Can one hear the shape of a drum ?

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Mark Kac
Can One Hear the Shape of a Drum?
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Outline

Section Content

Inverse Problems : Definition

Keller, 1976
Two problems are inverse to each other if the formulation of each of them requires all or partial knowledge of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former problem is called the direct problem, while the latter is called the inverse problem.

(Inverse Problems , Joseph B. Keller,
The American Mathematical Monthly, Vol. 83, No. 2. (Feb., 1976), pp. 107-118.)

Inverse Problems : Examples

Example 1

What are the questions to which the answers are

1. "Washington Irving" ?
2. "Nine W"?
3. "Chicken Sukiyaki"?

1. What is the capital of the United States, Max ?
2. Do you spell your name with a "V", Herr Wagner ?
3. What is the name of the sole surving Kamikaze pilot ?

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Example 2 (Differentiation)

Direct problem (DP): Given $\varphi \in C([0,1])$, solve

$$(T\varphi)(x) := \int_0^x \varphi(t)dt, \qquad x \in [0,1]$$

Inverse problem (IP) : Given $g \in C([0,1])$ with $g(0) = 0$, solve

$$T\varphi = g$$

Remark

(IP) has a solution $\varphi \in C([0,1])$ if and only if $g \in C^1([0,1])$.

Assume $g^\delta \in C([0,1])$ with $\|g^\delta - g\|_\infty \le \delta, \quad 0 < \delta < 1$.
Define $g_n^\delta(x) := g(x) + \delta \sin \frac{nx}{\delta}, \quad x \in [0,1]$
We have $(g_n^\delta)'(x) := g'(x) + n \cos \frac{nx}{\delta}, \quad x \in [0,1]$
It holds $\|(g_n^\delta)' - g'\|_\infty = n$.

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Example 3 (Backward Heat Conduction)

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}$$

in $D := [0,1] \times [0,T]$ with

$$(BC:) \quad u(0,t) = u(1,t) = 0, \quad 0 \le t \le T$$

$$(IC:) \quad u(x,0) = \varphi(x), \quad x \in [0,1]$$

Backward Heat Conduction II

Direct Problem :
Given the initial temperature $\varphi \in L^2([0,1])$,
find the final temperature $f := u(\cdot, T)$

$$u(x,t) = \sqrt{2}\sum_{n=1}^{\infty} \varphi_n e^{-\pi^2 n^2 t} \sin(n\pi x)$$

Inverse Problem :
Given the final temperature f, find the initial temperature φ.

$$(T_H\varphi)(x) := \int 2\sum_{n=1}^{\infty} \left(e^{-\pi^2 n^2 T} \sin(n\pi x) \sin(n\pi y) \right) \varphi(y) dy$$

$$T_H\varphi = f$$

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Hadamard's Postulation of Well-posedness

Hadamard (1902, 1923)
A problem is called well-posed, if it has the following properties

1. Existence of a solution.
2. Uniqueness of the solution.
3. (Stability) Continuous dependence of the solution on the data.

otherwise it is called ill-posed.

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Examples of Ill-Posed Problems

Example 4 (Cauchy Problem for the Laplace Equation)

Find a harmonic function u in $D := \mathbb{R} \times [0, \infty]$ satisfying the following initial conditions

$$u(\cdot, 0) = 0, \quad \frac{\partial}{\partial y} u(\cdot, 0) = f,$$

where f is a given continuous function.

Let $f_n(x) = \frac{1}{n} \sin nx, x \in \mathbb{R}$.
For $n \in \mathbb{N}$, we obtain the solution

$$u_n(x, y) = \frac{1}{n^2} \sin nx \sinh ny, \quad (x, y) \in D.$$

Clearly, $(f_n) \to 0$, but (u_n) doesn't converge in any reasonable norm.

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Example 5 (Fredholm Integral Equation of the First Kind)

$$A\varphi(x) := \int_a^b K(x,y)\varphi(y)dy, \quad x \in [c,d]$$

Solving $A\varphi = f$ is ill-posed if, for example, the kernel K is continuous. If K is continuous, then the operator A will be compact. In this case, the operator A will not have a bounded inverse.

