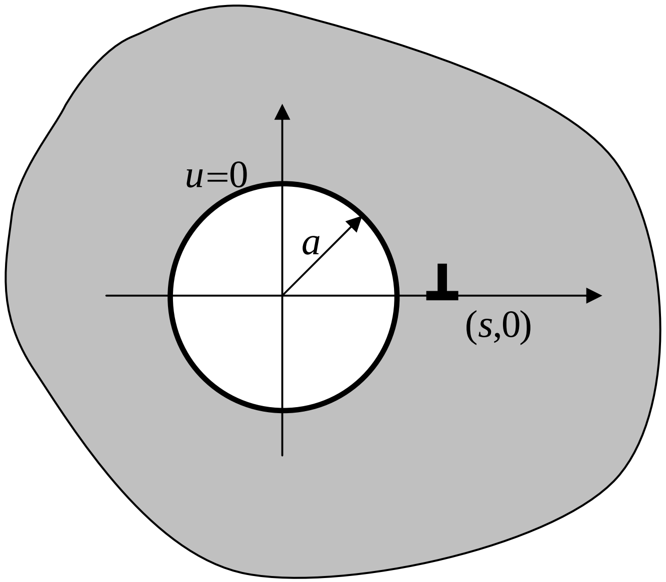
問題描述



- E. Smith 複變解析
- Image method
- Present method (Superposition)

E. Smith 複變解析

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$$

Exists some potential function $F(z)$

$$\mu_E w=\mathrm{Re}[F(z)]$$

$$F(z)=\frac{\mu_E\lambda}{2\pi i}\log(z-z_0)+\frac{\mu_E\lambda}{2\pi i}\log(\frac{a_0^2}{z}-\overline{z}_0)$$

$$=\frac{\mu_E\lambda}{2\pi i}(\ln\left|z-z_0\right|+i\arg(z-z_0))+\frac{\mu_E\lambda}{2\pi i}(\ln\left|\frac{a_0^2}{z}-\overline{z}_0\right|+i\arg(\frac{a_0^2}{z}-\overline{z}_0))$$

$$w_E=\frac{1}{\mu_E}\mathrm{Re}[F_E(z)]$$

$$=\frac{\lambda}{2\pi}[\arg(z-z_0)+\arg(\frac{a_0^2}{z}-\overline{z}_0)]$$

$$=\frac{\lambda}{2\pi}[\arg((x+iy)-(x_0+iy_0))+\arg(\frac{a_0^2}{x^2+y^2}(x-yi)-(x_0-iy_0))]$$

$$=\frac{\lambda}{2\pi}[\arg((x+iy)-(x_0+iy_0))+\arg(\frac{a_0^2}{\rho^2}(x-yi)-(x_0-iy_0))]$$

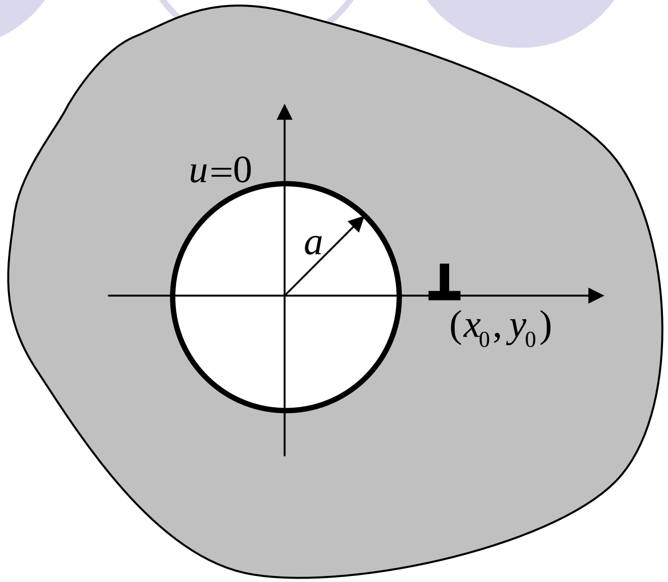
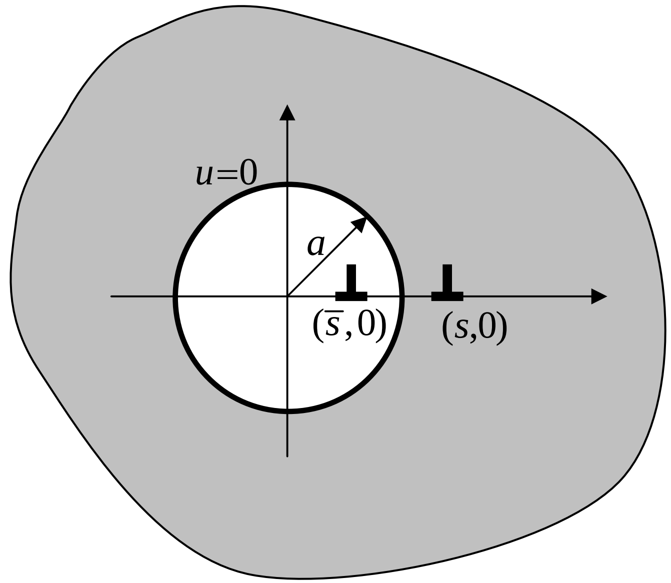


