Journal of Applied Mechanics

Brief Notes

A Brief Note is a short paper that presents a specific solution of technical interest in mechanics but which does not necessarily contain new general methods or results. A Brief Note should not exceed 1500 words *or equivalent* (a typical one-column figure or table is equivalent to 250 words; a one line equation to 30 words). Brief Notes will be subject to the usual review procedures prior to publication. After approval such Notes will be published as soon as possible. The Notes should be submitted to the Technical Editor of the JOURNAL OF APPLIED MECHANICS. Discussions on the Brief Notes should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, or to the Technical Editor of the JOURNAL OF APPLIED MECHANICS. Discussions on Brief Notes appearing in this issue will be accepted until two months after publication. Readers who need more time to prepare a Discussion should request an extension of the deadline from the Editorial Department.

On Bonded Circular Inclusions in Plane Thermoelasticity

C. K. Chao[1] and M. H. Shen[2]

A general solution to the thermoelastic problem of a circular inhomogeneity in an infinite matrix is provided. The thermal loadings considered in this note include a point heat source located either in the matrix or in the inclusion and a uniform heat flow applied at infinity. The proposed analysis is based upon the use of Laurent series expansion of the corresponding complex potentials and the method of analytical continuation. The general expressions of the temperature and stress functions are derived explicitly in both the inclusion and the surrounding matrix. Comparison is made with some special cases such as a circular hole under remote uniform heat flow and a circular disk under a point heat source, which shows that the results presented here are exact and general.

1 Introduction

The thermal stresses induced by an insulated hole in an isotropic medium was studied by Florence and Goodier (1959, 1960). The same problem was solved by Chen (1967) for an orthotropic medium containing a circular or elliptic hole. Hwu (1990) studied the thermal stresses in an anisotropic body under uniform heat flow disturbed by an insulated elliptic hole using the Stroh formalism (Stroh, 1958). Following the Lekhnitskii complex potential approach, Tarn and Wang (1993) found the thermal stresses in anisotropic bodies with a hole or a rigid inclusion. Recently, Kattis and Meguid (1995) gave a solution of thermoelastic problems of an elastic curvilinear inclusion embedded in an elastic matrix where all singularities are located in the matrix. In this note, we aim to provide a general solution to the elastic inclusion problem subjected to a point heat source or a uniform heat flow. A point heat source considered in this note resides either outside or inside the circular inclusion. The analysis is based upon the complex variable theory and the method of analytical continuation which allows us to express the general solutions of the temperature and stress functions in a compact form. Some special examples are solved in closed form and are compared with existing analytical solutions, such as a point heat source in the circular disk and an infinite matrix with a circular elastic inclusion under a remote uniform heat flow.

2 Problem Formulation

Consider a circular elastic inclusion perfectly bonded to an infinite matrix subjected to a point heat source located either in the matrix (including infinity) or in the inclusion and a uniform heat flow applied at infinity. The regions occupied by the elastic matrix ($|z| > a$) and the inclusion ($|z| < a$) will be referred to as regions S_1 and S_2, respectively, and the quantities associated with these regions will be denoted by the corresponding subscripts (see Fig. 1). A point heat source in the system or a uniform heat flow at infinity causes a thermal stress distribution as a result of the different thermoelastic properties of the two phases. For a two-dimensional heat conduction problem, the resultant heat flow Q_j and the temperature T_j can be expressed in terms of a single complex potential $g_j'(z)$ as

$$Q_j = \int (q_{xj}dy - q_{yj}dx) = -k_j \operatorname{Im}[g_j'(z)] \qquad (1)$$

$$T_j = \operatorname{Re}[g_j'(z)] \qquad (2)$$

where Re and Im denote the real part and imaginary part of the bracketed expression, respectively. The quantities q_{xj}, q_{yj} in (1) are the components of heat flux in the x and y-direction, respectively, and k_j stands for the heat conductivity with $j = 1$ for S_1 and $j = 2$ for S_2. Once the heat conduction problem is solved, the temperature function $g_j'(z)$ is determined. For a two-dimensional theory of thermoelasticity, the components of the displacement and traction force can be expressed in terms of two stress functions $\phi_j(z)$, $\psi_j(z)$ and a temperature function $g_j'(z)$ as (Bogdanoff, 1954)

