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Higher-Order Approximations in Multiple Scattering.* I. Two-Dimensional Scalar Case

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A formula is derived which expresses the perturbed scattering amplitudes of a combination of two arbitrary cylinders as a function of the unperturbed scattering amplitudes of the individual cylinders. The formula is valid when the spacing of the scatterers is large compared to their dimensions. It involves derivatives of the scattering amplitudes with respect to the angles of incidence and of observation. Interaction terms of degrees $d^{-\frac{1}{2}}$, d^{-1}, and $d^{-\frac{3}{2}}$ are taken into account, where d is the spacing. Verification is obtained in a special case. The result is employed to calculate the total scattering cross section.

1. INTRODUCTION

THE present paper deals with the diffraction of plane electromagnetic or acoustic waves by a pair of parallel cylinders of arbitrary shape. The diffraction by each cylinder, in isolation, is assumed known, and the diffraction by the configuration is calculated explicitly, in terms of these data. An approximation is involved which will be described below.

The question of multiple scattering has already been treated, but in less detail, by a number of writers. A brief sketch of the history of the problem follows, with emphasis on those treatments of the problem whose accuracy increases with the spacing. We recall the work of Reiche and Schaefer,[1] who were the first to have given a wave theoretical discussion of the finite grating of circular cylinders. These authors neglected the interaction between the cylinders; their work was therefore valid, in principle, in the limit of large spacing. A very general expression for the diffraction by an arbitrary assemblage of circular cylinders was given by Twersky,[2] who took all orders of interaction into account. The method depended heavily upon the separability of the circular geometry, and the most general form of the result was too complicated to be discussed. However, Twersky found that, if he proceeded to the limit of large spacing, he could simplify his result immensely and achieve a perspicuous discussion of the correction to single scattering. This work gave a correct account of terms of degree $d^{-\frac{1}{2}}$ and d^{-1} in the spacing.

This success led to a general investigation of the large spacing approximation by Karp,[3] who showed that for cylinders of arbitrary shape, the leading terms of the *interaction correction* could themselves be expressed explicitly in terms of *noninteraction* or *single scattering* results, the latter being regarded as given. In fact, the interaction term could be regarded as being composed of the response of each cylinder to a plane wave arriving from the direction of the other cylinders. The techniques of Karp[3] were exploited by Karp and Radlow[4] and by Karp[5] in the analysis of a grating of cylinders. Similar methods were used by Karp and Russek[6] in expressing the approximate solution to the problem of diffraction by a wide slit in terms of the well-known solution for the half-plane problem.

The purpose of the present paper is to extend the work of Karp[3] so as to take into account higher order terms. Just as in reference 3, the cylinders are arbitrary, and the scattering by each cylinder in isolation is assumed as given. But, it is found[7] that even the higher order correction terms can be calculated generally, simply, and explicitly in terms of the single scattering data used in reference 3 for calculation of the leading terms. This is the principal result of the present paper.

The general calculation was carried out so as to include all effects of order of magnitude $d^{-\frac{1}{2}}$, d^{-1}, and $d^{-\frac{3}{2}}$, where d is the spacing of the cylinders. For purpose of comparison, Twersky's calculation for a pair of circular cylinders was continued so as to include terms of order $d^{-\frac{3}{2}}$; this special calculation by the repeated application of additional theorems was then shown to agree with the general result of the present work, when the latter result is specialized to the case of circular cylinders. The result is used to calculate the total cross section for a pair of circular cylinders as a function of the spacing and the known unperturbed (or noninteraction) scattering amplitude functions.

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2. STATEMENT OF THE PROBLEM AND OF THE METHOD OF ANALYSIS

We would like to know the scattering pattern of a combination of two infinite parallel cylinders in terms of the scattering patterns which these cylinders would have if they were isolated from each other. In other words, we want to obtain a functional relationship between the unperturbed and the perturbed scattering patterns of the cylinders. Such a relation is desirable because it simplifies the calculation of the pattern for the combination. If we can calculate the unperturbed patterns, we need only insert them in this relation to obtain the perturbed patterns. The relation is useful, moreover, even if the shapes of the cylinders are so complicated that we cannot separate variables or if calculation by separation of variables is too tedious.[8] Also, in such cases, the unperturbed patterns might be measured experimentally for all angles of observation and these unperturbed patterns might then be substituted into the relation obtained here to yield the perturbed patterns.

The situation is the following: A plane wave of unit amplitude is incident upon the two parallel cylinders A and B (Fig. 1).

To avoid ambiguity in the definition of the spacing d, we define a coordinate system for each cylinder. Let A' and B' be circular cylinders circumscribed about A and B, respectively. Let a and b be the respective radii of A' and B'. We shall let Z_a be the axis of A' and Z_b be the axis of B'. The problem is two dimensional, and we shall operate in a plane perpendicular to the Z axes. The respective intersections of this plane with the Z_a and Z_b axes will then be the origins of the coordinate systems of A and B. We can now define the spacing d as the distance between the axes of A' and B', that is, the distance between the two origins (Fig. 1).

We make the following assumptions:

(1) The individual complex scattering pattern is known when each cylinder stands alone in space.

(2) $d \gg \lambda$ where λ is the wavelength of the incident wave.[9]

(3) $d \gg a$ and $d \gg b$.

Our object is to find a functional relation between the scattering pattern of the combination and those of the isolated component cylinders. This cannot be accomplished by simple superposition of the unperturbed patterns of the components since the individual pattern of each component cylinder is modified by the field scattered by the other cylinder. We must, therefore, consider the interaction.

[8] See reference 4 for the use of an even less accurate relation for theoretical purposes.

[9] Numerical calculation in reference 5, which are based on a less accurate procedure, showed that d could be as small as one wavelength without impairing the accuracy materially. The present result would, therefore, allow an even smaller spacing, provided the cylinders are kept sufficiently small compared to the spacing.

