Scattering by two penetrable cylinders at oblique incidence. I. The analytical solution

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The Mueller scattering matrix elements (S_{ij}) and the cross sections for the scattering of an electromagnetic plane wave from two infinitely long, parallel, circular cylinders at oblique incidence are derived. Each cylinder can be of arbitrary materials (any refractive index). The incident wave can be in any polarization state. To find the scattering coefficients needed for calculating S_{ij} and the cross sections, the multiple scatterings are taken into account for all orders. The formal solutions of the scalar wave equation are obtained for the three regions concerned (the region outside the two cylinders and the region inside each cylinder), and the scattering coefficients are found by satisfying the boundary conditions. The scattering coefficients for some special cases (normal incidence, small radii, perfectly conducting cylinders, and a single cylinder) are given and discussed. The results for these special cases are compared (numerically or analytically) with those obtained in other published works. To our knowledge, this is the first comprehensive study of the two-cylinder problem. Applications of this formalism, including calculations of S_{ij} and the cross sections, will be presented in part II of this series [J. Opt. Soc. Am. A **5**, 1097 (1988)].

1. INTRODUCTION

The scattering of electromagnetic waves from objects has attracted the attention of many scientists. Lord Rayleigh[1] was the first to obtain the solution to the problem of the scattering of a plane wave from an infinitely long, dielectric cylinder at normal incidence by using the original formalism of Maxwell. Other problems have since been solved by using somewhat different formalisms.

Adey[2] obtained the scattering coefficients for two coaxial dielectric cylinders at normal incidence by solving the scalar wave equation for the three different regions. The solution was obtained by matching the boundary conditions between adjacent regions.

The scattering of electromagnetic plane waves from an infinitely long, dielectric cylinder at oblique incidence was determined by Wait,[3] who obtained the scattering coefficients by solving the scalar wave equation in the regions inside and outside the cylinder. The boundary conditions were then applied on the surface of the cylinder. In the case of normal incidence, the two polarization modes, the TE and the TM modes (see Section 2 for definitions), are uncoupled. However, in the case of oblique incidence the two modes are coupled.

The problem of determining the scattering of electromagnetic waves from two or more cylinders at normal incidence has been treated by many others. Twersky[4] obtained an approximate solution for many parallel, conducting cylinders at normal incidence by taking into account various orders of multiple scatterings. In another paper, Twersky[5] applied this method to two conducting cylinders.

Row[6] solved the scattering of cylindrical and plane waves by two parallel conducting cylinders at normal incidence using the Green's-function approach. When the proper boundary conditions were applied, an infinite set of integral equations was obtained, from which the scattered fields were found. Row also did experimental work in the microwave region by measuring the electric fields in the near-field region. To our knowledge, Row's results are the only experimental data published on this subject.

Millar[7] used an approach similar to that used by Row: to find a solution for the scattering of an electromagnetic plane wave from an array of parallel, infinitely conducting cylinders of small radii.

Olaofe[8] derived expressions for the scattered fields from two infinitely long, identical, dielectric cylinders at normal incidence. He obtained his solution by solving the scalar wave equation and satisfying the boundary conditions.

Others extended or modified Olaofe's solution to other similar problems. Krill and Farrell[9] used Olaofe's approach to find the solution of the scattered electromagnetic plane waves from two perfectly conducting, infinitely long half-cylinders lying upon a perfectly conducting sheet. Ragheb and Hamid[10] derived the solution of the scattering of a plane wave from a number of infinitely long, identical, conducting cylinders.

The solution of the scattering of electromagnetic waves from two cylinders of arbitrary material and small radii at oblique incidence was obtained by Wait.[11] Wait solved the problem by finding the current distribution on the surface of each cylinder from which the scattered field was derived. He noted that the general case (cylinders of arbitrary material and radii) was a complicated one.

The general solution of the problem of scattering from one cylinder has been of great help to experimentalists. However, there is no known general solution for the two-cylinder problem in the literature. As mentioned above, the solution exists for only some special cases: small radii, a perfect dielectric, or perfectly conducting cylinders; the last two solutions were obtained for normal incidence only. There has been a great demand from experimentalists for a general solution for the two-cylinder problem expressed in terms of Stokes vectors and Mueller matrices. The purpose of this paper is to obtain an analytical solution for the scattering of

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two infinitely long, parallel cylinders of different radii and materials at oblique incidence, i.e., the completely general case.

We believe that the solution presented in this paper will have applications in biology, chemistry, atmospheric science, and engineering.

In Section 2, the scattering coefficients are found by solving the scalar wave equation and applying the boundary conditions. In Section 3, the Mueller scattering matrix elements (S_{ij}) are expressed in terms of the scattering coefficients. In Section 4, expressions for the scattering and the extinction cross sections are derived by using the Poynting vector. In Section 5 special cases are discussed. Finally, some discussion of the results is made in Section 6.

2. FORMALISM

The purpose of this section is to find the amplitude scattering coefficients of two parallel cylinders that are due to a plane-wave excitation.

Consider two parallel, infinitely long, circular cylinders C_1 and C_2. The radius of each is a_j ($j = 1, 2$). The permittivity, conductivity, and permeability of each cylinder are ϵ_j, σ_j, and μ_j, respectively, while those of the surrounding medium are ϵ_0, σ_0, and μ_0. The two cylinders are located in two separate, cylindrical coordinate systems O_1 and O_2. The coordinates of each system are ρ_j, ϕ_j, and z. The axis of each cylinder lies along the z axis. The distance between their centers is d.

The direction of propagation of the incident electromagnetic plane wave makes an angle θ_0 with the axis of each cylinder. The projection of the propagation vector k_0 on the x-y plane makes an angle ϕ_0 with the x axis, as shown in Fig. 1.

We define the TM and TE polarizations (modes) of the incident wave as follows: (a) The case in which the incident electric field is parallel to the incident plane, i.e., the plane that contains the z axis and the direction of propagation of the incident wave, is referred to as TM polarization. (b) The case in which the incident electric field is perpendicular to the incident plane is referred to as TE polarization.

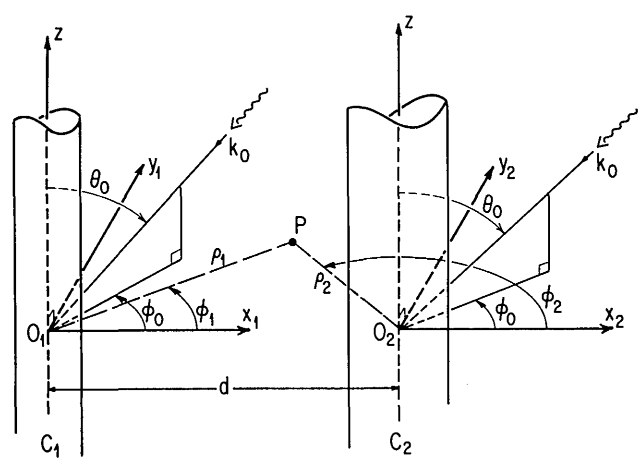

Fig. 1. Two infinitely long, circular cylinders, C_1 and C_2, are located in two separate cylindrical coordinate systems, O_1 and O_2. The coordinates of each system are ρ_j, ϕ_j, and z. The axis of each cylinder lies along the z axis; the distance between their centers is d. The direction of the incident field makes an angle θ_0 with the axis of each cylinder, and its projection on the x–y plane makes an angle ϕ_0 with the x axis.

Any incident wave (a circularly polarized one, for example) can be described as a superposition of these two polarizations, and we treat them separately.

A. Incident TM Waves

If we assume that $e^{i\omega t}$ represents the time dependence, the z component of the incident electric field ($E_{z_j}^{\text{inc}}$) in each of the two coordinate systems is given by

$$E_{z_j}^{\text{inc}} = E_0 \sin\theta_0 \exp[i\delta(j-1)] \sum_m \{i^m J_m(\lambda_0\rho_j) \times \exp[-im(\phi_j - \phi_0)]\}\exp(ik_0 z\cos\theta_0). \quad (2.1)$$

Here and in subsequent formulas, $j = 1, 2$, referring to each of the two cylinders. J_m is a Bessel function of the first kind. The incident wave is chosen to have a phase of zero at the center of cylinder 1, so that $\delta = k_0 d \cos\phi_0 \sin\theta_0$ is the phase of the incident wave at the center of cylinder 2. (The location of the incident wave phase is, of course, arbitrary. Alternatively, the phase can be chosen to be zero at the center of cylinder 2 or at any other point in space.) λ_0 and k_0 are given by

$$\lambda_0 = k_0 \sin\theta_0, \qquad k_0^2 = -i\mu_0\omega(\sigma_0 + i\omega\epsilon_0). \quad (2.2)$$

Since the cylinders are infinitely long, the electric field inside or outside each cylinder varies as $\exp(ik_0 z\cos\theta_0)$. In addition, the z component of the field (inside or outside each cylinder) satisfies the scalar wave equation. If u represents the z component of $\mathbf{E}$ or $\mathbf{H}$, then

$$\nabla^2 u + k^2 u = 0. \quad (2.3)$$

The z component of the scattered electric field from cylinder j can therefore be written as

$$E_{z_j}^{\text{sca}} = \sum_m a_{om}^{(j)} H_m^{(2)}(\lambda_0\rho_j)\exp[-im(\phi_j - \phi_0)]\exp(ik_0 z\cos\theta_0), \quad (2.4)$$

where $a_{om}^{(j)}$ are referred to as the TM scattering coefficients of cylinder j. The index o of these coefficients refers to the outside field. $H_m^{(2)}$ is the Hankel function of the second kind. The Hankel function of the second kind is required because it represents an outgoing cylindrical wave at infinity, consistent with the use of a time dependence of $e^{i\omega t}$.