$$2G_j(u_j + iv_j) = \kappa_j\phi_j(z) - z\overline{\phi_j'(z)} - \overline{\psi_j(z)} + 2G_j\beta_j \int g_j'(z)dz \qquad (3)$$

$$-Y_j + iX_j = \phi_j(z) + z\overline{\phi_j'(z)} + \overline{\psi_j(z)} \qquad (4)$$

where G_j is the shear modulus, and $\kappa_j = (3 - \nu_j)/(1 + \nu_j)$, β_j

[1] Department of Mechanical Engineering, National Taiwan Institute of Technology, Taipei, Taiwan 106, Republic of China.

[2] Department of Mechanical Engineering, Nankai Jr. College of Technology, Nantou, Taiwan, Republic of China.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Aug. 19, 1996; final revision, Sept. 19, 1996. Associate Technical Editor: X. Markenscoff.

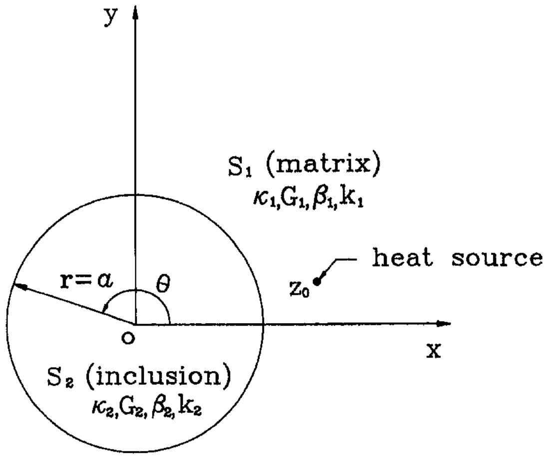

Fig. 1 A bonded circular inclusion subjected to a point heat source outside the inclusion

$= \alpha_j$ for plane stress and $\kappa_j = 3 - 4\nu_j$, $\beta_j = (1+\nu_j)\alpha_j$ for plane strain with ν_j being the Poisson's ratio and α_j the thermal expansion coefficients. Primes denote differentiation with respect to z and a superimposed bar denotes the complex conjugate. For the condition that both the stresses and displacements are single-valued either in the matrix or in the inclusion, the stress functions $\phi_j(z)$, $\psi_j(z)$ must take the form

$$\phi_j(z) = A_j z \ln z + B_j \ln z + \phi_j^*(z) \tag{5}$$

$$\psi_j(z) = C_j \ln z + \psi_j^*(z) \tag{6}$$

where A_j is a real constant and B_j, C_j are complex constants which are related by the following equations

$$(\kappa_j + 1)A_j z + \kappa_j B_j + \bar{C}_j = \frac{-2G_j\beta_j}{2\pi i} \oint_{c_j} g_j'(t)dt \tag{7}$$

$$B_j - \bar{C}_j = 0 \tag{8}$$

where c_j is any surrounding contour within the region S_j ($j = 1, 2$). Note that the singularity of the term $z \ln z$ appeared in the stress functions, Eq. (5), results from the logarithmic singularity of the temperature function induced by a point heat source. The two holomorphic functions $\phi_j^*(z)$ and $\psi_j^*(z)$ in (5) and (6), respectively, can be expressed in a series form as

$$\phi_1^* = \sum_{m=1}^{\infty} L_m z^{-m}, \quad \psi_1^* = \sum_{m=1}^{\infty} M_m z^{-m} \tag{9}$$

$$\phi_2^* = \sum_{m=1}^{\infty} N_m z^{m}, \quad \psi_2^* = \sum_{m=1}^{\infty} P_m z^{m} \tag{10}$$

where the constant coefficients L_m, M_m, N_m, and P_m may be determined from the interface continuity conditions.