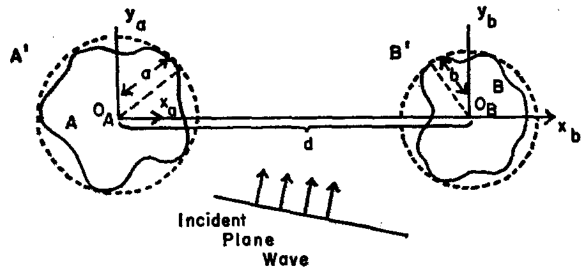

FIG. 1. Plane wave incident upon two parallel cylinders.

We shall assume that the response U_s of each cylinder to a plane wave is of the form

$$U_s = \frac{e^{ikr}}{r^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{f_n(\theta,\theta_0)}{r^n} \quad \text{for large } r, \tag{1}$$

where r is the distance from the axis of the circumscribed circular cylinder, θ is the angle of observation, and θ_0 is the angle of incidence. Both θ and θ_0 are measured from the x axis of each cylinder. The x axes are collinear with d.

As will be explained below, at a sufficiently large distance from the cylinder, the scattered field U_s resembles a plane wave. This approximate plane wave elicits a response from the other cylinder, perturbing its scattered field. This response also has the form (1) and in turn perturbs the field scattered by the first cylinder. We can carry out successive calculations for this process until the desired order of accuracy is obtained. When the perturbed patterns have been calculated, they can be superposed.

We shall deal, in this paper, with interaction terms of degrees $d^{-\frac{1}{2}}$, d^{-1}, and $d^{-\frac{3}{2}}$. The procedure involves a new kind of expansion of the waves scattered by each cylinder about the origin located in the other cylinder. In order to ensure that all terms up to order $d^{-\frac{3}{2}}$ are contained in the result, we must include them in the first expansion. We then find that the field scattered by a given cylinder can be represented, in the neighborhood of the other cylinder, as a plane wave, plus additional terms which are recognized as derivatives of a plane wave with respect to its angle of incidence. The simple way of expressing the higher-order excitations is what enables us to calculate the higher-order responses conveniently.

3. EXPANSION OF THE SCATTERED WAVES IN TERMS OF PLANE WAVES

A. Expansion of the Response of Cylinder A in a Neighborhood of Cylinder B

Let us consider what happens when a plane wave is incident upon A. The wave function for a plane wave (Fig. 2) is

$$U_i = \exp[ik(x_a \cos\theta_0 + y_a \sin\theta_0)]. \tag{2}$$

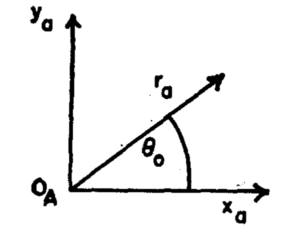

FIG. 2. Coordinate system for cylinder A.

The wave scattered by A in response to the plane wave is represented by an asymptotic solution of the reduced wave equation.

$$\text{D.E.} \qquad (\Delta+k^2)U=0. \tag{3}$$

The boundary conditions are such that the diffraction problem for each cylinder, and for the combination, is well posed. They may be, for example,

$$\text{B.C.} \qquad \begin{cases} \text{(a)} \ U=0 \\ \text{or (b)} \ \partial U/\partial n=0 \\ \text{or (c)} \ CU+D(\partial U/\partial n)=0, \end{cases} \tag{4}$$

where $\partial U/\partial n$ is the normal derivative of U, and C and D are constants. Alternatively, one or both of the cylinders may be filled with dielectric materials.

The radiation condition for an outgoing wave, i.e.,

$$\lim_{r\to\infty} r^{\frac{1}{2}}\left[\frac{\partial U}{\partial r}-ikU\right]=0, \tag{5}$$

is imposed on all scattered fields which occur.

The solution of (3) has the form

$$U=U_i+U_a, \tag{6}$$

where U_i is the incident field and U_a is the field scattered by cylinder A. We assume that U_a may be represented in the asymptotic form

$$U_a=\frac{e^{ikr_a}}{(r_a)^{\frac{1}{2}}}\left[f_0^{a0}(\theta_a,\theta_0)+\frac{f_1^{a0}}{r_a}(\theta_a,\theta_0)+\mathcal{O}(r_a^{-2})\right]. \tag{7}$$

The letter "a" signifies that the variable in question refers to cylinder A, and the superscript "$a0$" signifies that the pattern is unperturbed. If, as we have assumed, B is sufficiently small in relation to its distance from A, then the wave scattered by A is practically a plane wave in the neighborhood of B, and we may imagine such a wave incident on B. We can demonstrate an explicit representation of this approximately plane wave by expanding U_a in a neighborhood of B. We shall express the expansion in powers of $d^{-\frac{1}{2}}$ in a rectangular coordinate system with origin at the center of B' and carry out the calculation up to order $d^{-\frac{3}{2}}$. The calculation proceeds as follows. Let P be a point in the neighborhood of B (Fig. 3). Let r_a = the distance from the axis of A' to P. We see from Fig. 3 that $r_a=[(d+x_b)^2+y_b^2]^{\frac{1}{2}}$, and we shall henceforth omit the subscripts "a", "b" as a matter of convenience. Then

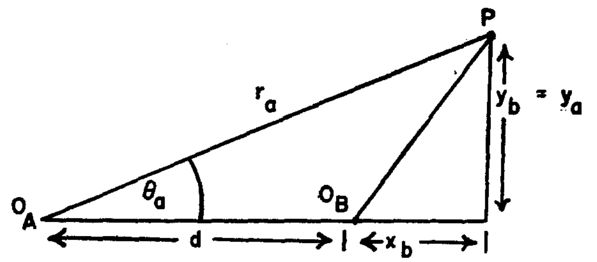

FIG. 3. Coordinate system for expansion in a neighborhood of cylinder B.