Similarly, the z component of the electric field inside cylinder j can be written as

$$E_{z_j}^{\text{inside}} = \sum_m a_{im}^{(j)} J_m(\lambda_j\rho_j)\exp[-im(\phi_j - \phi_0)]\exp(ik_0 z\cos\theta_0), \quad (2.5)$$

where $a_{im}^{(j)}$ are the TM internal (index i) coefficients of cylinder j. λ_j and k_j are given by

$$\lambda_j = (k_j^2 - k_0^2\cos^2\theta_0)^{1/2},$$
$$k_j^2 = -i\mu_j\omega(\sigma_j + i\epsilon_j\omega). \quad (2.6)$$

In the case of normal incidence, the scattered field and the internal field do not contain cross-polarized components.

In other words, the modes of the internal field and of the scattered field are the same as the mode of the incident field. For oblique incidence, the scattered wave and the internal wave are a superposition of the TE and the TM modes, regardless of the incident wave mode. Therefore the z component of the scattered magnetic field from cylinder j can be written as

$$H_{z_j}^{\text{sca}} = \sum_m b_{om}^{(j)} H_m^{(2)}(\lambda_j \rho_j) \exp[-im(\phi_j - \phi_0)] \exp(ik_0 z \cos\theta_0), \tag{2.7}$$

where $b_{om}^{(j)}$ are the TE scattering coefficients of cylinder j and the z component of the magnetic field inside cylinder j is given by

$$H_{z_j}^{\text{inside}} = \sum_m b_{im}^{(j)} J_m(\lambda_j \rho_j) \exp[-im(\phi_j - \phi_0)] \exp(ik_0 z \cos\theta_0), \tag{2.8}$$

with the TE internal coefficients $b_{im}^{(j)}$.

The effective incident field upon one cylinder is equal to the original incident field plus the scattered field from the other cylinder. If ψ^{inc} is the original incident field, ψ_1^{sca} is the scattered field from cylinder 1, and ψ_2^{sca} is the scattered field from cylinder 2, then the total field (ψ^{total}) outside the two cylinders is

$$\psi^{\text{total}} = \psi^{\text{inc}} + \psi_1^{\text{sca}} + \psi_2^{\text{sca}}. \tag{2.9}$$

With ψ being the electric field, the z component of the total electric field outside the two cylinders is found from Eq. (2.9), with ψ^{inc}, ψ_1^{sca}, and ψ_2^{sca} given by Eqs. (2.1) and (2.4), respectively:

$$\begin{aligned} E_z^{\text{total}} = \sum_m (\{i^m E_0 \sin\theta_0 \exp[i\delta(j-1)] J_m(\lambda_0 \rho_j) \\ + a_{om}^{(j)} H_m^{(2)}(\lambda_0 \rho_j)\} \exp[-im(\phi_j - \phi_0)] \\ + a_{om}^{(k)} H_m^{(2)}(\lambda_0 \rho_k) \exp[-im(\phi_k - \phi_0)]) \exp(ik_0 z \cos\theta_0), \end{aligned} \tag{2.10}$$

where $j, k = 1, 2$ or $j, k = 2, 1$.

Similarly, the z component of the total magnetic field outside the two cylinders is found from Eq. (2.9), with $\psi^{\text{inc}} = 0$ and with ψ_1^{sca} and ψ_2^{sca} given by Eq. (2.7):

$$\begin{aligned} H_z^{\text{total}} = \sum_m \{b_{om}^{(j)} H_m^{(2)}(\lambda_0 \rho_j) \exp[-im(\phi_j - \phi_0)] \\ + b_{om}^{(k)} H_m^{(2)}(\lambda_0 \rho_k) \exp[-im(\phi_k - \phi_0)]\} \exp(ik_0 z \cos\theta_0). \end{aligned} \tag{2.11}$$

In order for the boundary conditions to be applied, the electric field and the magnetic field outside cylinder j must be expressed in terms of the coordinates of cylinder j. This means that the scattered field from cylinder k must be transformed as an incident field upon cylinder j. In other words, (ρ_k, ϕ_k) in Eqs. (2.10) and (2.11) must be transformed to (ρ_j, ϕ_j). This transformation can be done by using the results from Appendix A. Therefore Eq. (2.10) can be written as

$$\begin{aligned} E_z^{\text{total}} = \sum_m (\{i^m E_0 \sin\theta_0 \exp[i\delta(j-1)] J_m(\lambda_0 \rho_j) \\ + a_{om}^{(j)} H_m^{(2)}(\lambda_0 \rho_j) + A_{om}^{(k)} J_m(\lambda_0 \rho_j)\} \\ \times \exp[-im(\phi_j - \phi_0)]) \exp(ik_0 z \cos\theta_0), \end{aligned} \tag{2.12}$$

where

$$A_{om}^{(k)} = \sum_l (-1)^{(m+l)(k-1)} a_{ol}^{(k)} H_{l-m}^{(2)}(\lambda_0 d) \exp[i(l-m)\phi_0]. \tag{2.13}$$

In a similar way, Eq. (2.11) can be transformed to

$$\begin{aligned} H_z^{\text{total}} = \sum_m \{[b_{om}^{(j)} H_m^{(2)}(\lambda_0 \rho_j) + B_{om}^{(k)} J_m(\lambda_0 \rho_j)] \\ \times \exp[-im(\phi_j - \phi_0)]\} \exp(ik_0 z \cos\theta_0), \end{aligned} \tag{2.14}$$

where

$$B_{om}^{(k)} = \sum_l (-1)^{(m+l)(k-1)} b_{ol}^{(k)} H_{l-m}^{(2)}(\lambda_0 d) \exp[i(l-m)\phi_0]. \tag{2.15}$$

It should be noted that the total electric field in Eq. (2.12) or the total magnetic field in Eq. (2.14) can be expressed in terms of the coordinates of cylinder 1 ($j = 1$, $k = 2$) or cylinder 2 ($j = 2$, $k = 1$). The two expressions can be shown to be equivalent. However, in the stage of applying the boundary conditions, both expressions are needed, as is shown at the end of this section.

All the incoming and outgoing fields of both cylinders are contained in our formalism; i.e., all orders of multiple scattering are included.

To apply the boundary conditions, expressions for E_ϕ and H_ϕ (inside and outside each cylinder) must be found in terms of E_z and H_z. They are obtained from Maxwell's equations,

$$E_\phi^m = \frac{1}{k^2 - k_0^2 \cos^2\theta_0} \left(\frac{m k_0 \cos\theta_0 E_z^m}{\rho} + i\mu\omega \frac{\partial H_z^m}{\partial \rho} \right), \tag{2.16}$$

$$H_\phi^m = \frac{1}{k^2 - k_0^2 \cos^2\theta_0} \left[\frac{m k_0 \cos\theta_0}{\rho} H_z^m - (\sigma + i\epsilon\omega) \frac{\partial E_z^m}{\partial \rho} \right], \tag{2.17}$$

where E_ϕ^m, E_z^m, H_ϕ^m, and H_z^m are the components of the fields for each mode m. The general solution of the wave equation is obtained by adding the contributions of all these modes,

$$E_\phi = \sum_m E_\phi^m, \qquad H_\phi = \sum_m H_\phi^m.$$

The ϕ component of the total outside electric field in the coordinate system of cylinder j is found by substituting Eqs. (2.12) and (2.14) into Eq. (2.16), with $k = k_0$, $\sigma = \sigma_0$, $\epsilon = \epsilon_0$, and $\rho = \rho_j$:

$$E_\phi^{\text{total}} = \sum_m \Bigg[\frac{\exp[-im(\phi_j - \phi_0)]}{k_0^2 - k_0^2 \cos^2\theta_0} \Bigg(\frac{mk_0 \cos\theta_0}{\rho_j} \{i^m E_0 \sin\theta_0 \times \exp[i\delta(j-1)] J_m(\lambda_0\rho_j) + a_{om}^{(j)} H_m^{(2)}(\lambda_0\rho_j) + A_{om}^{(k)} J_m(\lambda_0\rho_j)\} + i\mu_0\omega\lambda_0 [b_{om}^{(j)} H_m^{(2)\prime}(\lambda_0\rho_j) + B_{om}^{(k)} J_m'(\lambda_0\rho_j)] \Bigg) \Bigg] \exp(ik_0 z \cos\theta_0), \tag{2.18}$$

where a prime indicates the derivative with respect to the argument.