Image method

$$\varphi(s,x)=\begin{cases}\theta+\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\sin[m(\theta-\phi)], & R\geq\rho\\ \phi-\pi-\sum_{i=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\sin[m(\theta-\phi)], & R<\rho\end{cases}$$



滿足B.C.

$$\begin{cases}\varphi(s,x)=\theta+\sum_{m=1}^{\infty}\frac{1}{m}(\frac{a}{R})^m\sin[m(\theta-\phi)]\\ \varphi(\bar{s},x)=\phi-\pi-\sum_{i=1}^{\infty}\frac{1}{m}(\frac{\bar{R}_1}{a})^m\sin[m(\bar{\theta}-\phi)]\end{cases},\quad x\in B$$

Find the image point

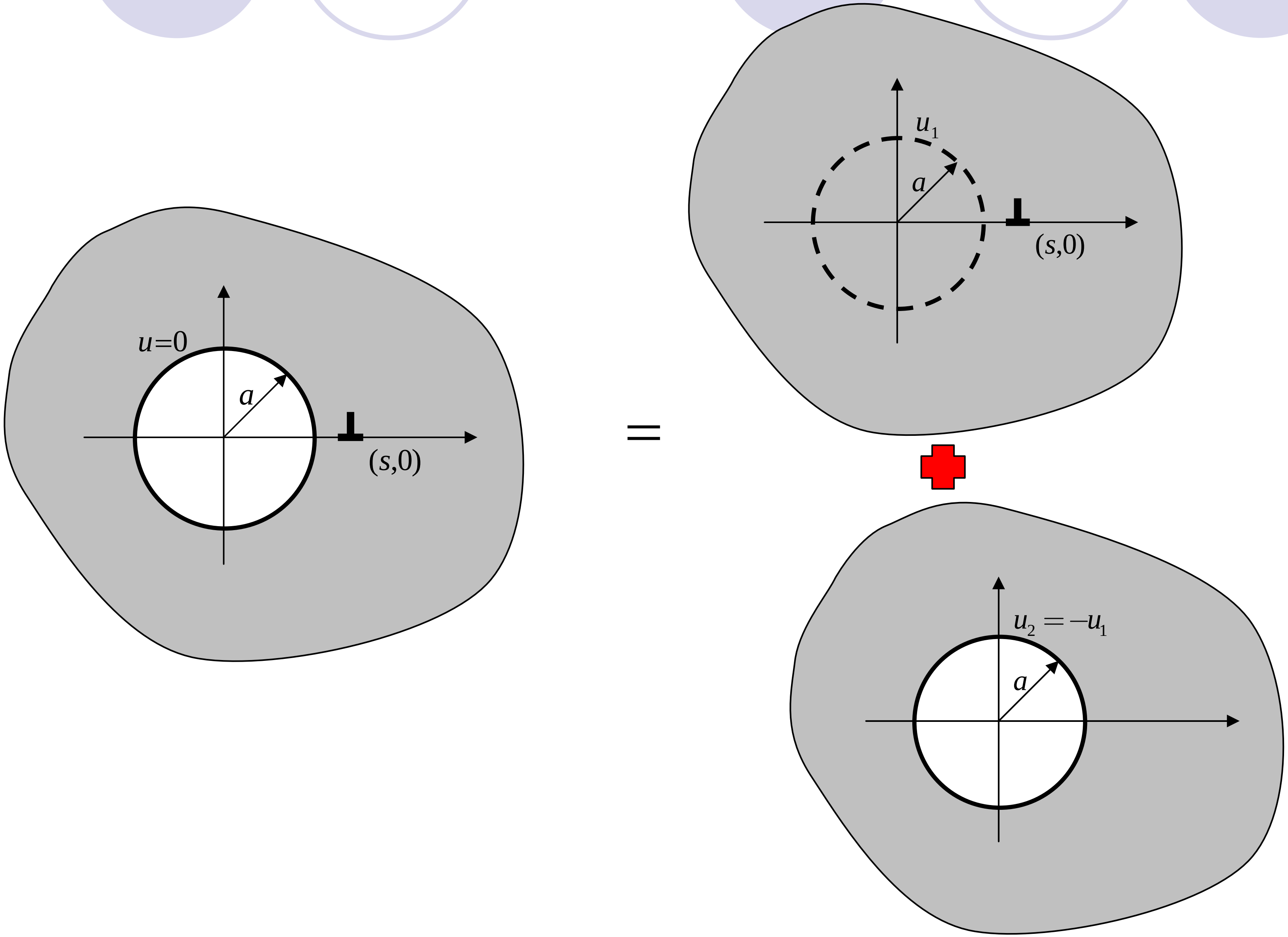
$$\theta=\bar{\theta}$$

$$\frac{R}{a}=\frac{a}{\bar{R}}\quad\Rightarrow\quad\bar{R}=\frac{a^2}{R}$$

$$u(x)=\varphi(s,x)+\varphi(s_1,x)-\phi+\pi$$

$$u(x)=\begin{cases}\theta+\sum_{i=1}^{n}\frac{1}{m}[(\frac{\rho}{R})^m-(\frac{a^2}{\rho R})^m]\sin[m(\theta-\phi)], & R\geq\rho\geq a\\ \phi-\pi-\sum_{i=1}^{n}\frac{1}{m}[(\frac{\rho}{R})^m+(\frac{a^2}{\rho R})^m]\sin[m(\theta-\phi)], & R<\rho<\infty\end{cases}$$

Present method (Superposition)



Present method (Superposition)

第一部分

$$u^1(x)=\begin{cases}\theta+\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\sin m(\theta-\phi), & \rho\le R\\ \phi-\pi-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\sin m(\theta-\phi), & \rho>R\end{cases},\quad x\in D$$

第二部分

$$u_2(s)=-u_1(s)=-\theta-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{a}{R})^m\sin m(\theta-\theta)$$

$$t_2(s)=a_0+\sum_{n=1}^{\infty}a_n\cos n\theta+b_n\sin n\theta$$

$$\int_B T(s,x)u_2(s)dB(s)-\int_B U(s,x)t_2(s)dB(s)=0,\qquad U(s,x)=\begin{cases}\ln R-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos m(\theta-\phi), & R\ge\rho\\ \ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos m(\theta-\phi), & R<\rho\end{cases}$$