Fredholm integral Equations 2. Kind

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}, \quad 0 \le x \le 1$$

Trapzoidal rule

n	$x=0$	$x=0.5$	$x=1$
4	-0.007146	-0.010816	-0.015479
8	-0.001788	-0.002711	-0.003882
16	-0.000447	-0.000678	-0.000971
32	-0.000112	-0.000170	-0.000243

Simpson's rule

n	$x=0$	$x=0.5$	$x=1$
4	-0.00006652	-0.00010905	-0.00021416
8	-0.00000422	-0.00000692	-0.00001366
16	-0.00000026	-0.00000043	-0.00000086
32	-0.00000002	-0.00000003	-0.00000005

Fredholm integral Equations 1. Kind

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 - e^{-(x+1)}, \quad 0 \le x \le 1$$

Trapzoidal rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	0.4057	0.3705	0.1704
8	-4.5989	14.6094	-4.4770
16	-8.5957	2.2626	-153.4805
32	3.8965	-32.2907	22.5570
64	-88.6474	-6.4484	-182.6745

Simpson's rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	0.0997	0.2176	0.0566
8	-0.5463	6.0868	-1.7274
16	-15.4796	50.5015	-53.8837
32	24.5929	-24.1767	67.9655
64	23.7868	-17.5992	419.4284

Ill-Posed Problems : Regularization

Definition 1 (Regularization)

Assume X, Y are normed spaces.
Let the operator $A : X \to Y$ be linear, bounded and injective.
A family of bounded linear operators $R_\alpha : Y \to X, \alpha > 0$ is called a regularization scheme for

$$A\varphi = f,$$

if it satisfies the following pointwise convergence

$$\lim_{\alpha \to 0} R_\alpha A\varphi = \varphi, \text{ for all } \varphi \in X$$

In this case, the parameter α is called the regularization parameter.

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Regularization : Error

Find a stable approximation to the equation

$$A\varphi = f$$

The regularized approximation

$$\varphi_\alpha^\delta := R_\alpha f^\delta$$

The total approximation error

$$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A\varphi - \varphi$$

We have

$$\|\varphi_\alpha^\delta - \varphi\| \le \delta \|R_\alpha\| + \|R_\alpha A\varphi - \varphi\|$$

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Regularization : Methods

How to choose the regularization parameter α ?

1. a priori choice based on some information of the solution.
   In general not available
2. a posteriori choice based on the data error level δ

Discrepancy Principle of Morozov :

$$\|AR_\alpha f^\delta - f^\delta\| = \gamma\delta, \qquad \gamma \geq 1$$

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Regularization : Example

X, Y Hilbert spaces.

Theorem 1

Assume $A : X \to Y$ compact and linear.
Then for every $\alpha > 0$, the operator

$$\alpha I + A^* A : X \to X$$

is bijective and has a bounded inverse.
Furthermore, if the operator A is injective, then

$$R_\alpha := (\alpha I + A^* A)^{-1} A^*, \qquad \alpha > 0$$

describes a regularization scheme with $\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}$.

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Tikhonov Regularization

Theorem 2

Let $A : X \to Y$ be a linear and bounded operator. Assme $\alpha > 0$. Then for each $f \in Y$ there exists a unique $\varphi_\alpha \in X$ such that

$$\|A\varphi_\alpha - f\| + \alpha\|\varphi_\alpha\| = inf_{\varphi \in X}\left\{\|A\varphi - f\|^2 + \alpha\|\varphi\|^2\right\}$$

The minimizer φ_α is given by the unique solution of the equation

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

and depends continuously on f.

Approximate Solution

Definition 2 (Minimum Norm Solution)

Let $A : X \to Y$ be a bounded linear operator and let $\delta > 0$. For a given $f \in Y$ an element $\varphi_0 \in X$ is called a minimum norm solution of $A\varphi = f$ with discrepancy δ if $\|A\varphi_0 - f\| \leq \delta$ and

$$\|\varphi_0\| = \inf_{\|A\varphi - f\| \leq \delta} \|\varphi\|$$

Remark

φ_0 is a minimal norm solution to $A\varphi = f$ with discrepancy δ if and only if φ_0 is a best approximation to the zero element of X with respect to $U_f := \{\varphi \in X : \|A\varphi - f\| \leq \delta\}$.