In a similar manner, the ϕ component of the electric field inside cylinder j is found by substituting Eqs. (2.7) and (2.8) into Eq. (2.16), with $k = k_j$, $\sigma = \sigma_j$, $\epsilon = \epsilon_j$, and $\rho = \rho_j$:

$$E_{\phi_j}^{\text{inside}} = \sum_m \Bigg\{ \frac{\exp[-im(\phi_j - \phi_0)]}{k_j^2 - k_0^2\cos^2\theta_0} \Bigg[\frac{mk_0\cos\theta_0}{\rho_j} a_{im}^{(j)} J_m(\lambda_j\rho_j) + i\mu_j\omega\lambda_j b_{im}^{(j)} J_m'(\lambda_j\rho_j) \Bigg] \Bigg\} \exp(ik_0 z\cos\theta_0). \tag{2.19}$$

By using Eq. (2.17), we obtain the ϕ components of the magnetic field outside and inside cylinder j:

$$H_\phi^{\text{total}} = \sum_m \Bigg[\frac{\exp[-im(\phi_j - \phi_0)]}{k_0^2 - k_0^2\cos^2\theta_0} \times \Bigg(\frac{mk_0\cos\theta_0}{\rho_j} [b_{om}^{(j)} H_m^{(2)}(\lambda_0\rho_j) + B_{om}^{(k)} J_m(\lambda_0\rho_j)] - (\sigma_0 + i\epsilon_0\omega)\lambda_0 \{i^m E_0 \sin\theta_0 \exp[i\delta(j-1)] J_m'(\lambda_0\rho_j) + a_{om}^{(j)} H_m^{(2)\prime}(\lambda_0\rho_j) + A_{om}^{(k)} J_m'(\lambda_0\rho_j)\} \Bigg) \Bigg] \times \exp(ik_0 z\cos\theta_0), \tag{2.20}$$

$$H_{\phi_j}^{\text{inside}} = \sum_m \Bigg\{ \frac{\exp[-im(\phi_j - \phi_0)]}{k_j^2 - k_0^2\cos^2\theta_0} \Bigg[\frac{mk_0\cos\theta_0}{\rho_j} b_{im}^{(j)} J_m(\lambda_j\rho_j) - (\sigma_j + i\epsilon_j\omega)\lambda_j a_{im}^{(j)} J_m'(\lambda_j\rho_j) \Bigg] \Bigg\} \exp(ik_0 z\cos\theta_0). \tag{2.21}$$

Application of the boundary conditions (the tangential components of E and H are continuous) at $\rho_j = a_j (j = 1, 2)$,

$$E_z^{\text{total}} = E_{z_j}^{\text{inside}}, \qquad H_z^{\text{total}} = H_{z_j}^{\text{inside}},$$
$$E_\phi^{\text{total}} = E_{\phi_j}^{\text{inside}}, \qquad H_\phi^{\text{total}} = H_{\phi_j}^{\text{inside}}, \tag{2.22}$$

yields eight coupled linear equations for the scattering coefficients $a_{om}^{(j)}$, $a_{im}^{(j)}$, $b_{om}^{(j)}$, and $b_{im}^{(j)}$. It is not possible to separate the eight equations into two groups so that one group contains the TE coefficients $b_{om}^{(j)}$ and $b_{im}^{(j)}$ and the other group contains the TM coefficients $a_{om}^{(j)}$ and $a_{im}^{(j)}$. It is possible, however, to simplify them further by eliminating the coefficients of the internal fields $a_{im}^{(1)}$, $a_{im}^{(2)}$, $b_{im}^{(1)}$, and $b_{im}^{(2)}$. The following expressions for the scattering coefficients are then obtained:

$$a_{om}^{(j)} = \frac{mk_0\cos\theta_0}{a_j S_j} [b_{om}^{(j)} H_m^{(2)}(\lambda_0 a_j) + B_{om}^{(l)} J_m(\lambda_0 a_j)]\lambda_{0j} - \frac{i}{\omega S_j} \Bigg[\frac{k_0^2}{\mu_0\lambda_0} J_m'(\lambda_0 a_j) - \frac{k_j^2}{\mu_j\lambda_j} J_m(\lambda_0 a_j) \frac{J_m'(\lambda_j a_j)}{J_m(\lambda_j a_j)} \Bigg] \times \{i^m E_0 \sin\theta_0 \exp[i\delta(j-1)] + A_{om}^{(l)}\}, \tag{2.23}$$

$$b_{om}^{(j)} = \frac{mk_0\cos\theta_0}{a_j y_j} \{i^m E_0 \sin\theta_0 \exp[i\delta(j-1)] J_m(\lambda_0 a_j) + a_{om}^{(j)} H_m^{(2)}(\lambda_0 a_j) + A_{om}^{(l)} J_m(\lambda_0 a_j)\}\lambda_{0j} - \frac{i\omega B_{om}^{(l)}}{y_j} \times \Bigg[\frac{\mu_i}{\lambda_j} \frac{J_m(\lambda_0 a_j)}{J_m(\lambda_j a_j)} J_m'(\lambda_j a_j) - \frac{\mu_0}{\lambda_0} J_m'(\lambda_0 a_j) \Bigg]. \tag{2.24}$$

where $j, l = 1, 2$ or $j, l = 2, 1$ and

$$S_j = \frac{i}{\omega} \Bigg[\frac{k_0^2}{\mu_0\lambda_0} H_m^{(2)\prime}(\lambda_0 a_j) - \frac{k_j^2}{\mu_j\lambda_j} J_m'(\lambda_j a_j) \frac{H_m^{(2)}(\lambda_0 a_j)}{J_m(\lambda_j a_j)} \Bigg],$$

$$y_j = i\omega \Bigg[\frac{\mu_i}{\lambda_j} \frac{H_m^{(2)}(\lambda_0 a_j)}{J_m(\lambda_j a_j)} J_m'(\lambda_j a_j) - \frac{\mu_0}{\lambda_0} H_m^{(2)\prime}(\lambda_0 a_j) \Bigg],$$

$$\lambda_{0j} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_j^2}.$$

It is clear from Eqs. (2.23) and (2.24) that the scattering coefficients of each cylinder are modified by the presence of the other cylinder. We may write these two equations as the independent scattering coefficients of one cylinder plus a correction term to account for multiple scatterings.

B. Incident TE Waves

The scattering coefficients for incident TE waves are obtained from the symmetry as we discuss here. Maxwell's equations are symmetric in the components of the electric and magnetic fields by the replacements

$$\mathbf{E} \to \mathbf{H}, \qquad \mathbf{E} \to -\mathbf{H}, \qquad (\sigma + i\epsilon\omega) \to i\mu\omega, \qquad i\mu\omega \to (\sigma + i\epsilon\omega).$$

Accordingly, the magnetic and electric scattering coefficients $c_{om}^{(j)}$ and $d_{om}^{(j)}$ are obtained from $a_{om}^{(j)}$ [Eq. (2.23)] and $b_{om}^{(j)}$ [Eq. (2.24)], respectively, by the replacements $a_{om}^{(j)} \to c_{om}^{(j)}$, $b_{om}^{(j)} \to -d_{om}^{(j)}$, and $i\mu\omega \leftrightarrows (\sigma + i\epsilon\omega)$ to get

$$c_{om}^{(j)} = \frac{mk_0\cos\theta_0}{a_j y_j} [d_{om}^{(j)} H_m^{(2)}(\lambda_0 a_j) + D_{om}^{(l)} J_m(\lambda_0 a_j)]\lambda_{0j} + \frac{i\omega}{y_j} \Bigg[\frac{\mu_0}{\lambda_0} J_m'(\lambda_0 a_j) - \frac{\mu_j}{\lambda_j} J_m(\lambda_0 a_j) \frac{J_m'(\lambda_j a_j)}{J_m(\lambda_j a_j)} \Bigg] \times \{i^m H_0 \sin\theta_0 \exp[i\delta(j-1)] + C_{om}^{(l)}\}, \tag{2.25}$$

$$d_{om}^{(j)} = \frac{mk_0\cos\theta_0}{a_j S_j} \{i^m H_0 \sin\theta_0 \exp[i\delta(j-1)] J_m(\lambda_0 a_j) + c_{om}^{(j)} H_m^{(2)}(\lambda_0 a_j) + C_{om}^{(l)} J_m(\lambda_0 a_j)\}\lambda_{0j} - \frac{iD_{om}^{(l)}}{\omega S_j} \times \Bigg[\frac{k_0^2}{\mu_0\lambda_0} J_m'(\lambda_0 a_j) - \frac{k_j^2}{\mu_j\lambda_j} J_m(\lambda_0 a_j) \frac{J_m'(\lambda_j a_j)}{J_m(\lambda_j a_j)} \Bigg], \tag{2.26}$$

where $C_{om}^{(l)}$ and $D_{om}^{(l)}$ are defined as in Eqs. (2.13) and (2.15), with $a_{ol}^{(k)}$ and $b_{ol}^{(k)}$ replaced by $c_{ol}^{(k)}$ and $d_{ol}^{(k)}$, respectively.

Equations (2.23) and (2.24) for TM polarization and Eqs. (2.25) and (2.26) for TE polarization provide an exact solution for the two-cylinder problem in terms of the scattering coefficients $a_{om}^{(j)}$, $b_{om}^{(j)}$, $c_{om}^{(j)}$, and $d_{om}^{(j)}$ ($j = 1, 2$). The solution applies in the near-field region as well as in the far-field region and for any separation.

The scattering coefficients for both cases (TE and TM) are functions of the radii, the refractive indices, the separation between the two cylinders, and the angles θ_0 and ϕ_0.

3. FAR-FIELD APPROXIMATION AND S_{ij}

The purpose of this section is to find the Mueller scattering matrix elements S_{ij} in the far-field region.

We begin the sequence of developing the analysis by finding expressions for the scattered electric fields parallel and perpendicular to the scattering plane (the plane containing the z axis and the direction of propagation of the scattered field) in the far-field region. From these expressions, we construct the amplitude scattering matrix, and, finally, we find the Mueller scattering matrix from the amplitude matrix by using the Stokes vectors.

One way to measure the scattered field is to move a detector in a circle around the scatterer. The center of this circle is, in our case, conveniently chosen to be at the center of cylinder 1. The plane of this circle is often called the plane of measurement. If the scatterer is two cylinders, then the plane of measurement may be chosen to be perpendicular to the z axis (the axis of each cylinder). In the following discussion this plane is chosen. Therefore the scattering plane is perpendicular to the plane of measurement. In the limit that the radius of the circle mentioned above is large, the following points are true:

(1) The observed scattered field on the circumference of the circle (or at any distance far away from the center of the circle) at any direction is the algebraic sum of the scattered fields from cylinders 1 and 2 along that direction. For example, the ρ component of the total scattered field at a large distance is the algebraic sum of the ρ component of the scattered field from cylinder 1 and the ρ component of the scattered field from cylinder 2. This is not true (except for the z component) at finite distances from the center of the circle because the unit vector associated with any component of the scattered field from cylinder 1 has a different direction from that scattered from cylinder 2.