$$e^{ikr}=e^{ik(d+x)}\left[1+\frac{iky^2}{2d}+\frac{ikxy^2-\frac{1}{4}k^2y^4}{d^2}+\mathcal{O}(d^{-3})\right], \tag{8}$$

$$r^{-\frac{1}{2}}=(d^2+2xd+x^2+y^2)^{-\frac{1}{4}} = (1/d^{\frac{1}{2}})-(x/2d^{\frac{3}{2}})+\mathcal{O}(d^{-\frac{5}{2}}), \tag{9}$$

$$r^{-\frac{3}{2}}=d^{-\frac{3}{2}}[1-\tfrac{3}{2}(x/d)+\mathcal{O}(d^{-2})] = (1/d^{\frac{3}{2}})+\mathcal{O}(d^{-\frac{5}{2}}), \tag{10}$$

$$\theta_a=\arctan(y/d+x) = (y/d)-(xy/d^2)+\mathcal{O}(d^{-3}), \tag{10a}$$

$$f_n^{a0}(\theta,\theta_0)=f_n^{a0}(0,\theta_0)+[D_\theta f_n^{a0}(0,\theta_0)]\theta +[D_\theta^2 f_n^{a0}(0,\theta_0)](\theta^2/2)+\cdots, \tag{10b}$$

where $D_\theta\equiv\partial/\partial\theta$. By using (10a), we can rewrite (10b) in the form

$$f_n^{a0}(\theta,\theta_0)=f_n^{a0}(0,\theta)+[D_\theta f_n^{a0}(0,\theta_0)](y/d) -[D_\theta f_n^{a0}(0,\theta_0)](xy/d^2) +[D_\theta^2 f_n^{a0}(0,\theta_0)](y^2/2d^2)+\mathcal{O}(d^{-3}). \tag{11}$$

On substituting Eqs. (8)–(11) into (7), we obtain

$$U_a^0=e^{ik(d+x)}\left[1+\frac{iky^2}{2d}+\frac{ikxy^2-\frac{1}{4}k^2y^4}{d^2}+\mathcal{O}(d^{-3})\right] \times\left\{\begin{array}{l}\left[\dfrac{1}{d^{\frac{1}{2}}}-\dfrac{x}{2d^{\frac{3}{2}}}+\mathcal{O}(d^{-\frac{5}{2}})\right] \\ \quad[f_0^{a0}(0,\theta_0)+D_\theta f_0^{a0}(0,\theta_0)(y/d)+\mathcal{O}(d^{-2})] \\ \qquad +\dfrac{1}{d^{\frac{3}{2}}}[f_1^{a0}(0,\theta_0)+\mathcal{O}(d^{-1})]\end{array}\right.$$

$$U_a^0=e^{ik(d+x)}\left[\frac{f_0^{a0}(0,\theta_0)}{d^{\frac{1}{2}}} +\frac{\frac{1}{2}(-x+iky^2)f_0^{a0}(0,\theta_0)+yD_\theta f_0^{a0}(0,\theta_0)+f_1^{a0}(0,\theta_0)}{d^{\frac{3}{2}}} +\mathcal{O}(d^{-\frac{5}{2}})\right]. \tag{12}$$

We have thus obtained the field scattered by A as it appears in a neighborhood of B. We must now carry out the corresponding expansion for the field initially scattered by B. Note that the leading term of (12) is a plane wave, i.e., a constant multiple of e^{ikx}, as we explained earlier.

B. Expansion in a Neighborhood of A for the Wave Scattered by B

The same type of procedure as that used above gives us the expansion in a neighborhood of A of the wave initially scattered by B. The expression here corresponding to (12) is

$$U_b{}^0=e^{ik(d-x)}\left[\frac{f_0{}^{b0}(\pi,\theta_0)}{d^{\frac{1}{2}}}+\frac{\frac{1}{2}(x+iky^2)f_0{}^{b0}(\pi,\theta_0)-yD_\theta f_0{}^{b0}(\pi,\theta_0)+f_1{}^{b0}(\pi,\theta_0)}{d^{\frac{3}{2}}}+O(d^{-\frac{5}{2}})\right]. \quad (13)$$

The unperturbed fields scattered by either cylinder, (12) or (13), are the excitations of the other cylinder. We assumed, to begin with, that we knew the unperturbed response of each cylinder to a plane-wave excitation for all angles of incidence. If (12) and (13) were plane waves, we could calculate the responses for the second scattering.[10] (12) and (13), however, are not plane waves, but this impediment does not prevent the calculation of the effect of further scattering. The reason is that (12) and (13) may be represented in terms of plane waves by appropriate substitutions. This representation, which will enable us to calculate successive scattering, will now be given.

C. Expression of the Scattered Waves in Terms of Plane Waves

We can reduce the further scattering of singly-scattered waves to the scattering of plane waves by expressing (12) and (13) in terms of plane waves and derivatives of plane waves. On noting that a plane wave is represented by

$$v(\theta_0)=\exp[ik(x\cos\theta_0+y\sin\theta_0)], \quad (14)$$

we observe that

$$ikye^{ikx}=v_{\theta_0}(0) \quad (15)$$

$$-ikye^{-ikx}=v_{\theta_0}(\pi) \quad (16)$$

$$ik(-x+iky^2)e^{ikx}=v_{\theta_0\theta_0}(0) \quad (17)$$

$$ik(x+iky^2)e^{-ikx}=v_{\theta_0\theta_0}(\pi). \quad (18)$$

Substitution of (15) and (17) in (12), and of (16) and (18) in (13) yields the following representation of the scattered fields in terms of plane waves:

$$U_a{}^0=e^{ikd}\left[\frac{v(0)f_0{}^{a0}(0,\theta_0)}{d^{\frac{1}{2}}}+\frac{(1/2ik)[D_{\theta_0}{}^2v(0)]f_0{}^{a0}(0,\theta_0)+(1/ik)[D_{\theta_0}v(0)]D_\theta f_0{}^{a0}(0,\theta_0)+v(0)f_1{}^{a0}(0,\theta_0)}{d^{\frac{3}{2}}}+O(d^{-\frac{5}{2}})\right] \quad (19)$$

$$U_b{}^0=e^{ikd}\left[\frac{v(\pi)f_0{}^{b0}(\pi,\theta_0)}{d^{\frac{1}{2}}}+\frac{(1/2ik)[D_{\theta_0}{}^2v(\pi)]f_0{}^{b0}(\pi,\theta_0)+(1/ik)[D_{\theta_0}v(\pi)]D_\theta f_0{}^{b0}(\pi,\theta_0)+v(\pi)f_1{}^{b0}(\pi,\theta_0)}{d^{\frac{3}{2}}}+O(d^{-\frac{5}{2}})\right]. \quad (20)$$

