr)\mathbf{Y}_{JlM}(\theta,\varphi) \tag{4.5}$$

and (4.4) becomes

$$\Psi_{JlM}(\mathbf{r}) = \sum_{\substack{J'l'M'\\ \lambda\mu}} c^{[1]}(JlM|J'l'M'|\lambda\mu) \times\Psi^+_{J'l'M'}(\mathbf{r}_<)\Psi_{\lambda\mu}(\mathbf{r}_>). \tag{4.6}$$

Here the orbital quantum numbers l and l' can take the values $J-1, J, J+1$ and $J'-1, J', J'+1$, respectively. It is clear from (3.4), (3.6), and (4.5) that $\Psi_{JlM}(\mathbf{r})$ is just a linear combination of $\mathbf{L}_{JM}(\mathbf{r})$, $\mathbf{M}_{JM}(\mathbf{r})$, and $\mathbf{N}_{JM}(\mathbf{r})$ waves. The addition theorems (3.8) and (3.10) therefore follow from (4.6) by simple operations with 3×3 matrices. We recall that the symmetry properties embodied in the form of the addition theorems (3.8) and (3.10) are due to the vector field relations (3.2) and (3.3). We may use these symmetry properties to simplify the expressions for the coefficients. Thus we find for the coefficient function $F^{JM}_{J'M'}(\boldsymbol{\rho})$,

$$F^{JM}_{J'M'}(\boldsymbol{\rho}) = \left[\frac{J(J+1)}{J'(J'+1)}\right]^{1/2} \times\sum_{\lambda\mu} c^{[1]}(JJM|J'J'M'|\lambda\mu)\Psi_{\lambda\mu}(\boldsymbol{\rho}) \tag{4.7}$$

and for the coefficient function $G^{JM}_{J'M'}(\boldsymbol{\rho})$,

$$G^{JM}_{J'M'}(\boldsymbol{\rho}) = i\left[\frac{J(2J+1)}{J'(J'+1)}\right]^{1/2} \times\sum_{\lambda\mu} c^{[1]}(J,J-1,M|J'J'M'|\lambda\mu)\Psi_{\lambda\mu}(\boldsymbol{\rho}). \tag{4.8}$$

Substituting (4.3) for $S=1$ and using expressions for the Wigner 6j symbols given by Edmonds,[15] we hence obtain the expressions (3.11) and (3.13).

V. DISCUSSION

An important feature of the addition theorems (3.9) and (3.10) is that the coefficient functions $F^{JM}_{J'M'}(\boldsymbol{\rho})$ and $G^{JM}_{J'M'}(\boldsymbol{\rho})$ themselves are solutions of the scalar wave equation. Langbein[5] has derived an addition theorem for vector spherical waves for the special case where $\boldsymbol{\rho}$ is directed along the z axis. His result (5.36) is of the form (3.10), with

$$F^{JM}_{J'M'}(-a\mathbf{e}_z) = f^M_{JJ'} V^{M*}_{JJ'}(ka)\delta_{MM'},\qquad G^{JM}_{J'M'}(-a\mathbf{e}_z) = f^M_{JJ'} W^{M*}_{JJ'}(ka)\delta_{MM'}, \tag{5.1}$$

with coefficients

$$F^M_{JJ'} = (-1)^{J+M} \times \left[(2J+1)(2J'+1)\frac{(J-M)!}{(J+M)!}\frac{(J'-M)!}{(J'+M)!}\right]^{1/2} \tag{5.2}$$

and functions

$$V^M_{JJ'}(\zeta) = U^M_{JJ'}(\zeta) - \{(J'-M+1)[(J'+1)(2J+1)]\}\zeta U^M_{J,J'+1}(\zeta) - \{(J'+M)/[J'(2J'+1)]\}\zeta U^M_{J,J'-1}(\zeta),$$

$$W^M_{JJ'}(\zeta) = iM[J'(J'+1)]^{-1}\zeta U^M_{JJ'}(\zeta), \tag{5.3}$$

with $U^M_{JJ'} = U^M_{J'J}$ and for $J \leqslant J'$,

$$U^M_{JJ'}(\zeta) = \left(\frac{2}{\zeta}\right)^M \sum_{\nu=0}^{J-M}(-1)^\nu \times \frac{\Gamma(J-\nu+\frac{1}{2})\Gamma(J'-\nu+\frac{1}{2})\Gamma(M+\nu+\frac{1}{2})}{\Gamma(J+J'-M-\nu+\frac{3}{2})\Gamma(M+\frac{1}{2})\Gamma(\frac{1}{2})} \times \frac{(J+J'-\nu)!}{(J-M-\nu)!(J'-M-\nu)!\nu!} \times (J+J'-M-2\nu+\tfrac{1}{2}) f_{J+J'-M-2\nu}(\zeta). \tag{5.4}$$

(We note that the functions V and W employed by Gérardy and Ausloos[6] are slightly different.)

Langbein's functions $U^M_{JJ'}(\zeta)$ are defined from his version of the scalar wave addition theorem. Comparing his Eq. (5.34) with (2.15) for $\boldsymbol{\rho} = -a\mathbf{e}_z$ we obtain

$$U^M_{JJ'}(\zeta) = [f^M_{JJ'}]^{-1}\sum_\lambda(-1)^\lambda\left(\frac{2\lambda+1}{4\pi}\right)^{1/2} \times c(JM|J'M|\lambda 0)f_\lambda(\zeta). \tag{5.5}$$

Cruzan's relations (16) and (19), when specialized to $\boldsymbol{\rho} = -a\mathbf{e}_z$, agree with (5.1) and (5.3) when (2.11) and the above expression (5.5) for $U^M_{JJ'}(\zeta)$ is used [except for the opposite sign in the function $W^M_{JJ'}(\zeta)$]. From (3.11) and (3.12) we find by specializing to $\boldsymbol{\rho} = -a\mathbf{e}_z$,

$$V^M_{JJ'}(\zeta) = [f^M_{JJ'}]^{-1}\sum_\lambda(-1)^\lambda\left[\frac{2\lambda+1}{4\pi}\right]^{1/2} \times f(JM|J'M|\lambda 0)f_\lambda(\zeta), \tag{5.6}$$

with $f^M_{JJ'}$ defined by (5.2) and $f(JM|J'M'|\lambda\mu)$ defined by (3.12).

The validity of (5.5) may be proven directly from (5.4). One first proves the identity for $M=J$ and then uses the recursion relation (40) derived by Gérardy and Ausloos[6] to obtain the relation for general values of M. Similarly it should be possible to show (5.6) directly from (5.3). The derivation of (5.5) and (5.6) from the addition theorem is more straightforward.

For general directions of the connecting vector $\boldsymbol{\rho}$ the coefficient functions appearing in the addition theorems (3.9) and (3.10) have the desirable property that they are solutions of the scalar wave equation. Gérardy and Ausloos[6] have generalized Langbein's[5] addition theorem to arbitrary direction of $\boldsymbol{\rho}$ by performing a rotation of axes. This more general form is complicated and it is not evident that the coefficient functions satisfy the scalar wave equation.

In conclusion, we note that the addition theorems (3.7)–(3.10) may be used to derive similar theorems for the solutions of the equations of linearized hydrodynamics[16] and elasticity.[17]

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