(2) It is permissible to replace the Hankel and Bessel functions with their limiting asymptotic expressions. The calculation of the fields at large distances from the scatterers (the scattering region) is often referred to as the far-field approximation.

Since we want to find expressions for the scattered electric fields in the far-field region, the two orthogonal polarizations considered in Section 2 are considered separately here as well.

A. Incident TM Waves

The Mueller matrix S_{ij} is defined in terms of the amplitude scattering matrix in the far-field region. The latter matrix is defined in terms of the components of the field that are perpendicular and parallel to the scattering plane. The scattered electric field is expressed in terms of the three cylindrical components E_ρ, E_ϕ, and E_z. The components E_ρ and E_z are parallel to the scattering plane, while E_ϕ is perpendicular to the scattering plane.

In the far-field region, the Hankel function of the second kind may be replaced by its asymptotic value

$$H_m^{(2)}(\lambda_0\rho) = \left(\frac{2}{\pi\lambda_0\rho}\right)^{1/2} \exp(-i\lambda_0\rho) i^m \exp(i\pi/4). \tag{3.1}$$

The z component of the scattered electric field from cylinder j in this region is found by substituting Eq. (3.1) into Eq. (2.4):

$$E_{z_j}^{\text{sca}} = G \sum_m i^m a_{om}^{(j)} \exp[-im(\phi_j - \phi_0)], \tag{3.2}$$

where $G = (2/\pi\lambda_0\rho)^{1/2} \exp(i\pi/4)\exp(-i\lambda_0\rho)\exp(ik_0 z \cos\theta_0)$. Similarly, the z component of the scattered magnetic field from cylinder j is found by substituting Eq. (3.1) into Eq. (2.7):

$$H_{z_j}^{\text{sca}} = G \sum_m i^m b_{om}^{(j)} \exp[-im(\phi_j - \phi_0)]; \tag{3.3}$$

$E_z/\rho \to 0$ in the scattering region. Therefore Eq. (2.16) reduces to

$$E_{\phi_j} \cong \frac{i\mu_0\omega}{k_0^2 \sin^2\theta_0} \frac{\partial H_{z_j}}{\partial \rho_j}. \tag{3.4}$$

The ϕ component of the scattered electric field from cylinder j is found by substituting Eq. (3.3) into relation (3.4), and we obtain

$$E_{\phi_j}^{\text{sca}} = \frac{G\mu_0\omega\lambda_0}{k_0^2 \sin^2\theta_0} \sum_m i^m b_{om}^{(j)} \exp[-im(\phi_j - \phi_0)]. \tag{3.5}$$

In the far-field region, $\phi = \phi_1 = \phi_2$, and $\rho_2 = \rho_1 - d\cos\phi$. Therefore the ϕ component of the total scattered electric field, $E_\phi^{\text{sca}} = E_{\phi 1}^{\text{sca}} + E_{\phi 2}^{\text{sca}}$, is

$$E_\phi^{\text{sca}} = \frac{G\mu_0\omega\lambda_0}{k_0^2 \sin^2\theta_0} \sum_m \{i^m [b_{om}^{(1)} + b_{om}^{(2)} \exp(i\lambda_0 d \cos\phi)] \times \exp[-im(\phi - \phi_0)]\}. \tag{3.6}$$

The factor $\exp(i\lambda_0 d\cos\phi)$ in Eq. (3.6) accounts for the phase difference between the scattered waves from the two cylinders.

Similarly, the z component of the total scattered electric field (the sum of the z components of the scattered electric fields from cylinders 1 and 2) in the far-field region is

$$E_z^{\text{sca}} = G \sum_m \{i^m [a_{om}^{(1)} + a_{om}^{(2)} \exp(i\lambda_0 d\cos\phi)] \times \exp[-im(\phi - \phi_0)]\}. \tag{3.7}$$

The ρ component of the scattered electric field is given by

$$E_\rho^{\text{sca}} = \frac{\cos\theta_0 E_z^{\text{sca}}}{\sin\theta_0}. \tag{3.8}$$

The total scattered electric field in the far-field region is

$$\mathbf{E}^{\rm sca} = \hat{e}_\rho E_\rho^{\rm sca} + \hat{e}_z E_z^{\rm sca} + \hat{e}_\phi E_\phi^{\rm sca}, \tag{3.9}$$

where $\hat{e}_z$, $\hat{e}_\rho$, and $\hat{e}_\phi$ are unit vectors along the z, ρ, and ϕ directions, respectively.

Equation (3.9) can be written as the sum of two components, the first one being parallel to the scattering plane and the second being perpendicular to it:

$$\mathbf{E}^{\rm sca} = \hat{e}_{\rm par} E_{\rm par}^{\rm sca} + \hat{e}_{\rm per} E_{\rm per}^{\rm sca}, \tag{3.10}$$

with $\hat{e}_{\rm par} = \sin\theta_0 \hat{e}_z + \cos\theta_0 \hat{e}_\rho$ and $\hat{e}_{\rm per} = \hat{e}_\phi$. We then get

$$E_{\rm par}^{\rm sca} = \sum_m \left\{ \frac{1}{\sin\theta_0} [a_{om}^{(1)} + a_{om}^{(2)} \exp(i\lambda_0 d \cos\phi)]\phi_m \right\}, \tag{3.11}$$

$$E_{\rm per}^{\rm sca} = \sum_m \left\{ \frac{\mu_0 \omega}{k_0 \sin\theta_0} [b_{om}^{(1)} + b_{om}^{(2)} \exp(i\lambda_0 d \cos\phi)]\phi_m \right\}, \tag{3.12}$$

where

$$\phi_m = \left(\frac{2}{\pi\lambda_0\rho}\right)^{1/2} i^m \exp(i\pi/4)\exp[-im(\phi - \phi_0)] \times \exp(ik_0 z \cos\theta_0)\exp(-i\lambda_0\rho).$$

Two elements of the amplitude scattering matrix are given by Eqs. (3.11) and (3.12). To find the other two elements, the following polarization must be considered.

B. Incident TE Waves

From the symmetry of the problem, the scattered electric fields parallel and perpendicular to the scattering plane can be found from Eqs. (3.11) and (3.12). If $b_{om}^{(j)}$ is replaced by $c_{om}^{(j)}$, $a_{om}^{(j)}$ is replaced by $d_{om}^{(j)}$, and the right-hand sides of Eqs. (3.11) and (3.12) are multiplied by $k_0/\mu_0\omega$ [this factor ($k_0/\mu_0\omega$) makes the incident electric field perpendicular to the incident plane of unit amplitude], then the following expressions for the fields are obtained:

$$E_{\rm par}^{\rm sca} = \sum_m \left\{ \frac{k_0}{\mu_0\omega \sin\theta_0} [d_{om}^{(1)} + d_{om}^{(2)} \exp(i\lambda_0 d \cos\phi)]\phi_m \right\}, \tag{3.13}$$

$$E_{\rm per}^{\rm sca} = \sum_m \left\{ \frac{1}{\sin\theta_0} [c_{om}^{(1)} + c_{om}^{(2)} \exp(i\lambda_0 d \cos\phi)]\phi_m \right\}. \tag{3.14}$$

Equations (3.11)–(3.14) can be written in a matrix form,

$$\begin{bmatrix} E_{\rm par}^{\rm sca} \\ E_{\rm per}^{\rm sca} \end{bmatrix} = \left(\frac{2}{\pi\lambda_0\rho}\right)^{1/2} \exp(-i\lambda_0\rho)\exp(i\pi/4)\exp(ik_0 z\cos\theta_0) \times \begin{bmatrix} T_1 & T_3 \\ T_4 & T_2 \end{bmatrix} \begin{bmatrix} E_{\rm par}^{\rm inc} \\ E_{\rm per}^{\rm inc} \end{bmatrix}, \tag{3.15}$$

where the elements of the amplitude scattering matrix T_1, T_2, T_3, and T_4 are given by

$$T_1 = \sum_m \left\{ \frac{i^m}{\sin\theta_0} [a_{om}^{(1)} + a_{om}^{(2)} \exp(i\lambda_0 d\cos\phi)] \times \exp[-im(\phi - \phi_0)] \right\}, \tag{3.16}$$

$$T_2 = \sum_m \left\{ \frac{i^m}{\sin\theta_0} [c_{om}^{(1)} + c_{om}^{(2)} \exp(i\lambda_0 d\cos\phi)] \times \exp[-im(\phi - \phi_0)] \right\}, \tag{3.17}$$

$$T_3 = \sum_m \left\{ \frac{i^m}{\sin\theta_0} \frac{k_0}{\mu_0\omega} [d_{om}^{(1)} + d_{om}^{(2)} \exp(i\lambda_0 d\cos\phi)] \times \exp[-im(\phi - \phi_0)] \right\}, \tag{3.18}$$

$$T_4 = \sum_m \left\{ \frac{i^m}{\sin\theta_0} \frac{\mu_0\omega}{k_0} [b_{om}^{(1)} + b_{om}^{(2)} \exp(i\lambda_0 d\cos\phi)] \times \exp[-im(\phi - \phi_0)] \right\}. \tag{3.19}$$

It is seen from Eq. (3.15) that the scattered electric field from the two cylinders is a superposition of both the TM and the TE modes even though the mode of the incident field is either TM ($E_{\rm per}^{\rm inc} = 0, E_{\rm par}^{\rm inc} = 1$) or TE ($E_{\rm per}^{\rm inc} = 1, E_{\rm par}^{\rm inc} = 0$). The two cylinders change the polarization state of the incident field.