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Theorem 3

Let $A : X \to Y$ be a linear and bounded operator with dense range. For $\delta > 0$, there exists for every $f \in Y$ a unique minimal norm solution of $A\varphi = f$ with discrepancy δ.
Furthermore, the parameter α can be so chosen, that φ_0 is the solution of

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

with $\|A\varphi_0 - f\| = \delta$.

Theorem 4

Assume $A : X \to Y$ is a linear, bounded and injective operator with dense range. $\delta > 0, f \in A(X)$. For $f^\delta \in Y$ with $\|f^\delta - f\| \leq \delta$ and $\delta < \|f^\delta\|$ we have

$$\varphi^\delta \to A^{-1}f, \qquad \delta \to 0,$$

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Section Content

4 main types of integral equations

- Fredholm Integral Equations
  - 1. kind

$$\int_a^b K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

  - 2. kind

$$\varphi(x) + \int_a^b K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

- Volterra Integral Equations
  - 1. kind

$$\int_a^x K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

  - 2. kind

$$\varphi(x) + \int_a^x K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

Historical Remarks I

- Maxime Bôcher 1908
  The theory of integral equations may be regarded as dating back at least as far as the discovery by Fourier of the theorem concerning integrals which bears his name; for, though this was not the point of view of Fourier, this theorem may be regarded as a statement of the solution of a certain integral equation of the first kind.
  Fourier's inversion formula

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(x\xi) f(\xi) d\xi$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(x\xi) g(\xi) d\xi$$

Historical Remarks II

- Abel's Integral 1826
  - a mechanical problem : a Tautochrone
  - the general accepted begin of the theory of integral equations
  - actually an inverse problem
  - The problem is to find the unknown path in the plane along which a particle will fall, under the influence of gravity alone, so that at each instant the time of fall is a known function of the distance fallen.

$$g(t) = \int_0^t \frac{f(y)}{\sqrt{2a(t-y)}} dy$$

Historical Remarks III

- Joachimstahl's attraction problem 1861
  - also an inverse problem
  - find the law of attraction if one knows the attraction force

$$\frac{g(h)}{2h} = \int_h^\infty \frac{f(r)}{\sqrt{r^2 - h^2}} dr$$

- at the turn of 20. century :
  - Volterra, Fredholm, Hilbert, Schmidt, ...
  - Introduction of Hilbert spaces
  - functional analysis

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Solutions of Integral Equations

- Over 99.99 . . . % of integral equations do not have a closed form solution.
- The solvability of integral equations is ensured by functional analytic approach.
- Numerical approximate solutions.

Linear Operators

- linear operator
  X, Y linear spaces. $A : X \to Y$ is linear iff for all $\alpha, \beta \in \mathbb{C}$

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g), \quad \forall f, g \in X$$

- bounded operator
  X, Y are normed spaces. A is bounded if there exists a constant $C > 0$ such that

$$\|Af\| \leq C\|f\|, \quad \forall f \in X$$

- compact operator
  A is compact if it maps a bounded set to a relatively compact set.

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Integral Operators

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$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy, \quad x \in G \subset \mathbb{R}^m$$

where K is called the kernel of the integral operator.

- K is called weakly singular iff there exists a constant $M > 0$ and $\alpha \in (0, m]$ such that

$$|K(x,y)| \leq M|x-y|^{\alpha-m}, \quad \forall x, y \in G \subset \mathbb{R}^m, x \neq y$$

- A is compact if K is continuous or weakly singular.

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Riesz Theory I

Consider the following integral equation of the second kind with a compact $A : X \to X$

$$\varphi - A\varphi = f$$

Let $L := I - A$.

- First Riesz Theorem
  The nullspace of L is a finite-dimensional subspace.
- Second Riesz Theorem
  The range of the operator L is a closed linear subspace.