3 Temperature Field

3.1 A Point Heat Source in the Matrix. Consider a point heat source is located outside the inclusion (see Fig. 1), the temperature functions in the matrix and in the inclusion, respectively, can be written as

$$g_1'(z) = g_0'(z) + \mathbf{g}_1'(z) \tag{11}$$

$$g_2'(z) = \mathbf{g}_2'(z) \tag{12}$$

where $g_0'(z)$ represents the function associated with the unperturbed field which is related to the solutions of homogeneous media and is holomorphic in the entire domain except a singular point under a point heat source, and the points at zero or infinity. $\mathbf{g}_1'(z)$ (or $\mathbf{g}_2'(z)$) is the function corresponding to the perturbed field of matrix (or inclusion) and is holomorphic in region S_1 (or S_2) except some singular points. In the present study, the temperature function $g_0'(z)$ is given as (Ozisik, 1980)

$$g_0'(z) = -\frac{q}{2\pi k_1} \ln (z - z_0) \tag{13}$$

for a point heat source with the strength q located at the point $z = z_0$ in the matrix, and

$$g_0'(z) = \tau e^{-i\lambda} z \tag{14}$$

for a remote uniform heat flow with the constant temperature gradient τ directed at an angle λ with respect to the positive x-axis (see Fig. 2). $\mathbf{g}_1'(z)$ and $\mathbf{g}_2'(z)$ in (11) and (12), respectively, will be determined from the interface continuity conditions, i.e., $T_1 = T_2$ and $Q_1 = Q_2$ along the interface $z = \sigma = ae^{i\theta}$. Using the above boundary conditions and applying the method of analytical continuation (Chao and Lee, 1996), we obtain the final results as

$$\mathbf{g}_1'(z) = g_0'(z) + \frac{k_1 - k_2}{k_1 + k_2} \bar{g}_0'\left(\frac{a^2}{z}\right) \tag{15}$$

$$\mathbf{g}_2'(z) = \frac{2k_1}{k_1 + k_2} g_0'(z) \tag{16}$$

3.2 A Point Heat Source in the Inclusion. If a point heat source is located inside the inclusion (see Fig. 3), the temperature functions can be written as

$$g_1'(z) = \frac{-q}{2\pi k_1} \ln \frac{z}{a} + \mathbf{g}_1'(z) \tag{17}$$

$$g_2'(z) = \frac{-q}{2\pi k_2} \ln \frac{z}{a} + g_0'(z) + \mathbf{g}_2'(z) \tag{18}$$

where $g_0'(z)$ is given by

$$g_0'(z) = \frac{-q}{2\pi k_2} \ln \left(a - \frac{a z_0}{z}\right). \tag{19}$$

Using the interface continuity conditions as mentioned above and the method of analytical continuation (Chao and Lee, 1996), the final expression for the temperature functions becomes

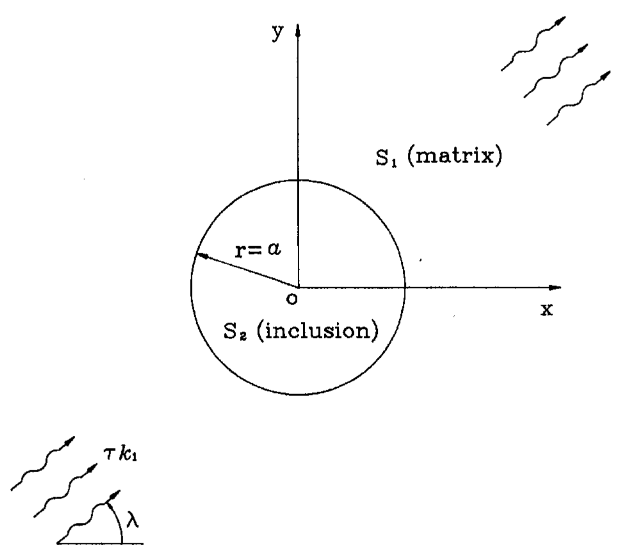

Fig. 2 A bonded circular inclusion subjected to a remote uniform heat flow

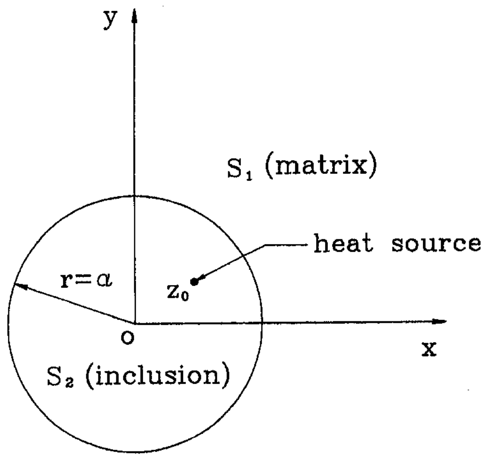

Fig. 3 A bonded circular inclusion subjected to a point heat source inside the inclusion