D. Elimination of f_1

We note that the numerators of (19) and (20) are sums of terms, consisting of plane waves and their derivatives, namely, the v's, and coefficients which are independent of θ, namely, the f's. We see that these formulas are expressed in terms of both f_0 and f_1. (For the meaning of f_0 and f_1 see reference 7.) An advantage would result from the elimination of f_1, since we could then express the result in terms of the scattering amplitude of the far field without having to know the scattering amplitudes of further asymptotic terms.

We eliminate f_1 by expressing it in terms of f_0. This can be done by means of a recursion formula. The recursion is obtained by substitution into (3) of the assumed asymptotic[11] representation (7) of any radiating solution of the reduced-wave equation. When we equate the corresponding inverse powers of r, we find that

$$f_n=(1/2ikn)[(n-\tfrac{1}{2})^2f_{n-1}+D_\theta{}^2f_{n-1}]. \quad (21)$$

This recursion is useful, also, for calculations of higher degree than we are considering here. Since we want to express f_1 in terms of f_0, we need use it only for the value $n=1$.

$$f_1=(1/2ik)[\tfrac{1}{4}f_0+D_\theta{}^2f_0]. \quad (22)$$

This is a different type of recursion from that obtained for large k by Keller, Lewis, and Seckler[12] although it is similar in form.

If we now substitute (22) into (19) and (20), we obtain

[10] This would be the method used in reference 3.

[11] The representation is asymptotic for large r, θ, and k being held fixed.

[12] J. B. Keller, R. M. Lewis, and B. D. Seckler, Communs. Pure and Appl. Math. 9, 207 (1956).

$$U_a{}^0=e^{ikd}\left[\frac{v(0)f_0{}^{a0}(0,\theta_0)}{d^{\frac{1}{2}}}+\frac{(1/2ik)[D_{\theta_0}{}^2v(0)]f_0{}^{a0}(0,\theta_0)+(1/ik)[D_{\theta_0}v(0)]D_\theta f_0{}^{a0}(0,\theta_0)+(1/2ik)v(0)[\frac{1}{4}f_0{}^{a0}(0,\theta_0)+D_\theta{}^2f_0{}^{a0}(0,\theta_0)]}{d^{\frac{3}{2}}}+\mathcal{O}(d^{-\frac{5}{2}})\right],\tag{23}$$

$$U_b{}^0=e^{ikd}\left[\frac{v(\pi)f_0{}^{b0}(\pi,\theta_0)}{d^{\frac{1}{2}}}+\frac{(1/2ik)[D_{\theta_0}{}^2v(\pi)]f_0{}^{b0}(\pi,\theta_0)+(1/ik)[D_{\theta_0}v(\pi)]D_\theta f_0{}^{b0}(\pi,\theta_0)+(1/2ik)v(\pi)[\frac{1}{4}f_0{}^{b0}(\pi,\theta_0)+D_\theta{}^2f_0{}^{b0}(\pi,\theta_0)]}{d^{\frac{3}{2}}}+\mathcal{O}(d^{-\frac{5}{2}})\right].\tag{24}$$

Expressions (23) and (24) represent the responses of the cylinders to the original incident plane wave and these responses are given near the other cylinder in terms of the plane waves. Expressions (23) and (24) are also excitations for the second scattering. Successive application of these formulas will yield the desired degree of interaction.

4. CALCULATION OF THE INTERACTION

We have expressed in (23) and (24), the fields singly scattered by A and by B, as they appear in a neighborhood of the second scatterer and have, moreover, expressed them in terms of plane waves and derivatives of plane waves. We may now imagine this combination of plane waves and their derivatives to be incident upon the second scatterer. The linearity of these expressions enables us to say that the responses of A and B to incident derivatives of plane waves are equal to the derivatives of the responses of A and B to the incident plane waves. Since we already know, by assumption, the unperturbed responses of A and B to an incident plane wave, we have reduced the second scattering to the previous case, namely, the first scattering.

If we carry out this process to the extent of three successive scatterings, we can obtain interaction terms of degrees $d^{-\frac{1}{2}}$, d^{-1}, and $d^{-\frac{3}{2}}$. The terms of degree $d^{-\frac{1}{2}}$ result from double scattering, those of degree d^{-1} result from triple scattering, and those of degree $d^{-\frac{3}{2}}$ result partly from double and partly from quadruple scattering.

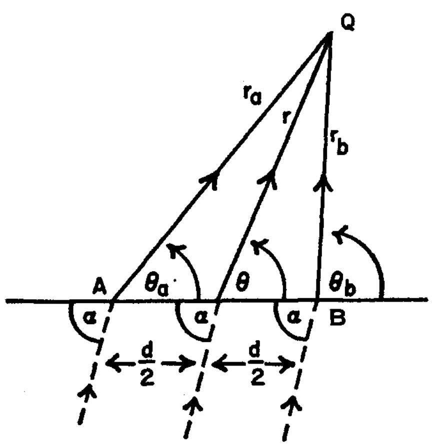

FIG. 4. Normalized coordinate systems.

For the sake of simplicity, the following results will be expressed in a normalized coordinate system (Fig. 4) with a common origin midway between the origins of the coordinate systems located in the scatterers. As a matter of convenience, we shall omit the subscript zero from the f's.