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Riesz Theory II

- Third Riesz Theorem
  There exists a uniquely determined nonnegative integer r, called the Riesz number of A such that

$$\{0\} = N(L^0) \overset{\subset}{\neq} N(L^1) \overset{\subset}{\neq} \cdots \overset{\subset}{\neq} N(L^r) = N(L^{r+1}) = \dots,$$

and

$$X = L^0(X) \overset{\supset}{\neq} L^1(X) \overset{\supset}{\neq} \cdots \overset{\supset}{\neq} L^r(X) = L^{r+1}(X) = \dots,$$

Furthermore, we have the direct sum

$$X = N(L^r) \oplus L^r(X).$$

Fundamental Result of Riesz Theory

Theorem 5

Let $A : X \to X$ be a compact operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective, then its inverse operator $(I - A)^{-1}$ is bounded.

Solvability of a Second Kind Equation I

Theorem 6

If the homogeneous equation

$$\varphi - A\varphi = 0$$

has ony the trivial solution $\varphi = 0$*, then for each* $f \in X$ *the inhomogeneous equation*

$$\varphi - A\varphi = f$$

has a unique solution $\varphi \in X$ *and this solution depends continuously on* f*.*

Solvability of a Second Kind Equation II

Theorem 7

If the homogeneous equation

$$\varphi - A\varphi = 0$$

has nontrivial solution $\varphi \neq 0$*, then it has only a finite number m of linearly independent solutions* $\varphi_1, \varphi_2, \ldots, \varphi_m \in X$ *and the inhomogeneous equation is either unsolvable or its general solution is of the form*

$$\varphi = \tilde{\varphi} + \sum_{i=1}^{m} \alpha_i \varphi_i$$

where $\tilde{\varphi}$ *is a particular solution of the inhomogeneous equation.*

Remarks

- Reduction of the solvability of the equation to the solvability of the simpler homogeneous equation $\varphi - A\varphi = 0$.
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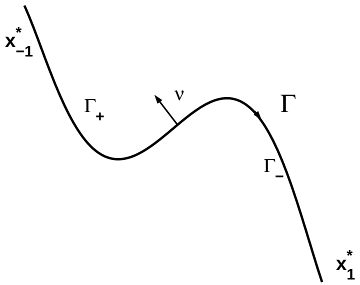
Section Content



$$\Gamma_0 := \Gamma \setminus \{x^*_{-1}, x^*_1\}$$

Direct Neumann problem

Given: C^3-open arc Γ
Find: $u \in C^2(\mathbb{R}^2 \setminus \Gamma) \cap C(\mathbb{R}^2 \setminus \Gamma_0)$, a solution to the Helmholtz equation

$$\Delta u + k^2 u = 0, \quad \text{in } \mathbb{R}^2 \setminus \Gamma, k \in \mathbb{R}. \tag{1}$$

which satisfies the Neumann boundary condition(NBC)

$$\frac{\partial u_\pm}{\partial \nu} = f \qquad \text{on } \Gamma_0 \tag{2}$$

for $f \in C^{0,\alpha}(\Gamma)$ and the Sommerfeld radiation condition(SRC)

$$\lim_{r\to\infty} \sqrt{r} \left(\frac{\partial u}{\partial \nu} - iku \right) = 0, \qquad r := |x| \tag{3}$$

uniformly for all directions $\hat{x} := \frac{x}{|x|}$

Well-posedness of DP

Theorem 8

The direct Neumann problem is well-posed. That is, for every $f \in C^{0,\alpha}(\Gamma)$*, the solution of the direct Neumann problem is given by*

$$u(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma. \tag{4}$$

with the density function $\varphi \in C^{1,\alpha,*}(\Gamma)$ *which is the unique solution to*

$$\frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y) = f(x) \tag{5}$$

for $x \in \Gamma_0$*, and* u *depends continuously on* f*.*

Main idea of the proof

- Functional analytical approach
- Uniqueness by Rellich Lemma and Radiation condition
- Existence and stability by Riesz Theory

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Inverse Problem

Definition 3 (IP)

Determine the scatterer Γ if the far field pattern $u_\infty(\cdot, d)$ is known for all incident directions d and for one wave number $k \in \mathbb{R}$.