$$g_1'(z) = \frac{-q}{2\pi k_1} \ln \frac{z}{a} - \frac{q}{\pi(k_1 + k_2)} \ln \left(a - \frac{a z_0}{z} \right) \quad (20)$$

$$g_2'(z) = \frac{-q}{2\pi k_2} \ln (z - z_0) - \frac{(k_2 - k_1) q}{2\pi(k_1 + k_2) k_2} \ln \left(a - \frac{z \bar{z}_0}{a} \right). \quad (21)$$

4 Thermal Stress Field

Having the temperature functions as derived previously, the general solutions for the stress and displacement fields can be obtained in terms of the complex potentials given in Eqs. (5) and (6) in which the constant coefficients A_j, B_j, and C_j may be determined from (7) and (8) while the two holomorphic functions $\phi_j^*(z)$ and $\psi_j^*(z)$ will be obtained from the interface continuity conditions.

4.1 A Point Heat Source in the Matrix. We now consider a heat source located in the matrix for which the temperature functions $g_1'(z)$, $g_2'(z)$ have been given in (15) and (16), respectively. Substituting (15) and (16) into (7) and using (8), one obtains

$$A_1 = \frac{G_1 \beta_1 q}{\pi k_1 (1 + \kappa_1)}, \quad B_1 = \frac{-G_1 \beta_1 q z_0}{\pi k_1 (1 + \kappa_1)}, \quad \text{for} \quad |z| > |z_0| \quad (22)$$

$$A_1 = B_1 = 0, \quad \text{for} \quad a < |z| < |z_0| \quad (23)$$

and

$$A_2 = B_2 = 0, \quad \text{for} \quad |z| < a. \quad (24)$$

Since the inclusion and the matrix are assumed to be perfectly bonded along the interface, the displacements and surface tractions at the interface must be continuous, i.e., $u_1 + iv_1 = u_2 + iv_2$ and $-Y_1 + iX_1 = -Y_2 + iX_2$ along the interface $z = \sigma = ae^{i\theta}$. Using the above boundary conditions and applying the method of analytical continuations (Chao and Lee, 1996), the constant coefficients appeared in (9) and (10) are obtained as

$$L_m = -\frac{2G_1 G_2 \beta_1}{G_1 + G_2 \kappa_1} b_m \quad (25)$$

$$N_1 = \frac{-2G_1 G_2 (\beta_2 c_1 - \beta_1 a_1)}{(\kappa_2 G_1 + 2G_2 - G_1)} \quad (26)$$

$$N_m = -\frac{2G_1 G_2}{G_2 + G_1 \kappa_2} (\beta_2 c_m - \beta_1 a_m), \quad \text{for} \quad m \geq 2 \quad (27)$$

$$M_1 = [2N_1 - A_1(1 + \ln a^2)] a^2 \quad (28)$$

$$M_2 = \bar{N}_2 a^4 - B_1 a^2 \quad (29)$$

$$M_m = (m - 2) a^2 L_{m-2} + a^{2m} \bar{N}_m, \quad \text{for} \quad m \geq 3 \quad (30)$$

$$P_m = a^{-2m} \bar{L}_m - (m + 2) a^2 N_{m+2}. \quad (31)$$

Having the solutions in (25)–(31), the final expression of the stress functions can then be determined by substituting (9)–(10) and (22)–(24) into (5) and (6).