In order to find the Mueller matrix, we must first define the Stokes vectors for the incident and scattered fields. Therefore we discuss them briefly.

The Stokes parameters are a set of four quantities (written as a column matrix) that describes the polarization state of the radiation. Following the nomenclature of Walker,[12] we denote them by I, Q, U, and V, and they are related to the electric fields by

$$I = \frac{k_0}{2\omega\mu_0} \langle E_{\rm par}E_{\rm par}^* + E_{\rm per}E_{\rm per}^* \rangle, \tag{3.20a}$$

$$Q = \frac{k_0}{2\omega\mu_0} \langle E_{\rm par}E_{\rm par}^* - E_{\rm per}E_{\rm per}^* \rangle, \tag{3.20b}$$

$$U = \frac{k_0}{2\omega\mu_0} \langle E_{\rm par}E_{\rm per}^* + E_{\rm per}E_{\rm par}^* \rangle, \tag{3.20c}$$

$$V = -i\frac{k_0}{2\omega\mu_0} \langle E_{\rm par}E_{\rm per}^* - E_{\rm per}E_{\rm par}^* \rangle, \tag{3.20d}$$

where $E_{\rm par}$ and $E_{\rm per}$ are the electric fields parallel and perpendicular to the incident plane or to the scattering plane if the Stokes parameters of the incident field or the scattered field, respectively, are to be found. The angle brackets $\langle\ \rangle$ signify the time average. I represents the total intensity, while Q, U, and V together represent the polarization state. The Stokes parameters are related by the following inequality: $I^2 \geq Q^2 + U^2 + V^2$, where $\geq$ is replaced by $=$ for completely polarized light.

It is worthwhile to note that Eqs. (3.20a)–(3.20c) are the same if either $e^{i\omega t}$ or $e^{-i\omega t}$ is used for the time dependence, whereas V is dependent on this choice. Thus Eq. (3.20d) can be interpreted as the difference between the irradiances of the right-handed circular polarization and the left-handed circular polarization. (We chose the sense of rotation of the electric field to be counterclockwise for right-handed circular polarization as looking toward the source; i.e., $e^{i\omega t}$ was chosen.)

The relation between the incident and the scattered Stokes vectors can be written in terms of the 4×4 Mueller matrix S with elements S_{ij}:

$$\begin{bmatrix} I^s \\ Q^s \\ U^s \\ V^s \end{bmatrix} = \left(\frac{2}{\pi\lambda_0\rho}\right) \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} I^i \\ Q^i \\ U^i \\ V^i \end{bmatrix}. \tag{3.21}$$

This equation defines the Mueller matrix and can be considered a mathematical model that describes the interaction between the incident field and the optical device (the two cylinders in our case).

The definitions of the Stokes parameters [Eqs. (3.20)] and the T matrix [Eq. (3.21)] give the relations among the Mueller scattering matrix elements (S_{ij}) and the amplitude scattering matrix elements:

$$S_{11} = \tfrac{1}{2}[|T_1|^2 + |T_2|^2 + |T_3|^2 + |T_4|^2],$$
$$S_{12} = \tfrac{1}{2}[|T_1|^2 - |T_2|^2 + |T_4|^2 - |T_3|^2],$$
$$S_{13} = \mathrm{Re}[T_1T_3^* + T_4T_2^*],$$
$$S_{14} = \mathrm{Im}[T_3T_1^* + T_2T_4^*],$$
$$S_{21} = \tfrac{1}{2}[|T_1|^2 - |T_2|^2 - |T_4|^2 + |T_3|^2],$$
$$S_{22} = \tfrac{1}{2}[|T_1|^2 + |T_2|^2 - |T_4|^2 - |T_3|^2],$$
$$S_{23} = \mathrm{Re}[T_1T_3^* - T_2T_4^*],$$
$$S_{24} = \mathrm{Im}[T_3T_1^* + T_4T_2^*],$$
$$S_{31} = \mathrm{Re}[T_1T_4^* + T_2T_3^*],$$
$$S_{32} = \mathrm{Re}[T_1T_4^* - T_2T_3^*],$$
$$S_{33} = \mathrm{Re}[T_2T_1^* + T_3T_4^*],$$
$$S_{34} = \mathrm{Im}[T_2T_1^* + T_3T_4^*],$$
$$S_{41} = \mathrm{Im}[T_1T_4^* + T_3T_2^*],$$
$$S_{42} = \mathrm{Im}[T_1T_4^* + T_2T_3^*],$$
$$S_{43} = \mathrm{Im}[T_1T_2^* + T_3T_4^*],$$
$$S_{44} = \mathrm{Re}[T_1T_2^* - T_4T_3^*],$$

where Re and Im represent the real and imaginary parts, respectively, of the quantities of interest.

A comprehensive discussion of the Mueller matrix and the Stokes vectors and their physical meanings can be found in Ref. 13. An excellent description of the experimental procedures to determine them was given by Bickel and Bailey.[14]

We also point out that the 16 elements of the Mueller matrix are not independent. There are nine independent relationships among these elements.[15]

4. CROSS SECTIONS IN THE FAR-FIELD REGION

In this section, expressions for the total scattering and extinction cross sections per unit length are obtained for the two polarizations of the incident wave (the TM and TE polarizations considered in Section 2).

We express the scattering cross sections in terms of the scattering coefficients and the extinction cross section in terms of the forward-scattering amplitude.

We begin by finding the rate at which energy is extracted from the incident beam because of *scattering*. The scattering cross section can then be found. In a similar manner the rate at which energy is extracted from the incident beam because of *absorption* is calculated. The extinction cross section can then also be found.

The scattered or absorbed energy per unit time will be found from the real part of the Poynting vector.

In the discussion in Subsection 4.A we refer to the two cylinders as the scatterer. Since the scatterer is of an infinite length, the cross section per unit length is the quantity of physical interest.

A. Scattering Cross Section Per Unit Length

The scattering cross section per unit length is the total power scattered in all directions by a length (l) of the scatterer when the irradiance of the incident beam is assumed to be unity. Therefore the scattering cross section per unit length (C_{sca}) may be written as

$$C_{\mathrm{sca}} = \lim_{l\to\infty} \frac{P_{\mathrm{sca}}}{lI_i}, \tag{4.1}$$

where P_{sca} is the scattered power in watts, l is the length of the scatterer in meters, and I_i is the irradiance of the incident plane waves in watts per meter. It is clear from Eq. (4.1) that the SI units of the scattering cross section per unit length are meters.

The total scattered power can be found by integrating the radial component of the scattered Poynting vector (S_ρ) over a large surface that encloses the scatterer. It is natural for this problem to choose a cylindrical surface of length l and radius ρ. The advantage of choosing such a surface is that the radial component of the Poynting vector is perpendicular to it. Therefore Eq. (4.1) can be written as

$$C_{\mathrm{sca}} = \lim_{l\to\infty} \frac{\int S_\rho \mathrm{d}A}{lI_i}, \tag{4.2}$$

where $\mathrm{d}A$ is the differential element of the cylindrical surface.

The Poynting vector is related to the electric field (**E**) and the magnetic field (**H**) by

$$\mathbf{S} = \tfrac{1}{2}\mathbf{E} \times \mathbf{H}^*, \tag{4.3}$$

where H^* is the complex conjugate of the magnetic field (H).

The radial component of the Poynting vector (S_ρ) can be expressed in terms of the components of the fields; this follows directly from Eq. (4.3):

$$S_\rho = \tfrac{1}{2}\,\mathrm{Re}[E_\phi H_z^* - E_z H_\phi^*]. \tag{4.4}$$

The total scattered power may be found by integrating Eq. (4.4) over the cylindrical surface. This gives

$$P_{sca} = \tfrac{1}{2}\,\mathrm{Re}\int_{-l/2}^{l/2}\int_0^{2\pi} (E_\phi^{sca}H_z^{*sca} - E_z^{sca}H_\phi^{*sca})\mathrm{d}\phi\mathrm{d}z, \qquad (4.5)$$

where E_ϕ^{sca} and E_z^{sca} are the ϕ component and the z component, respectively, of the total scattered electric field and are given by Eqs. (3.6) and (3.7). H_z^{sca} is the z component of the total scattered magnetic field in the far-field region and is obtained by adding the contribution from each cylinder. From Eq. (3.3) we find that

$$H_z^{sca} = G\sum_m \{i^m[b_{om}^{(1)} + b_{om}^{(2)}\exp(i\lambda_0 d\cos\phi)] \times \exp[-im(\phi - \phi_0)]\}, \qquad (4.6)$$

for which G is defined as for Eq. (3.2).

The ϕ component of the scattered magnetic field H_ϕ^{sca} in the far-field region can be obtained by using Eq. (2.17), i.e.,

$$H_\phi^{sca} \cong -i\,\frac{\epsilon_0\omega}{k_0^2\sin^2\theta_0}\,\frac{\partial E_z^{sca}}{\partial\rho}. \qquad (4.7)$$

This result is obtained by dropping the first term of Eq. (2.17) in the far-field scattering region and setting $\sigma = 0$, $\epsilon = \epsilon_0$, and $k = k_0$. We consider the medium outside the cylinders to be nonabsorbing to simplify the analysis.