Theorem 9 (Uniqueness)

If Γ_1 and Γ_2 are the solutions to the inverse problem with the same far field pattern for a fixed wave number k, then $\Gamma_1 = \Gamma_2$.

Newton Method

Solve the nonlinear ill-posed far field equation:

$$F(\Gamma) = u_\infty \tag{6}$$

Linearization:

$$F(\gamma) + F'(\gamma)h = u_\infty \tag{7}$$

Newton iteration:

$$\gamma_{\nu+1} = \gamma_\nu + h, \qquad \nu = 0, 1, 2, \dots$$

Regularization is needed in solving (7).

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Advantages:

- Conceptionally simple
- Very good reconstructions

Disadvantages

- Convergence ?
- a priori information
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A new method

Recall Equation (5) in Theorem 8

$$\frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y) = f(x), \qquad x \in \Gamma_0.$$

Denote the left hand side of (5) by $(W_\Gamma \varphi)(x)$, we have

Theorem 10

(IP) is equivalent to the system:

$$\begin{cases} W_\Gamma \varphi = f \\ F_{\Gamma,\infty} \varphi = u_\infty \end{cases} \tag{8}$$

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Newton Method

Rewrite the equation (8) in Theorem 10 in operator form:

$$F(\gamma,\varphi)=g(\gamma) \tag{9}$$

Newton's Method

$$F(\gamma,\varphi)+F'_\gamma(\gamma,\varphi)h+F(\gamma,\chi)=g(\gamma)+g'(\gamma)h \tag{10}$$

n-th Iteration Step:

$$\gamma_{n+1}=\gamma_n+h_n,\quad \varphi_{n+1}=\varphi_n+\chi_n \tag{11}$$

Solution Space

We treat curves which have a form $\Gamma = \{(x, f(x)) | x \in [-1, 1]\}$. As solution space for Γ, we choose

$$V_m = \text{span}\{T_0, T_1, \ldots, T_m\}$$

where $T_k's$ are the k-th Chebyshev monomials $T_k(x) = \cos(k \cos^{-1} x)$. For the density function φ, we choose the trigonometric interpolation space

$$P_n = \text{span}\{\sin x, \sin 2x, \ldots, \sin(n-1)x\}$$

Tikhonov Regularization

Minimizing the Tikhonov functional:

$$\|F(\gamma_n,\varphi_n)+F'_\gamma(\gamma_n,\varphi_n)h+F(\gamma_n,\chi)-g(\gamma_n)-g'(\gamma_n)h\|^2+\alpha\|h\|^2+\beta\|\chi\|^2$$

for $h \in V_m$ and $\chi \in P_n$.

Stopping criterion:

$$\|h\|_2/\|\gamma_n\|_2 \leq 10^{-5}$$

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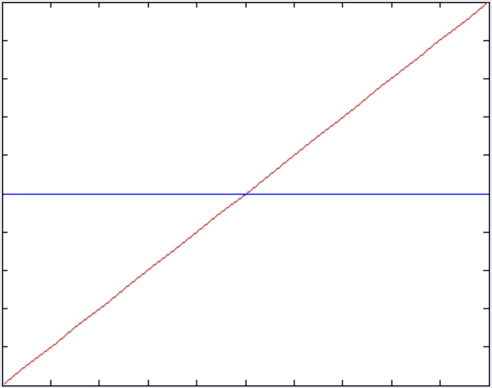
$y = x$