4.2 A Point Heat Source in the Inclusion. If a heat source is located in the inclusion, the constant coefficients in (5) and (6) can be determined by substituting (20) and (21) into (7) and (8) as

$$A_1 = \frac{G_1 \beta_1 q}{\pi k_1 (1 + \kappa_1)}, \quad B_1 = \frac{-G_1 \beta_1 q z_0}{\pi (k_1 + k_2)(1 + \kappa_1)}, \quad \text{for} \quad |z| > a \quad (32)$$

$$A_2 = \frac{G_2 \beta_2 q}{\pi k_2 (1 + \kappa_2)}, \quad B_2 = \frac{-G_2 \beta_2 q z_0}{\pi k_2 (1 + \kappa_2)}, \quad \text{for} \quad |z_0| < |z| < a \quad (33)$$

and

$$A_2 = B_2 = 0, \quad \text{for} \quad |z| < |z_0|. \quad (34)$$

By using the interface continuity conditions and the method of analytical continuation (Chao and Lee, 1996), we finally obtain

$$L_m = -\frac{2G_1 G_2}{G_1 + G_2 \kappa_1} (\beta_1 e_m - \beta_2 h_m) \quad (35)$$

$$N_1 = \frac{-(G_2 A_2 - G_1 A_2)(1 + \ln a^2) - 2G_1 G_2 (\beta_2 f_1 - \beta_1 d_1)}{(\kappa_2 G_1 + 2G_2 - G_1)} \quad (36)$$

$$N_2 = -\frac{2G_1 G_2}{G_2 + G_1 \kappa_2} \beta_2 f_2 - \frac{G_2 \bar{B}_2 - G_1 \bar{B}_2}{(G_2 + G_1 \kappa_2) a^2} \quad (37)$$

$$N_m = -\frac{2G_1 G_2}{G_2 + G_1 \kappa_2} \beta_2 f_m, \quad \text{for} \quad m \geq 3 \quad (38)$$

$$M_1 = [2N_1 + (A_2 - A_1)(1 + \ln a^2)] a^2 \quad (39)$$

$$M_2 = \bar{N}_2 a^4 + (B_2 - B_1) a^2 \quad (40)$$

$$M_m = (m - 2) a^2 L_{m-2} + a^{2m} \bar{N}_m, \quad \text{for} \quad m \geq 3 \quad (41)$$

$$P_m = a^{-2m} \bar{L}_m - (m + 2) a^2 N_{m+2}. \quad (42)$$

With the results in (35)–(42) the general solutions for the stress functions can then be obtained by substituting (32)–(34) and (9)–(10) into (5) and (6).

5 Examples

5.1 Elastic Circular Inclusion under Remote Heat Flow. As our first example, we consider a circular elastic inclusion perfectly bonded to a matrix which is subjected to a uniform heat flux with the temperature gradient τ directed at an angle λ with respect to the positive x-axis (see Fig. 2). The solution of temperature functions can be easily obtained by substituting (14) into (15) and (16) as

$$g_1'(z) = \tau e^{-i\lambda} z + \frac{\tau(k_1 - k_2)}{(k_1 + k_2)} e^{i\lambda} \frac{a^2}{z} \quad (43)$$

$$g_2'(z) = \frac{2k_1\tau}{(k_1 + k_2)} e^{-i\lambda} z. \quad (44)$$

Applying the formulae given in Section 4.1, the final solutions for the stress field can be given by

$$\phi_1(z) = \frac{-2G_1\beta_1\tau}{(1+\kappa_1)} \frac{(k_1 - k_2)}{(k_1 + k_2)} a^2 e^{i\lambda} \ln z \quad (45)$$

$$\psi_1(z) = \frac{-2G_1\beta_1\tau}{(1+\kappa_1)} \frac{(k_1 - k_2)}{(k_1 + k_2)} a^2 e^{-i\lambda} \ln z - \frac{G_1G_2\tau}{(G_1\kappa_2 + G_2)} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1\right) a^4 e^{i\lambda} \frac{1}{z^2} + \frac{2G_1\beta_1\tau}{(1+\kappa_1)} \frac{(k_1 - k_2)}{(k_1 + k_2)} a^4 e^{i\lambda} \frac{1}{z^2} \quad (46)$$

$$\phi_2(z) = -\frac{2G_1G_2\tau}{G_1\kappa_2 + G_2} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1\right) e^{-i\lambda} z^2 \quad (47)$$