By substituting Eq. (3.7) into Eq. (4.7), we obtain the following expression for H_ϕ^{sca}:

$$H_\phi^{sca} = -\frac{\epsilon_0\omega}{k_0\sin\theta_0}\,G\sum_m \{i^m[a_{om}^{(1)} + a_{om}^{(2)}\exp(i\lambda_0 d\cos\phi)] \times \exp[-im(\phi - \phi_0)]\}. \qquad (4.8)$$

In Eq. (4.5) the integrand does not depend on z. Therefore it reduces to

$$P_{sca} = \frac{l}{2}\,\mathrm{Re}\int_0^{2\pi} (E_\phi^{sca}H_z^{*sca} - E_z^{sca}H_\phi^{*sca})\mathrm{d}\phi. \qquad (4.9)$$

If the expressions for the fields (E_ϕ, H_z, E_z, and H_ϕ) are substituted into Eq. (4.9) and if the integration over ϕ is performed, then, by using some of the properties of Bessel functions, we obtain the following expression for the cross section per unit length for the TM polarization of the incident wave:

$$C_{sca}^{TM} = \frac{4\eta\mu_0\omega}{\lambda_0^2}\,C_{sca}^{TMII} + \frac{4\eta\epsilon_0\omega}{\lambda_0^2}\,C_{sca}^{TMI}, \qquad (4.10)$$

where

$$C_{sca}^{TMI} = \sum_m \left\{|a_{om}^{(1)}|^2 + |a_{om}^{(2)}|^2 + \mathrm{Re}\sum_n \gamma[e^{-i\beta}a_{om}^{(1)}a_{on}^{*(2)} + e^{i\beta}a_{om}^{(2)}a_{on}^{*(1)}]\right\}, \qquad (4.11)$$

$$C_{sca}^{TMII} = \sum_m \left\{|b_{om}^{(1)}|^2 + |b_{om}^{(2)}|^2 + \mathrm{Re}\sum_n \gamma[e^{-i\beta}b_{om}^{(1)}b_{on}^{*(2)} + e^{i\beta}b_{om}^{(2)}b_{on}^{*(1)}]\right\}, \qquad (4.12)$$

$$\gamma = (-1)^n i^{m+n}\exp[-i(n-m)\phi_0]J_{n-m}(\lambda_0 d),$$

$$\beta = (n-m)\pi/2,$$

and η is the impedance of the medium outside the cylinders. TMI and TMII refer to the scattered TM and TE polarizations, respectively.

In Eq. (4.2), $I_i = 1/2\eta$ is the irradiance of the incident wave. This last result is a consequence of our original assumption that the amplitude of the incident wave for the TM polarization case is 1.

The cross section per unit length for the incident TE polarization may be found from that of the TM polarization. If $a_{om}^{(1)}$ and $a_{om}^{(2)}$ are replaced by $d_{om}^{(1)}$ and $d_{om}^{(2)}$ and if $b_{om}^{(1)}$ and $b_{om}^{(2)}$ are replaced by $c_{om}^{(1)}$ and $c_{om}^{(2)}$, then the following expression is obtained:

$$C_{sca}^{TE} = \frac{4\mu_0\omega}{\lambda_0^2\eta}\,C_{sca}^{TEI} + \frac{4\epsilon_0\omega}{\lambda_0^2\eta}\,C_{sca}^{TEII}, \qquad (4.13)$$

where

$$C_{sca}^{TEI} = \sum_m \left\{|c_{om}^{(1)}|^2 + |c_{om}^{(2)}|^2 + \mathrm{Re}\sum_n \gamma[e^{-i\beta}c_{om}^{(1)}c_{on}^{*(2)} + e^{i\beta}c_{om}^{(2)}c_{on}^{*(1)}]\right\}, \qquad (4.14)$$

$$C_{sca}^{TEII} = \sum_m \left\{|d_{om}^{(1)}|^2 + |d_{om}^{(2)}|^2 + \mathrm{Re}\sum_n \gamma[e^{-i\beta}d_{om}^{(1)}d_{on}^{*(2)} + e^{i\beta}d_{om}^{(2)}d_{on}^{*(1)}]\right\}. \qquad (4.15)$$

TEI and TEII refer to the scattered TE and TM polarizations, respectively.

The irradiance (I_i) of the incident wave is now equal to $\eta/2$ because of our original assumption that the amplitude of the incident magnetic field is 1.

B. Extinction Cross Section per Unit Length

If the scatterer has a finite conductivity, some of the incident energy is absorbed. The absorbed energy is converted into other forms, such as heat. The absorbed power (P_{abs}) is related to the incident irradiance (I_i) by

$$P_{abs} = Q_{abs}I_i, \qquad (4.16)$$

where Q_{abs} is the absorption cross section.

Since scattering and absorption remove energy from the incident field, the incident wave is attenuated. The extinction cross section is designed to account for the attenuation of the incident waves. Therefore we may define this cross section as the power removed from the incident radiation by scattering and absorption (P_{ext}) when the irradiance of the incident field is assumed to be unity, i.e.,

$$P_{ext} = Q_{ext}I_i, \qquad (4.17)$$

where Q_{ext} is the extinction cross section.

The extinction cross section is the sum of the absorption cross section and the scattering cross section:

$$Q_{ext} = Q_{abs} + Q_{sca}. \qquad (4.18)$$

The above statement is another way of stating that energy is conserved.

To find the extinction cross section, we must find the total energy per unit time that is removed from the incident beam by scattering and absorption. This can be calculated by integrating the radial component of the Poynting vector (S_ρ^{ext}) over the cylindrical surface (described in Subsection 4.A).

$$S_\rho^{\text{ext}} = \tfrac{1}{2}\,\text{Re}(E_\phi^i H_z^{*s} - E_z^i H_\phi^{*s} + E_\phi^s H_z^{*i} - E_z^s H_\phi^{*i}), \qquad (4.19)$$

where i and s refer to incident and scattered fields, respectively. Discussion of Eq. (4.19) can be found in many references. See, for example, Ref. 16.

It is clear from Eq. (4.19) that the radial component of the Poynting vector (S_ρ^{ext}) depends on the polarization state of the incident wave; therefore we must find the extinction cross sections for the two independent polarizations (the TM and TE polarizations).

1. *TM Polarization*

Since the incident magnetic field has no component along the z axis, i.e., $H_z^{\text{inc}} = 0$, Eq. (4.19) reduces to

$$S_\rho^{\text{ext}} = \tfrac{1}{2}\,\text{Re}(E_\phi^i H_z^{*s} - E_z^i H_\phi^{*s} - E_z^s H_\phi^{*i}). \qquad (4.20)$$

The power that is due to extinction may be found by integrating Eq. (4.20) over the cylindrical surface, which we described in Subsection 4.A:

$$P_{\text{ext}} = \lim_{l\to\infty} \int_{-l/2}^{l/2} \int_{-\pi}^{\pi} S_\rho^{\text{ext}} \rho \mathrm{d}\phi \mathrm{d}z. \qquad (4.21)$$

Since the integrand has no z dependence, Eq. (4.21) reduces to

$$P_{\text{ext}} = \lim_{l\to\infty} l \int_{-\pi}^{\pi} S_\rho^{\text{ext}} \rho \mathrm{d}\phi. \qquad (4.22)$$

By substituting Eq. (4.20) into Eq. (4.22), we obtain the extinction power per unit length ($P_{\text{norm}} = P_{\text{ext}}/l$):

$$P_{\text{norm}} = \frac{1}{2}\,\text{Re} \int_{-\pi}^{\pi} (E_\phi^i H_z^{*s} - E_z^i H_\phi^{*s} - E_z^s H_\phi^{*i}) \rho \mathrm{d}\phi. \qquad (4.23)$$

If the appropriate expressions for E_z^i, H_ϕ^s, E_z^s, and H_ϕ^i are substituted into Eq. (4.23), then the first integral is zero, and the second integral is the complex conjugate of the third integral. Therefore Eq. (4.23) may be written as

$$P_{\text{norm}} = -\text{Re} \int_{-\pi}^{\pi} E_z^i H_\phi^{*s} \rho \mathrm{d}\phi. \qquad (4.24)$$

If the limit is divided into two parts [the first part being $(-\pi, 0)$ and the second being $(0, \pi)$], then Eq. (4.24) may be written as

$$P_{\text{norm}} = -\text{Re} \int_{\pi}^{0} E_z^i(-\phi) H_\phi^{*s}(-\phi) \rho \mathrm{d}\phi + \text{Re} \int_{\pi}^{0} E_z^i(\phi) H_\phi^{*s}(\phi) \rho \mathrm{d}\phi. \qquad (4.25)$$

Each of the integrals [in Eq. (4.25)] may be taken by using the stationary phase method, which is given in Ref. 17, p. 749. By using Eq. (4.17), we can obtain the extinction cross section $C_{\text{ext}} = Q_{\text{ext}}/l$:

$$C_{\text{ext}} = -\frac{4}{k_0}\,\text{Re}[T_1(\phi = \pi + \phi_0)]. \qquad (4.26)$$

2. *TE Polarization*

For the TE polarization, the expression for the extinction cross section per unit length can be written from the symmetry relation [T_1 is replaced by T_2 in Eq. (4.26)]:

$$C_{\text{ext}} = -\frac{4}{k_0}\,\text{Re}[T_2(\phi = \pi + \phi_0)]. \qquad (4.27)$$

Equations (4.26) and (4.27) also result from the well-known optical theorem.