Example 6

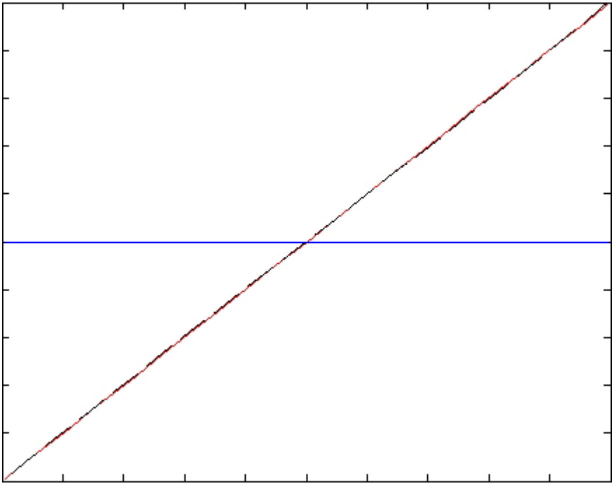


$$k = 1$$

$$\alpha = 4^{-3}$$

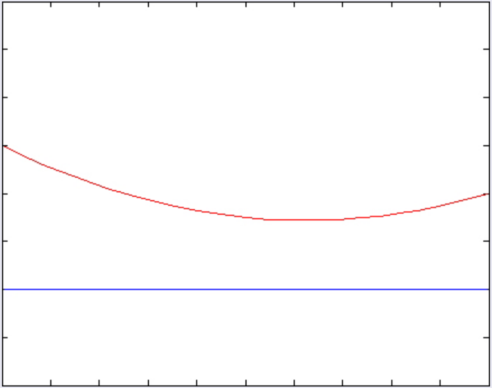
$$\beta = 4^{-5}$$

10 iterations



$y = 0.2x^2 - 0.1x + 0.3$

Example 7

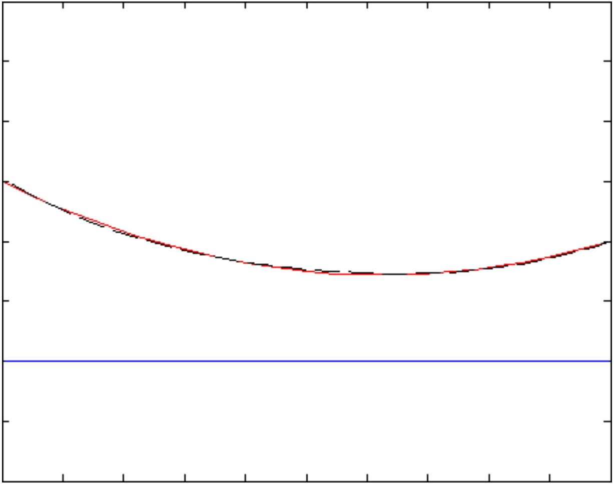


$$k = 1$$

$$\alpha = 4^{-5}$$

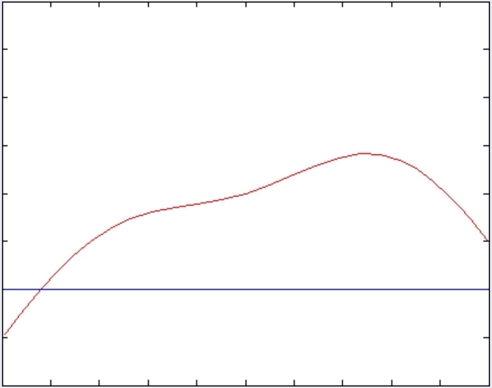
$$\beta = 4^{-11}$$

18 iterations



$y = 0.2\cos(\frac{\pi x}{2}) + 0.5\sin(\frac{\pi x}{2}) - 0.1\cos(\frac{3\pi x}{2})$