$$\psi_2(z) = 0. \quad (48)$$

The interfacial stresses along the inclusion boundary can be performed by using field solutions of the matrix or inclusion as

$$\sigma_{rr} = -\frac{G_1G_2\tau}{(G_1\kappa_2 + G_2)} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1\right) a \cos(\theta - \lambda)$$

$$\sigma_{r\theta} = -\frac{G_1G_2\tau}{(G_1\kappa_2 + G_2)} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1\right) a \sin(\theta - \lambda)$$

$$(\sigma_{\theta\theta})_1 = \left[\frac{-8G_1\beta_1\tau(k_1 - k_2)}{(1+\kappa_1)(k_1 + k_2)} + \frac{2G_1G_2\tau}{(G_1\kappa_2 + G_2)} \times \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1\right)\right] a \cos(\theta - \lambda)$$

$$(\sigma_{\theta\theta})_2 = -\frac{12G_1G_2\tau}{(G_1\kappa_2 + G_2)} \left(\frac{2k_1\beta_2}{(k_1 + k_2)} - \beta_1\right) a \cos(\theta - \lambda).$$

When the inclusion is assumed to be an insulated and traction-free hole, the hoop stress along the hole boundary can be obtained by letting $k_2 = 0$ and $G_2 = 0$ as

$$\sigma_{\theta\theta} = -\frac{8G\beta\tau}{(1+\kappa)} a \cos(\theta - \lambda)$$

which is in agreement with the result of Florence and Goodier (1959). For a special case of $k_1 = k_2$, $G_1 = G_2$ and $\beta_1 = \beta_2$, the solutions of the corresponding homogeneous problem is trivially given as

$$\phi(z) = \psi(z) = 0. \quad (49)$$

This is expected that there is no thermal stresses induced by a homogeneous body under the condition of free expansion.

5.2 A Point Heat Source at the Center of the Inclusion. As a second example we consider the inclusion subjected to a point heat source acting at the origin. The temperature functions can be obtained by putting $z_0 = 0$ into (20) and (21) as

$$g_1'(z) = -\frac{q}{2\pi k_1} \ln \frac{z}{a} - \frac{q}{\pi(k_1 + k_2)} \ln a \quad (50)$$

$$g_2'(z) = -\frac{q}{2\pi k_2} \ln z - \frac{(k_2 - k_1)q}{2\pi(k_1 + k_2)k_2} \ln a. \quad (51)$$

A direct application of the formulae given in Section 4.2, the stress functions can be obtained as

$$\phi_1(z) = \frac{G_1\beta_1 q}{\pi k_1(1+\kappa_1)} z \ln z \quad (52)$$

$$\psi_1(z) = M_1/z \quad (53)$$

$$\phi_2(z) = \frac{G_2\beta_2 q}{\pi k_2(1+\kappa_2)} z \ln z + N_1 z \quad (54)$$

$$\psi_2(z) = 0 \quad (55)$$

where N_1 and M_1 are

$$N_1 = \frac{q}{\kappa_2 G_1 + 2G_2 - G_1} \left\{ \frac{G_2\beta_2(G_1 - G_2)(1 + \ln a^2)}{\pi k_2(1+\kappa_2)} - \frac{G_1G_2}{\pi} \left[\frac{\beta_2}{k_2} + \frac{(k_1 - k_2)\beta_2 \ln a}{(k_1 + k_2)k_2} - \frac{\beta_1}{k_1}(1 + \ln a) + \frac{2\beta_1 \ln a}{(k_1 + k_2)} \right] \right\}$$

$$M_1 = 2N_1 a^2 + \frac{q}{\pi} \left[\frac{G_2\beta_2}{k_2(1+\kappa_2)} - \frac{G_1\beta_1}{k_1(1+\kappa_1)} \right] (1 + \ln a^2) a^2.$$

Note that, the stresses would not be bounded either at zero or at infinity due to the presence of the singular term $z \ln z$ appeared in the stress functions induced by a point heat source. Nevertheless, the solutions are useful as the outer boundary of a body remains finite.