Let α be the angle of incidence of the original plane wave. The perturbed patterns will then be

$$\begin{aligned}f^a(\theta,\alpha)=&e^{-ik\frac{1}{2}d\cos\alpha}f^{a0}(\theta,\alpha)\\&+\frac{e^{ik\frac{1}{2}d\cos\alpha}e^{ikd}f^{b0}(\pi,\alpha)f^{a0}(\theta,\pi)}{d^{\frac{1}{2}}}\\&+\frac{e^{-ik\frac{1}{2}d\cos\alpha}e^{ik2d}f^{a0}(0,\alpha)f^{b0}(\pi,0)f^{a0}(\theta,\pi)}{d}\\&+\frac{e^{ik\frac{1}{2}d\cos\alpha}e^{ik3d}}{d^{\frac{3}{2}}}f^{b0}(\pi,\alpha)f^{a0}(0,\pi)f^{b0}(\pi,0)f^{a0}(\theta,\pi)\\&+\frac{e^{ik\frac{1}{2}d\cos\alpha}e^{ikd}}{d^{\frac{3}{2}}2ik}(f^{b0}(\pi,\alpha)D_{\theta_0}{}^2f^{a0}(\theta,\pi)\\&+2D_\theta f^{b0}(\pi,\alpha)D_{\theta_0}f^{a0}(\theta,\pi)+[\tfrac{1}{4}f^{b0}(\pi,\alpha)\\&\qquad+D_\theta{}^2f^{b0}(\pi\alpha)]f^{a0}(\theta,\pi))+\mathcal{O}(d^{-\frac{5}{2}}),\end{aligned}\tag{25}$$

and

$$\begin{aligned}f^b(\theta,\alpha)=&e^{ik\frac{1}{2}d\cos\alpha}f^{b0}(\theta,\alpha)+\frac{e^{-ik\frac{1}{2}d\cos\alpha}e^{ikd}}{d^{\frac{1}{2}}}f^{a0}(0,\alpha)f^{b0}(\theta,0)\\&+\frac{e^{ik\frac{1}{2}d\cos\alpha}e^{ik2d}}{d}f^{b0}(\pi,\alpha)f^{a0}(0,\pi)f^{b0}(\theta,0)\\&+\frac{e^{-ik\frac{1}{2}d\cos\alpha}e^{ik3d}}{d^{\frac{3}{2}}}f^{a0}(0,\alpha)f^{b0}(\pi,0)f^{a0}(0,\pi)f^{b0}(\theta,0)\\&+\frac{e^{-ik\frac{1}{2}d\cos\alpha}e^{ikd}}{d^{\frac{3}{2}}2ik}(f^{a0}(0,\alpha)D_{\theta_0}{}^2f^{b0}(\theta,0)\\&+2D_\theta f^{a0}(0,\alpha)D_{\theta_0}f^{b0}(\theta,0)+[\tfrac{1}{4}f^{a0}(0,\alpha)\\&\qquad+D_\theta{}^2f^{a0}(0,\alpha)]f^{b0}(\theta,0))+\mathcal{O}(d^{-\frac{5}{2}}).\end{aligned}\tag{26}$$

We note that the successive powers of $d^{-\frac{1}{2}}$ in (25) and (26) represent the various degrees of interaction. The scattered far fields resulting from the interaction will have the form

$$U_a=\frac{\exp[ik(r+\frac{1}{2}d\cos\theta)]}{r^{\frac{1}{2}}}f^a(\theta,\alpha), \qquad (27)$$

and

$$U_b=\frac{\exp[ik(r-\frac{1}{2}d\cos\theta)]}{r^{\frac{1}{2}}}f^b(\theta,\alpha), \qquad (28)$$

where $f^a(\theta,\alpha)$ and $f^b(\theta,\alpha)$ are the perturbed scattering amplitudes given by (25) and (26). The sum of the scattered fields has the following far-field representation which is the sum of (27) and (28).

$$U_s=U_a+U_b=(e^{ikr}/r^{\frac{1}{2}})(2/\pi k)^{\frac{1}{2}}e^{-i(\pi/4)}F(\theta,\alpha), \qquad (29)$$

where $F(\theta,\alpha)$ is the scattering amplitude of the combination of cylinders. We see by comparison with (27) and (28) that

$$F(\theta,\alpha)=(\pi k/2)^{\frac{1}{2}}e^{i(\pi/4)}[\exp(ik\tfrac{1}{2}d\cos\theta)f^a(\theta,\alpha)+\exp(-ik\tfrac{1}{2}d\cos\theta)f^b(\theta,\alpha)]. \qquad (30)$$

A more explicit formula for the far field is obtained by combining (25), (26), (29), and (30). The far field can then be written as follows:

$$u\cong\frac{e^{ikr}}{r^{\frac{1}{2}}}\Bigg\{\exp[ik\tfrac{1}{2}d(\cos\theta-\cos\alpha)]f^{a0}(\theta,\alpha)+\exp[-ik\tfrac{1}{2}d(\cos\theta-\cos\alpha)]f^{b0}(\theta,\alpha)$$
$$+\frac{e^{ikd}}{d^{\frac{1}{2}}}[\exp[ik\tfrac{1}{2}d(\cos\theta+\cos\alpha)]f^{b0}(\pi,\alpha)f^{a0}(\theta,\pi)+\exp[-ik\tfrac{1}{2}d(\cos\theta+\cos\alpha)]f^{a0}(0,\alpha)f^{b0}(\theta,0)]$$
$$+\frac{e^{ik2d}}{d}[\exp[ik\tfrac{1}{2}d(\cos\theta-\cos\alpha)]f^{a0}(0,\alpha)f^{b0}(\pi,0)f^{a0}(\theta,\pi)+\exp[-ik\tfrac{1}{2}d(\cos\theta-\cos\alpha)]f^{b0}(\pi,\alpha)f^{a0}(0,\pi)f^{b0}(\theta,0)]$$
$$+\frac{e^{ik3d}}{d^{\frac{3}{2}}}[\exp[ik\tfrac{1}{2}d(\cos\theta+\cos\alpha)]f^{b0}(\pi,\alpha)f^{a0}(0,\pi)f^{b0}(\pi,0)f^{a0}(\theta,\pi)$$
$$+\exp[-ik\tfrac{1}{2}d(\cos\theta+\cos\alpha)]f^{a0}(0,\alpha)f^{b0}(\pi,0)f^{a0}(0,\pi)f^{b0}(\theta,0)]$$
$$+\frac{e^{ikd}}{2ikd^{\frac{3}{2}}}\left[\begin{array}{l}\exp[ik\tfrac{1}{2}d(\cos\theta+\cos\alpha)]\left(\begin{array}{r}f^{b0}(\pi,\alpha)D_{\theta0}{}^2f^{a0}(\theta,\pi)+2D_\theta f^{b0}(\pi,\alpha)D_{\theta0}f^{a0}(\theta,\pi)\\+\{\tfrac{1}{4}f^{b0}(\pi,\alpha)+D_\theta{}^2f^{b0}(\pi,\alpha)\}f^{a0}(\theta,\pi)\end{array}\right)\\+\exp[-ik\tfrac{1}{2}d(\cos\theta+\cos\alpha)]\left(\begin{array}{r}f^{a0}(0,\alpha)D_{\theta0}{}^2f^{b0}(\theta,0)+2D_\theta f^{a0}(0,\alpha)D_{\theta0}f^{b0}(\theta,0)\\+\{\tfrac{1}{4}f^{a0}(0,\alpha)+D_\theta{}^2f^{a0}(0,\alpha)\}f^{b0}(\theta,0)\end{array}\right)\end{array}\right]+O(d^{-\frac{5}{2}})\Bigg\}. \qquad (31)$$