5. SPECIAL CASES AND LIMITING CASES

A. Perpendicular Incidence

The case of perpendicular incidence corresponds to $\theta_0 = \pi/2$. The scattering of electromagnetic waves from two parallel cylinders at normal incidence becomes a two-dimensional problem (the fields do not vary in the z direction). The scattering coefficients reduce to

$$a_{om}^{(j)} = -\frac{\dfrac{k_0}{\mu_0} J_m'(k_0 a_j) J_m(k_j a_j) - \dfrac{k_i}{\mu_j} J_m'(k_j a_j) J_m(k_0 a_j)}{\dfrac{k_0}{\mu_0} H_m^{(2)\prime}(k_0 a_j) J_m(k_j a_j) - \dfrac{k_i}{\mu_j} J_m'(k_j a_j) H_m^{(2)}(k_0 a_j)} \times \{i^m E_0 \exp[i\delta(j-1)] + A_{om}^{(l)}\}. \qquad (5.1)$$

The cross modes reduce to zero; i.e., $b_{om}^{(1)} = b_{om}^{(2)} = 0$. This result is similar to that for a single cylinder (no coupling between the TE mode and the TM mode for perpendicular incidence).

Equation (5.1) shows explicitly that the scattering coefficient of each cylinder is modified by the presence of the other. $A_{om}^{(l)}$ (which is given in Section 2) represents the modification to the single-cylinder coefficient. The first term on the right-hand side of Eq. (5.1) (containing E_0) is identical to the single-scattering coefficient for cylinder j. We refer to it as such. The second term (containing $A_{om}^{(l)}$ is due to multiple scattering. We may look at this second term as a correction to the zero-order scattering coefficient. The expression for $A_{om}^{(l)}$ includes a Hankel function, which results in the multiple scattering becoming unimportant, as it should, if the separation between the two cylinders becomes large. $a_{om}^{(1)}$ and $a_{om}^{(2)}$ then reduces to the correct expressions for the single cylinders at normal incidence.

The expressions for the scattering coefficients given by Eq. (5.1) are more general than those given by Olaofe,[8] because Olaofe's expressions are valid for only two dielectric, identical cylinders. Our equations cannot be put in a form such that they are identical to those given by Olaofe, because we use $e^{i\omega t}$ for the time dependence in our solution, whereas Olaofe based his solution on the $e^{-i\omega t}$ factor. Consequently, our equations involve the Hankel function of the second kind, and Olaofe's equations involve the Hankel function of the first kind. However, we verified by numerical comparisons that his expressions and ours give exactly the same results.

B. Low-Frequency Approximation

We consider the case in which $\lambda_0 a_1$ and $\lambda_0 a_2$ are $\ll 1$. (This limit can be achieved either with low frequency or with small radii.)

In this limit, the only nonvanishing term is $m = 0$. In other words, $a_{om}^{(j)} = 0$ if $m \neq 0$. Therefore the general expression for $a_{om}^{(j)}$ reduces to

$$a_{00}^{(1)} = -E_0 \sin\theta_0 \left\{ \frac{1 + \beta H_0^{(2)}(\lambda_0 d) e^{i\delta}}{\beta[H_0^{(2)}(\lambda_0 d)]^2 - \dfrac{1}{\alpha}} \right\}, \tag{5.2}$$

$$a_{00}^{(2)} = -E_0 \sin\theta_0 \left\{ \frac{\alpha H_0^{(2)}(\lambda_0 d) + e^{i\delta}}{\alpha[H_0^{(2)}(\lambda_0 d)]^2 - \dfrac{1}{\beta}} \right\}, \tag{5.3}$$

where

$$\alpha = -\frac{\dfrac{k_0^2}{\mu_0\lambda_0} J_0(\lambda_1 a_1) J_0'(\lambda_0 a_1) - \dfrac{k_1^2}{\mu_1\lambda_1} J_0'(\lambda_1 a_1) J_0(\lambda_0 a_1)}{\dfrac{k_0^2}{\mu_0\lambda_0} H_0^{(2)\prime}(\lambda_0 a_1) J_0(\lambda_1 a_1) - \dfrac{k_1^2}{\mu_1\lambda_1} J_0'(\lambda_1 a_1) H_0^{(2)}(\lambda_0 a_1)},$$

$$\beta = -\frac{\dfrac{k_0^2}{\mu_0\lambda_0} J_0(\lambda_2 a_2) J_0'(\lambda_0 a_2) - \dfrac{k_2^2}{\mu_2\lambda_2} J_0'(\lambda_2 a_2) J_0(\lambda_0 a_2)}{\dfrac{k_0^2}{\mu_0\lambda_0} H_0^{(2)\prime}(\lambda_0 a_2) J_0(\lambda_2 a_2) - \dfrac{k_2^2}{\mu_2\lambda_2} J_0'(\lambda_2 a_2) H_0^{(2)}(\lambda_0 a_2)},$$

and $b_{om}^{(1)} = b_{om}^{(2)} = 0$.

The assumption made, that the $m = 0$ term is the leading term, is easily justified. The induced current on the surface of each cylinder does not depend on the angle ϕ if the radius is small. As a consequence, the scattered field from each cylinder does not depend on the angle ϕ. It is worth noticing that the previous approximation uncoupled the linear equations of the scattering coefficients even though this problem is still a three-dimensional one.

For perfectly conducting cylinders Eqs. (5.2) and (5.3) simplify considerably. It is easy to write the following expressions for the scattering coefficients {Bessel functions of the first kind [$J_0(\lambda_0 a_1)$ and $J_0(\lambda_0 a_2)$] are set to 1, since their arguments are small}:

$$a_{00}^{(1)} = -E_0 \sin\theta_0 \left\{ \frac{H_0^{(2)}(\lambda_0 a_2) - H_0^{(2)}(\lambda_0 d) e^{i\delta}}{H_0^{(2)}(\lambda_0 a_1) H_0^{(2)}(\lambda_0 a_2) - [H_0^{(2)}(\lambda_0 d)]^2} \right\}, \tag{5.4}$$

$$a_{00}^{(2)} = -E_0 \sin\theta_0 \left\{ \frac{H_0^{(2)}(\lambda_0 a_1) e^{i\delta} - H_0^{(2)}(\lambda_0 d)}{H_0^{(2)}(\lambda_0 a_1) H_0^{(2)}(\lambda_0 a_2) - [H_0^{(2)}(\lambda_0 d)]^2} \right\}. \tag{5.5}$$

Equations (5.4) and (5.5) are essentially those given by Wait.[11] However, Wait expressed his results in terms of the modified Bessel function, which is related to the Hankel function of the second kind (which we used in our solution) as follows:

$$K_0(\gamma a) = -\tfrac{1}{2}\pi i H_0^{(2)}(\lambda_0 a),$$

where

$$\gamma = i\lambda_0.$$

With this substitution, Eqs. (5.4) and (5.5) reduce exactly to Wait's results.

In the case of a large separation with respect to all the radii, Eqs. (5.4) and (5.5) can be expanded to zero order in $[H_0^{(2)}(\lambda_0 d)]^2/H_0^{(2)}(\lambda_0 a_1)H_0^{(2)}(\lambda_0 a_2)$. The following expressions for $a_{00}^{(1)}$ and $a_{00}^{(2)}$ are then found:

$$a_{00}^{(1)} = -\frac{E_0 \sin\theta_0}{H_0^{(2)}(\lambda_0 a_1)} + \frac{E_0 \sin\theta_0}{H_0^{(2)}(\lambda_0 a_2)} \frac{H_0^{(2)}(\lambda_0 d)}{H_0^{(2)}(\lambda_0 a_1)} e^{i\delta}, \tag{5.6}$$

$$a_{00}^{(2)} = -\frac{E_0 \sin\theta_0 e^{i\delta}}{H_0^{(2)}(\lambda_0 a_2)} + \frac{E_0 \sin\theta_0}{H_0^{(2)}(\lambda_0 a_1)} \frac{H_0^{(2)}(\lambda_0 d)}{H_0^{(2)}(\lambda_0 a_2)}. \tag{5.7}$$

The scattering pattern of a single cylinder of radius a is the same whether the incident wave is planar or cylindrical if $b/a > 10$, where b is the distance between the cylinder and the source of the cylindrical waves (the radius of curvature of the cylindrical wave).[18] Therefore, for the two cylinders, the scattered field from one cylinder in the neighborhood of the other resembles a plane wave when $d/a_j \gg 1$.

Equations (5.6) and (5.7) reveal the fact that each cylinder is excited by the incident field and by the scattered field from the other cylinder. The first term on the right-hand side of Eq. (5.6) represents the zero-order scattering coefficient (the response of cylinder 1 as if it were by itself in the field). The second term represents the response of cylinder 1 to the scattered field from cylinder 2, which is $E_0 \sin\theta_0 H_0^{(2)}(\lambda_0 d)e^{i\delta}/H_0^{(2)}(\lambda_0 a_2)$.

The signs justify further discussion. The first term represents a direct excitation from the incident field; therefore the scattered field suffers one reflection (scattering), which explains the minus in that term. (The zero-order scattered field from cylinder 1 is out of phase with the incident field in the neighborhood of cylinder 1 apart from the phase that is due to the complex scattering coefficient). The second term represents indirect excitation of cylinder 1 (the incident field hits cylinder 2 first and then cylinder 1). Therefore the wave suffers two reflections (scattering), which explains the plus in that term.

$H_0^{(2)}(\lambda_0 d)$ in the second term takes into account the strength of the scattered field from cylinder 1 in the neighborhood of cylinder 2, as well as the phase of the wave. Careful examination of the argument of the Hankel function of the separation reveals the fact that it represents the phase difference between the incident field and the zero-order scattered field from cylinder 2 in the neighborhood of cylinder 1. (The phase factor in the Hankel function is due to the cylindrical character of the wave, and it has nothing to do with the scattering process.) A similar discussion also applies to Eq. (5.7).