Example 8

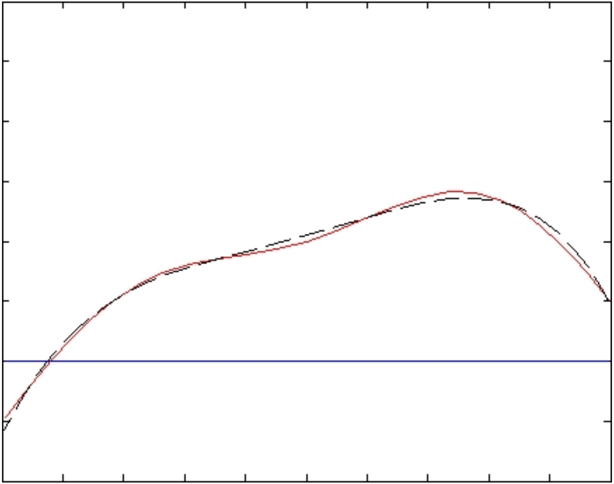


$$k = 1$$

$$\alpha = 4^{-3}$$

$$\beta = 4^{-7}$$

25 iterations



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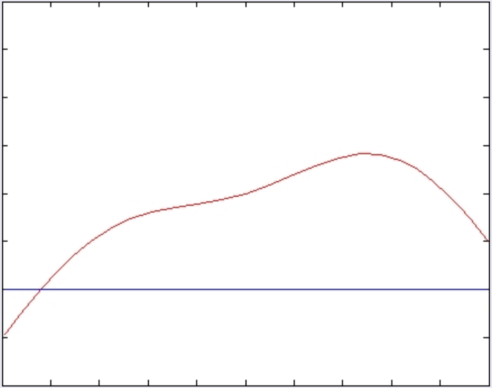
Example 9

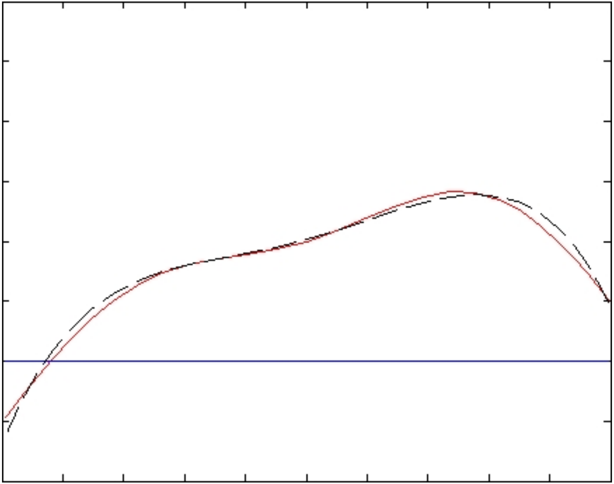


$k = 3$

$\alpha = 4^{-3}$

$\beta = 4^{-7}$

22 iterations



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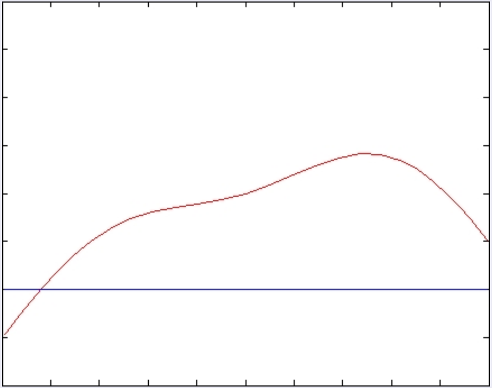
Example 10



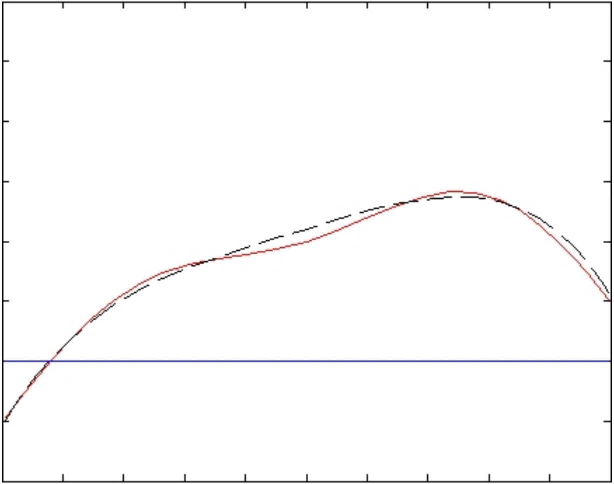
3% noise

$$k = 1$$

$$\alpha = 4^{-3}$$

$$\beta = 4^{-7}$$

30 iterations



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- Solve the system with regularized Newton's method.

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References

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