We have presented in (31) a relation between the scattered far field of the combination and those of the component cylinders. We wish to point out that a similar relation holds between the corresponding fields at *all* points of space. But this relation will not be detailed here.

5. SPECIAL CASE. SCATTERING BY TWO CONDUCTING CIRCULAR CYLINDERS

The abstract relations obtained above can be verified in the special case of scattering by two parallel, arbitrary circular cylinders A and B with corresponding sets of parameters a and b. For a single cylinder, say A, we have

$$U_a{}^0=\sum_{n=-\infty}^{\infty}i^nC_n{}^aH_n{}^{(1)}(kr_a)e^{in(\theta_a-\alpha)}, \qquad (32)$$

where $C_n{}^a$ is the appropriate scattering coefficient for any of the usual boundary conditions; e.g., if the field vanishes on the cylinder's surface $r=a$, then $C_n{}^a=-J_n(ka)/H_n{}^{(1)}(ka)$; if the normal derivative vanishes, we replace the functions $J_n(ka)$, $H_n{}^{(1)}(ka)$ by their derivatives with respect to their arguments, etc. The use of the asymptotic form of $H_n{}^{(1)}(kr_a)$ in (32) yields the far field

$$U_a{}^0=\frac{e^{ikr_a}}{(r_a)^{\frac{1}{2}}}\left\{e^{-i\pi/4}\left(\frac{2}{\pi k}\right)^{\frac{1}{2}}\sum_n C_n{}^ae^{in(\theta_a-\alpha)}\right\}=\frac{e^{ikr_a}}{(r_a)^{\frac{1}{2}}}f^{a0}(\theta_a,\alpha), \qquad (33)$$

where the function in braces is the unperturbed pattern. Similarly, for cylinder B (whose radius and boundary conditions differ in general from those of A), we replace a by b in (32) and (33).

The corresponding perturbed pattern may be obtained by specializing the general series of "Neumann type" in Eq. (3) of reference 2; see reference 13 for a derivation of the series and for a discussion of its physical significance. If we now expand this result in inverse powers of d and retain all terms of degree $d^{-\frac{3}{2}}$, we find (with reference to an origin midway between those of the cylinders)

$$
\begin{aligned}
f_0{}^a(\theta,\alpha)=&\exp[-ik\tfrac{1}{2}d\cos\alpha]e^{-i\pi/4}\left(\frac{2}{\pi k}\right)^{\frac{1}{2}}\sum_n C_n{}^a e^{in(\theta-\alpha)}\\
&+\frac{\exp[ik\frac{1}{2}d\cos\alpha]}{d^{\frac{1}{2}}}e^{-i\pi/2}\left(\frac{2}{\pi k}\right)\sum_n(-1)^{n-1}C_n{}^b e^{-in\alpha}\sum_{n'}(-1)C_{n'}{}^a e^{in'(\theta-\pi)}\\
&+\frac{\exp[-ik\frac{1}{2}d\cos\alpha]e^{ik2d}}{d}e^{-i3\pi/4}\left(\frac{2}{\pi k}\right)^{\frac{3}{2}}\sum_n C_n{}^a e^{-in\alpha}\sum_{n'}(-1)^{n'-1}C_{n'}{}^b\sum_{n''}e^{in''(\theta-\pi)}\\
&+\frac{\exp[ik\frac{1}{2}d\cos\alpha]e^{ik3d}}{d^{\frac{3}{2}}}e^{-i\pi}\left(\frac{2}{\pi k}\right)^{2}\sum_n(-1)^{n-1}C_n{}^b e^{-in\alpha}\sum_{n'}(-1)^{n'-1}C_{n'}{}^a\sum_{n''}(-1)^{n''-1}C_{n''}{}^b\\
&\times\sum_{n'''}(-1)C_{n'''}{}^a e^{in'''(\theta-\pi)}\\
&\quad+\frac{\exp[ik\frac{1}{2}d\cos\alpha]e^{ikd}}{d^{\frac{3}{2}}}\frac{e^{-i\pi/2}}{2ik}\left(\frac{2}{\pi k}\right)\sum_n(-1)^nC_n{}^b e^{in\alpha}\sum_{n'}(-1)[(n-n')^2-\tfrac{1}{4}]C_{n'}{}^a e^{in'(\theta-\pi)}+O(d^{-\frac{5}{2}}),
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
f_0{}^b(\theta,\alpha)=&\exp[ik\tfrac{1}{2}d\cos\alpha]e^{-i\pi/2}\left(\frac{2}{\pi k}\right)^{\frac{1}{2}}\sum_n C_n{}^b e^{in(\theta-\alpha)}\\
&+\frac{\exp[ik\frac{1}{2}d\cos\alpha]e^{ikd}}{d^{\frac{1}{2}}}e^{-i\pi/2}\left(\frac{2}{\pi k}\right)\sum_n(-1)C_n{}^a e^{-in\alpha}\sum_{n'}(-1)C_{n'}{}^b e^{in'\theta}\\
&+\frac{\exp[ik\frac{1}{2}d\cos\alpha]e^{ik2d}}{d}e^{-i3\pi/4}\left(\frac{2}{\pi k}\right)^{\frac{3}{2}}\sum_n(-1)^nC_n{}^b e^{-in\alpha}\sum_{n'}(-1)^{n'-1}C_{n'}{}^a\sum_{n''}(-1)C_{n''}{}^b e^{in''\theta}\\
&+\frac{\exp[-ik\frac{1}{2}d\cos\alpha]e^{ik3d}}{d^{\frac{3}{2}}}e^{-i\pi}\left(\frac{2}{\pi k}\right)^{2}\sum_n(-1)C_n{}^a e^{-in\alpha}\sum_{n'}(-1)^{n'-1}C_{n'}{}^b\sum_{n''}(-1)^{n''-1}C_{n''}{}^a\\
&\times\sum_{n'''}(-1)C_{n'''}{}^b e^{in'''\theta}\\
&\quad+\frac{\exp[-k\frac{1}{2}d\cos\alpha]e^{ikd}}{d^{\frac{3}{2}}}\frac{e^{-i\pi/2}}{2ik}\left(\frac{2}{\pi k}\right)\sum_n C_n{}^a e^{-in\alpha}\sum_{n'}(-1)[(n'-n)^2-\tfrac{1}{4}]C_{n'}{}^b e^{in\theta}+O(d^{-\frac{5}{2}}),
\end{aligned}
\tag{35}
$$