C. Perfectly Conducting Cylinders

The case of perfectly conducting cylinders does not appear to have been treated previously; as a consequence of infinite conductivity, there are no fields present within the cylinders. The expressions for the scattering coefficients reduce to

$$a_{om}^{(1)} = -i^m E_0 \sin\theta_0 \frac{J_m(\lambda_0 a_1)}{H_m^{(2)}(\lambda_0 a_1)} - \frac{J_m(\lambda_0 a_1)}{H_m^{(2)}(\lambda_0 a_1)} A_{om}^{(2)}, \quad (5.8)$$

$$a_{om}^{(2)} = -i^m E_0 \sin\theta_0 \frac{J_m(\lambda_0 a_2)}{H_m^{(2)}(\lambda_0 a_2)} - \frac{J_m(\lambda_0 a_2)}{H_m^{(2)}(\lambda_0 a_2)} A_{om}^{(1)}. \quad (5.9)$$

In addition, $b_{om}^{(1)} = b_{om}^{(2)} = 0$.

Equations (5.8) and (5.9) are general equations for the case of perfectly conducting cylinders and may be used for any separation. The scattering coefficients for perfectly conducting cylinders at oblique incidence are simpler than those for the perpendicular incidence of an arbitrary material. In the former case, the scaling technique may be used (replace k_0 with $k_0 \sin\theta$), whereas this simple scaling technique cannot be used in the latter case.

The first term on the left-hand side of Eq. (5.8) or (5.9) represents the zero-order scattering coefficient of cylinder j (j = 1 or 2), and the second term represents the multiple scattering coefficient (the interaction) between the two cylinders.

Numerical results for the scattering amplitude as a function of the observation angle in the far-field region were compared with those obtained by Ragheb and Hamid at normal incidence[6]; a perfect agreement was obtained.

D. Single Cylinder

We permit the scattering coefficients of the second cylinder [$a_{om}^{(2)}$ and $b_{om}^{(2)}$] to be zero. This implies that the second cylinder does not exist and leads to $A_{om}^{(2)} = B_{om}^{(2)} = 0$. Our equations then reduce to two linear equations with two unknowns, $a_{om}^{(1)}$ and $b_{om}^{(1)}$. By using the relations

$$J_m(\lambda_0 a_1) H_{m-1}^{(2)}(\lambda_0 a_1) - H_m^{(2)}(\lambda_0 a_1) J_{m-1}(\lambda_0 a_1) = \frac{2}{\pi i \lambda_0 a_1},$$

$$H_{m-1}^{(2)}(\lambda_0 a_1) + H_{m+1}^{(2)}(\lambda_0 a_1) = \frac{2m}{\lambda_0 a_1} H_m^{(2)}(\lambda_0 a_1),$$

$$H_m^{(2)\prime}(\lambda_0 a_1) = \tfrac{1}{2} H_{m-1}^{(2)}(\lambda_0 a_1) - \tfrac{1}{2} H_{m+1}^{(2)}(\lambda_0 a_1),$$

Eqs. (10) and (11) of Ref. 3 are obtained. The comparison with Wait's equations described above, which was made by letting the scattering coefficients of the second cylinder be zero, is equivalent to letting the radius of the second cylinder be zero. This case cannot be studied numerically because of the singularity of the Hankel function. In order to study this limit numerically, we instead let the separation between the two cylinders become large. The scattering coefficients of two independent cylinders (noninteracting cylinders) were obtained. These results were compared with Eqs. (10) and (11) of Ref. 3 (in which the equations are programmed separately), and exact agreement was obtained. The square of the scattering amplitude was plotted as a function of the observation angle for selected values of the tilt angles in order to compare the results with those of Kerker *et al.*,[19] and we found exact agreement (the two curves are indistinguishable). Also, we found that the cross-polarized components of the scattered field are equal to each other and that they vanish in the forward and the backward directions, as they should for a single cylinder. In the same way, we were able to verify the results obtained by Lind and Greenberge[20] for the extinction cross section as a function of size parameter. Finally, we compared the Mueller scattering elements for a single cylinder at normal incidence with those obtained by Bohren and Huffman[13]; we found them also to be in exact agreement.

An alternative method to obtain the calculations for a single cylinder is to let the refractive index of the second cylinder be the same as the medium outside the cylinders. The validity of this method was also verified. The details of these comparisons and derivations can be found in the doctoral dissertation of Yousif.[21]

6. SUMMARY AND DISCUSSION

The analytical solution of the scattering of electromagnetic plane waves by two cylinders is obtained. This solution is in terms of the scattering coefficients from which the scattered fields, the cross sections, and the Mueller scattering matrix elements (S_{ij}) are constructed by using fundamental principles of electromagnetic theory.

In order to derive the scattering coefficients, the solutions of the wave equation inside and outside each cylinder are constructed, and the boundary conditions are applied at the surface of each cylinder. The effective incident field upon one cylinder includes not only the plane-wave incident field to make the solution an exact one but also the scattered field from the other cylinder. This solution is exact, as expressed by Eqs. (2.10), (2.11), etc. This proof is accomplished by using the translational theorem of Bessel functions (see Appendix A).

For each incident polarization state (TM and TE), four coupled linear equations for the scattering coefficients are obtained [see Eqs. (2.23), (2.24), (2.25), and (2.26)].

The expressions [Eqs. (4.26) and (4.27)] for the extinction cross sections could have been obtained directly from the optical theorem. The derivation in Section 4, however, provides an excellent opportunity to test the consistency of our solution.

This investigation was programmed on a high-speed computer. The results of the calculations are presented in part II of this series.[22]

APPENDIX A: TRANSLATIONAL THEOREM OF BESSEL FUNCTIONS

The Hankel function of the second kind ($H_m^{(2)} = J_m - iY_m$) can be constructed from Eqs. (5) and (6) on p. 361 of Ref. 23:

$$H_m^{(2)}(\varpi)\exp(\mp im\psi) = \sum_l H_{l+m}^{(2)}(Z) J_l(z)\exp(\mp il\phi). \quad (A1)$$

Comparing the triangle on p. 361 of Ref. 18 with the triangle O_1O_2P in Fig. 1, it is easy to see that

$$\psi = \phi_1, \qquad \phi = \pi - \phi_2, \qquad (\varpi) = \lambda_0 \rho_1,$$

$$z = \lambda_0 \rho_2, \qquad Z = \lambda_0 d;$$

therefore Eq. (A1) may be written as

$$H_m^{(2)}(\lambda_0 \rho_1)\exp(\mp im\phi_1) = \sum_l [(-1)^l H_{m+l}^{(2)}(\lambda_0 d) J_l(\lambda_0 \rho_2) \times \exp(\pm il\phi_2)]. \quad (A2)$$

Similarly,

$$H_m^{(2)}(\lambda_0\rho_2)\exp(\mp im\phi_2) = (-1)^m \sum_l H_{m+l}^{(2)}(\lambda_0 d)J_l(\lambda_0\rho_1) \times \exp(\pm il\phi_1). \tag{A3}$$

In the solution (Section 2) we have the following expressions:

$$\sum_m C_m^{(j)}H_m^{(2)}(\lambda_0\rho_j)\exp[-im(\phi_j - \phi_0)], \tag{A4}$$

where $C_m^{(j)}$ $(j = 1, 2)$ is the scattering coefficient of cylinder j.

Equation (A4) can be expressed in terms of the coordinates of cylinder 1 or cylinder 2. This can be done by using Eqs. (A2) and (A3):

$$\sum_m C_m^{(1)}H_m^{(2)}(\lambda_0\rho_1)\exp[-im(\phi_1 - \phi_0)] = \sum_m \left[C_m^{(1)} \sum_l (-1)^l H_{l+m}^{(2)}(\lambda_0 d)J_l(\lambda_0\rho_2) \times \exp(il\phi_2)\exp(im\phi_0) \right], \tag{A5}$$

$$\sum_m C_m^{(2)}H_m^{(2)}(\lambda_0\rho_2)\exp[-im(\phi_2 - \phi_0)] = \sum_m \left[(-1)^m C_m^{(2)} \sum_l H_{l+m}^{(2)}(\lambda_0 d)J_l(\lambda_0\rho_1) \times \exp(il\phi_1)\exp(im\phi_0) \right]. \tag{A6}$$

By interchanging the order of the summations on the right-hand sides of Eqs. (A5) and (A6), as well as the indices, and assuming that they converge uniformly, we may interchange the order of the summations again. This leads to the following equations:

$$C_m^{(1)}H_m^{(2)}(\lambda_0\rho_1)\exp[-im(\phi_1 - \phi_0)] = J_m(\lambda_0\rho_2)\exp[-im(\phi_2 - \phi_0)] \times \sum_l C_l^{(1)}H_{l-m}^{(2)}(\lambda_0 d)\exp[i(l - m)\phi_0], \tag{A7}$$

$$C_m^{(2)}H_m^{(2)}(\lambda_0\rho_2)\exp[-im(\phi_2 - \phi_0)] = J_m(\lambda_0\rho_1)\exp[-im(\phi_1 - \phi_0)] \times \sum_l (-1)^{l+m} C_l^{(2)}H_{l-m}^{(2)}(\lambda_0 d)\exp[i(l - m)\phi_0]. \tag{A8}$$

Equations (A7) and (A8) are the expressions required for translating the scattered field of one cylinder as incident upon the other.

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