5.2.1. Circular Disk. Consider a circular disk where the boundary surface is assumed to be free of traction and remain zero temperature. The corresponding temperature function and stress functions, respectively, can be obtained by letting $k_1 = \infty$ in (51) and $G_1 = 0$ in (54) as

$$g'(z) = -\frac{q}{2\pi k} \ln \frac{z}{a} \quad (56)$$

$$\phi(z) = \frac{G\beta q}{\pi k(1+\kappa)} z \ln \frac{z}{a} - \frac{G\beta q}{2\pi k(1+\kappa)} z \quad (57)$$

$$\psi(z) = 0. \quad (58)$$

Accordingly, the stress components are given by

$$\sigma_{rr} = \frac{2G\beta q}{\pi k(1+\kappa)} \ln \frac{r}{a}$$

$$\sigma_{\theta\theta} = \frac{2G\beta q}{\pi k(1+\kappa)} \ln \frac{r}{a} + \frac{2G\beta q}{\pi k(1+\kappa)}$$

$$\sigma_{r\theta} = 0$$

which is the same as the results obtained by Parkus (1968) essentially by guessing.

Acknowledgment

The authors would like to thank the support by the National Science Council, Republic of China, through grant no. NSC. 84-2212-E011-022.

References

Bogdanoff, J. L., 1954, "Note on Thermal Stress," ASME JOURNAL OF APPLIED MECHANICS, Vol. 21, p. 88.

Chao, C. K., and Lee, J. Y., 1996, "Interaction Between a Crack and a Circular Elastic Inclusion Under Remote Uniform Heat Flow," *Int. J. Solids Structures*, Vol. 33, pp. 3865–3880.
Chen, W. T., 1967, "Plane Thermal Stress at an Insulated Hole Under Uniform Heat Flow in an Orthotropic Medium," ASME JOURNAL OF APPLIED MECHANICS, Vol. 34, pp. 133–136.
Florence, A. L., and Goodier, J. N., 1959, "Thermal Stress at Spherical Cavities and Circular Holes in Uniform Heat Flow," ASME JOURNAL OF APPLIED MECHANICS, Vol. 26, pp. 293–294.
Florence, A. L., and Goodier, J. N., 1960, "Thermal Stress Due to Disturbance of Uniform Heat Flow by an Insulated Ovaloid Hole," ASME JOURNAL OF APPLIED MECHANICS, Vol. 27, pp. 635–639.
Hwu, C., 1990, "Thermal Stresses in an Anisotropic Plate Disturbed by an Insulated Elliptic Hole or Crack," ASME JOURNAL OF APPLIED MECHANICS, Vol. 57, pp. 916–922.
Kattis, M. A., and Meguid, S. A., 1995, "Two-Phase Potentials for the Treatment of an Elastic Inclusion in Plane Thermoelasticity," ASME JOURNAL OF APPLIED MECHANICS, Vol. 62, pp. 7–12.
Ozisik, M. N., 1980, *Heat Conduction*, John Wiley and Sons, New York.
Parkus, H., 1968, *Thermoelasticity*, Blaisdell, Waltham, MA.
Stroh, A. N., 1958, "Dislocations and Cracks in Anisotropic Elasticity," *Phil. Mag.*, Vol. 7, pp. 625–646.
Tarn, J. Q., and Wang, Y. M., 1993, "Thermal Stresses in Anisotropic Bodies with a Hole or a Rigid Inclusion," *J. Thermal Stresses*, Vol. 16, pp. 455–471.

Influence of the Interface Periodic Array of Coplanar Cracks on the Free-Surface Oscillations

M. Ciarletta[3] and M. A. Sumbatyan[4]

The works of Angel and Achenbach (1985a, b) and Mikata and Achenbach (1988) are devoted to a scattering by the periodic array of cracks in an infinite homogeneous elastic plane for coplanar/inclined cracks with a normal/oblique incidence of a longitudinal/transverse wave. The method developed in the recent work of Mikata (1993) has considerable merits since it permits for the coplanar cracks derivation of integral equations with the kernels where there is no need for numerical integrations.