which we shall first compare with the expansion of the closed-form approximation given in Eq. (6) of reference 2. That approximation was first obtained by keeping only the largest term of each order of scattering and hence its expansion is not quite as accurate as (34) and (35). To carry out the comparison, we specialize (34) and (35) to the case of two identical cylinders ($a=b$). Then, comparison shows agreement so far as the non-interaction terms and the terms of degrees $d^{-\frac{1}{2}}$ and d^{-1} are concerned. The first of our terms of degree $d^{-\frac{3}{2}}$ agrees with the term of order $d^{-\frac{3}{2}}$ in the expansion of Twersky's result. But our second term of degree $d^{-\frac{3}{2}}$ is new. This is to be expected for the following reason. The closed form referred to above is obtained by summing the leading terms only, of the successive orders of scattering, that is, the successive bounces. Our term of degree $d^{-\frac{3}{2}}$, on the other hand, contains higher-order contributions from the second bounce, in addition to the leading term of the fourth bounce.

The procedure used above for obtaining (34) and (35) is long and tedious. These results need not be obtained by that procedure. The use of the abstract formulas (25) and (26) simplifies the calculation considerably and should do the same in any other case where the scattering problems for the component cylinders are separable. All we need do is substitute the specific unperturbed patterns $f^{a0}(\theta,\alpha)$ and $f^{b0}(\theta,\alpha)$ into (25) and (26), respectively. The results follow immediately and agree with (34) and (35), respectively.

[13] V. Twersky, J. Acoust. Soc. Am. 24, 42 (1952).

6. EXPLANATION OF THE RESULT

Expression (31) for the scattered far field of the combination of two cylinders appears, at first glance, to be complicated. However, it is not as abstruse as it seems. A closer examination of these expressions reveals the significance of these various terms and factors. We note first that the terms are grouped in increasing orders of accuracy. We note also that they are grouped in pairs. The first members of each pair represent fields ultimately scattered by cylinder A, while the second members of the pairs represent fields ultimately scattered by cylinder B. The first pair corresponds to single scattering while the other pairs correspond to multiple scattering. The factors

$$\exp(\pm ik\tfrac{1}{2}d\cos\theta)$$

represent the phase differences for the scatterers relative to the point of observation, while the factors

$$\exp(\pm ik\tfrac{1}{2}d\cos\alpha)$$

take into account the phase of the incident wave at the center of a scatterer when the incident wave has zero phase at the origin. The f's containing a θ dependence are scattering patterns, whereas the f's containing specific values for θ are excitation factors accumulated in the multiple scattering. We note also that the factors of the form e^{iknd} where $n=0, 1, 2, 3$ refer to the increase ikd in the phase of a wave in going from one scatterer to another and that n signifies the number of bounces.

As an illustration of the above remarks, let us consider some of the terms in more detail. The term

$$(e^{ikr}/r^{\frac{1}{2}})\exp[ik\tfrac{1}{2}d(\cos\theta-\cos\alpha)]f^{a0}(\theta,\alpha)$$

would occur in the case of no interaction, i.e., in the limit of infinite spacing. The factor $\exp[\frac{1}{2}ikd(\cos\theta)]$ takes account of the fact that the origin is not at the center of A. The positive sign preceding $\cos\theta$ shows that the scattered wave came from cylinder A. The negative sign in the exponent of the factor $\exp(-ik\frac{1}{2}d\times\cos\alpha)$ shows that cylinder A received the initial excitation. It gives the phase of the incident wave at A. The term

$$(e^{ikr}/r^{\frac{1}{2}})\exp[ik\tfrac{1}{2}d(\cos\theta+\cos\alpha)]f_{(\pi,\alpha)}{}^{b0}f_{(\theta,\pi)}{}^{a0}$$

differs from the first term in the following respects. The positive sign in the factor $\exp(+\frac{1}{2}ikd\cos\alpha)$ shows that cylinder B received the original excitation as does the factor $f_{(\pi,\alpha)}{}^{b0}$. The phase factor e^{ikd} represents the increase of the phase of the wave in going from B to A.

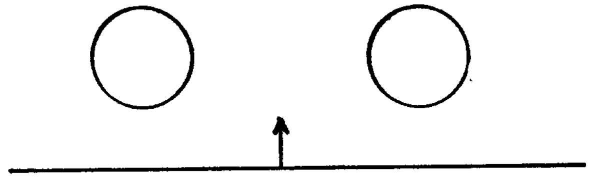

FIG. 5. Plane wave normally incident upon two parallel circular cylinders.

We shall now explain a typical term of the last square bracket. The first term in the last square bracket of (31) represents the scattering by A of a term of order $d^{-\frac{1}{2}}$ initially scattered by B. The phase factor e^{ikd} takes account of the travel of this wave from B to A.

The positive sign preceding $\cos\alpha$ shows that cylinder B was excited initially. The differentiation of the scattering amplitudes shows the effect of a higher-order excitation of A by the field initially scattered by B, since the higher-order response of B, (which acts as an excitation for A), is representable, near A, as a derivative of a plane wave with respect to angle of incidence.

The explanation of further terms in the final result proceeds on the same lines as the explanations given above. We omit these explanations for the sake of brevity.

7. TOTAL SCATTERING CROSS SECTION OF A COMBINATION OF TWO IDENTICAL CIRCULAR CYLINDERS

We consider a plane wave normally incident on a pair of identical circular cylinders. The circumstances are illustrated in Fig. 5.

The computation of the total scattering cross section σ of the two identical circular cylinders is facilitated by the use of (30) in conjunction with the following well-known theorem[14,15]:

$$\sigma=-(4/k)\,\mathrm{Re}F(\pi/2,\pi/2), \tag{36}$$

where $F(\theta,\alpha)$ is the scattering amplitude (30) of the combination. The phase factors are simplified for the values $\theta=\pi/2$, $\alpha=\pi/2$. The fact that the cylinders are identical enables us to write $f^0=f^{a0}=f^{b0}$. Further simplifications result from the geometrical symmetry of the problem, namely,

$$f^0(0,\pi)=f^0(\pi,0),\quad f^0(\pi/2,0)=f^0(0,\pi/2)$$
$$=f^0(\pi,\pi/2)=f^0(\pi/2,\pi).$$

The total scattering cross section in terms of the

[14] V. Twersky, J. Appl. Phys. 25, 859 (1954).
[15] C. H. Papas, J. Appl. Phys. 21, 318 (1950).

spacing is then

$$\sigma=-(4/k)\,\mathrm{Re}\left[(2\pi k)^{\frac{1}{2}}e^{i\pi/4}\left\{\begin{aligned}&f^0\left(\frac{\pi}{2},\frac{\pi}{2}\right)+\frac{e^{ikd}}{d^{\frac{1}{2}}}\left[f^0\left(0,\frac{\pi}{2}\right)\right]^2+\frac{e^{ik2d}}{d}\left[f^0\left(0,\frac{\pi}{2}\right)\right]^2 f^0(\pi,0)\\&+\frac{e^{ik3d}}{d^{\frac{3}{2}}}\left[f^0\left(0,\frac{\pi}{2}\right)\right]^2[f^0(\pi,0)]^2\\&+\frac{e^{ikd}}{d^{\frac{3}{2}}}\frac{1}{2ik}\left(f^0\left(0,\frac{\pi}{2}\right)D_{\theta_0}{}^2f^0\left(0,\frac{\pi}{2}\right)+2D_\theta f^0\left(0,\frac{\pi}{2}\right)D_{\theta_0}f_\phi{}^0\left(0,\frac{\pi}{2}\right)\right.\\&\qquad\left.+\left[\tfrac{1}{4}f^0\left(0,\frac{\pi}{2}\right)+D_\theta{}^2f^0\left(0,\frac{\pi}{2}\right)\right]f^0\left(0,\frac{\pi}{2}\right)\right)+\mathcal{O}(d^{-\frac{5}{2}})\end{aligned}\right\}\right] \tag{37}$$

8. ADDITIONAL REMARKS

It is clear that interactions of degree greater than $d^{-\frac{5}{2}}$ can be computed by the inclusion of more terms in the expansions used to obtain these results.

It is also clear that the method applies to cases of more than two scatterers, but the computations would be more tedious than in the case of two scatterers. The computations might be simplified by using a "consistency" method employed in reference 3 rather than tracing the successive scattering in detail. This method involves a steady-state point of view. The response of each cylinder is expanded in a neighborhood of each of the other cylinders. Each cylinder will then be excited by the incident plane wave and by an approximately plane wave from each of the other cylinders. These considerations introduce certain undetermined coefficients which can be determined by imposing the requirement that the fields scattered by the various cylinders be consistent with one another. Evaluation of the coefficients will provide the solution.

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Higher-Order Approximations in Multiple Scattering.* II. Three-Dimensional Scalar Case

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The method of Part I is extended to cover the three-dimensional scalar problem for two bodies of arbitrary shape. All interaction terms of order d^{-1} and d^{-2} are given.

1. STATEMENT OF THE PROBLEM

THE method employed previously in the case of multiple scattering of plane waves by two widely spaced cylinders of arbitrary shape can be applied, also, to the corresponding three-dimensional scalar problem for two bodies of arbitrary shape. The assumptions of spacing large compared to the wavelength and the dimensions of the bodies and that the individual responses of the scatterers are known apply also to this case.

The situation is the following. A plane wave of unit amplitude

$$u=\exp[ik(x\sin\theta_0\cos\phi_0+y\sin\theta_0\sin\phi_0+z\cos\theta_0)], \tag{1}$$

where θ_0 and ϕ_0 are the angles of incidence (see Fig. 1), is incident upon the combination of two bodies. The response of each body in isolation to the incident plane wave is of the form

$$U_s=\frac{e^{ikr}}{r}\sum_{n=0}^{\infty}\frac{f_n(\theta,\theta_0,\phi,\phi_0)}{r^n}. \tag{2}$$

Here we are referring to spherical coordinates, r, θ, ϕ,

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