England (1965), Erdogan (1965), Comninou (1977), and some other authors aimed at development of analytical techniques in the static problems when there is a single finite-length interface crack on a boundary between two different elastic half-planes. This problem can be reduced to a system of two integral equations of the second kind with the singular kernels of Cauchy type, which can be explicitly solved.

We study here the problem that is contiguous to that considered by Yang and Bogy (1985).

Let on the boundary surface between two elastic media be a periodic array of coplanar cracks. The distance between two neighbour cracks is $2b$, the step of the array is $2a$. The thickness of the upper medium is d the lower medium is a half-space.

For both the media we apply the Lame representation for displacements and the wave potentials satisfy the wave equations:

$$\Delta\varphi + k_p^2\varphi = 0, \quad \Delta\psi + k_s^2\psi = 0, \tag{1}$$

with the velocity components in this two-dimensional problem being

$$u_x = \partial\varphi/\partial x + \partial\psi/\partial y, \; u_y = \partial\varphi/\partial y - \partial\psi/\partial x, \; u_z = 0. \tag{2}$$

At last, the boundary conditions must be taken into consideration:

$$\sigma_{xx} = \sigma_{xy} = 0, \quad x = \pm 0, \quad y \in \text{cracks}, \tag{3a}$$

$$\sigma_{xx} = \sigma_{xy} = 0, \quad x = d, \quad -\infty < y < \infty. \tag{3b}$$

Let φ_0 be an amplitude of the incident longitudinal wave. Then due to a symmetry of the problem with respect to the axis y, we have the following representation for the lower and the upper medium, respectively:

$$\begin{cases} \varphi_1 = \varphi_0 e^{ik_{p_1}x} + \mathrm{R}e^{-ik_{p_1}x} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n}{a}y\right)e^{q_{1n}x} & (4a)\\ \qquad\qquad -\infty < x \le 0 \\ \psi_1 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{\pi n}{a}y\right)e^{r_{1n}x} & (4b)\end{cases}$$

$$\begin{cases} \varphi_2 = W\sin k_{p_2}(x-d) + \sum_{n=1}^{\infty}\left[C_n \,\mathrm{ch}\, q_{2n}(x-d) - \frac{r_{2n}^2 + \left(\frac{\pi n}{a}\right)^2}{2\frac{\pi n}{a}q_{2n}} \times D_n \,\mathrm{sh}\, q_{2n}(x-d)\right]\cos\left(\frac{\pi n}{a}y\right) & (5a)\\ \qquad\qquad 0 \le x \le d \\ \psi_2 = \sum_{n=1}^{\infty}\left[D_n \,\mathrm{ch}\, r_{2n}(x-d) - \frac{2\left(\frac{\pi n}{a}\right)^2 - k_{s_2}^2}{2\frac{\pi n}{a}r_{2n}} \times C_n \,\mathrm{sh}\, r_{2n}(x-d)\right]\sin\left(\frac{\pi n}{a}y\right) & (5b)\end{cases}$$

(sh and ch = hyperbolic sine and cosine).
Here

$$q_{1n} = \left[\left(\frac{\pi n}{a}\right)^2 - k_{p_1}^2\right)\Bigg]^{1/2}, \; r_{1n} = \left[\left(\frac{\pi n}{a}\right)^2 - k_{s_1}^2\right]^{1/2}, \tag{6a}$$

$$q_{2n} = \left[\left(\frac{\pi n}{a}\right)^2 - k_{p_2}^2\right]^{1/2}, \quad r_{2n} = \left[\left(\frac{\pi n}{a}\right)^2 - k_{s_2}^2\right]^{1/2}, \tag{6b}$$

k_{p_1}, k_{s_1} are the wave numbers for the first (lower) medium and k_{p_2}, k_{s_2} for the second (upper) one.

[3] Professor, DIMMA, University of Salerno, 84084 Fisciano (SA), Italy.
[4] Head, Laboratory of Diffraction Theory, Research Institute of Diffraction Theory, Research Institute of Mechanics and Applied Mathematics, 200/1 Stachki Prospect, Rostov-on-Don 344090, Russia.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Dec. 5, 1996; final revision, May 7, 1997. Associate Technical Editor: X. Markenscoff.