$$2\pi\theta+\sum_{m=1}^{\infty}\frac{\pi}{m}(\frac{a}{R})^m(\sin m\theta\cos m\phi-\cos m\theta\sin m\phi)=2\pi a\ln a a_0-\sum_{m=1}^{\infty}\frac{a\pi}{m}(a_m\cos m\phi+b_m\sin m\phi)\quad\Rightarrow\quad\begin{cases}a_0=\frac{\theta}{a\ln a}\\ a_m=-\frac{a^{m-1}}{R^m}\sin m\theta\\ b_m=\frac{a^{m-1}}{R^m}\cos m\theta\end{cases}$$

Present method (Superposition)

$$2\pi u_2(x)=\int_B Tu-UtdB(s)$$

$$=\int_0^{2\pi}-[-\sum_{m=1}^{\infty}\frac{R^{m-1}}{\rho^m}\cos m(\theta-\phi)][-\theta-\sum_{n=1}^{\infty}\frac{1}{n}(\frac{a}{\overline{R}})^n\sin n(\theta-\theta)]$$

$$-[\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos m(\theta-\phi)][\frac{\theta}{a\ln a}-\sum_{n=1}^{\infty}\frac{a^{m-1}}{R^m}\sin m(\theta-\theta)]dB(s)$$

$$=-\sum_{m=1}^{\infty}\frac{\pi}{m}(\frac{a^2}{\overline{R}\rho})^m\sin m(\theta-\phi)-2\pi a\ln\rho a_0+\sum_{m=1}^{\infty}\frac{\pi}{m}\frac{a^{m+1}}{\rho^m}(a_m\cos m\phi+b_m\sin m\phi)$$

$$u_2(x)=-\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}(\frac{a^2}{R\rho})^m\sin m(\theta-\phi)-a\ln\rho\frac{\theta}{a\ln a}+\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}(\frac{a^2}{\rho R})^m(-\sin m\theta\cos m\phi+\cos m\theta\sin m\phi)$$

$$u_2(x)=-\frac{\ln\rho}{\ln a}\theta-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{a^2}{R\rho})^m\sin m(\theta-\phi)$$

$$u(x)=u_1(x)+u_2(x)=\begin{cases}\theta-\dfrac{\ln\rho}{\ln a}\theta+\displaystyle\sum_{m=1}^{\infty}\frac{1}{m}[(\frac{\rho}{R})^m-(\frac{a^2}{R\rho})^m]\sin m(\theta-\phi), & R\geq\rho\geq a\\ \phi-\pi-\dfrac{\ln\rho}{\ln a}\theta-\displaystyle\sum_{m=1}^{\infty}\frac{1}{m}[(\frac{R}{\rho})^m+(\frac{a^2}{R\rho})^m]\sin m(\theta-\phi), & R<\rho<\infty\end{cases},\quad x\in D$$

Compare three solution

E. Simth

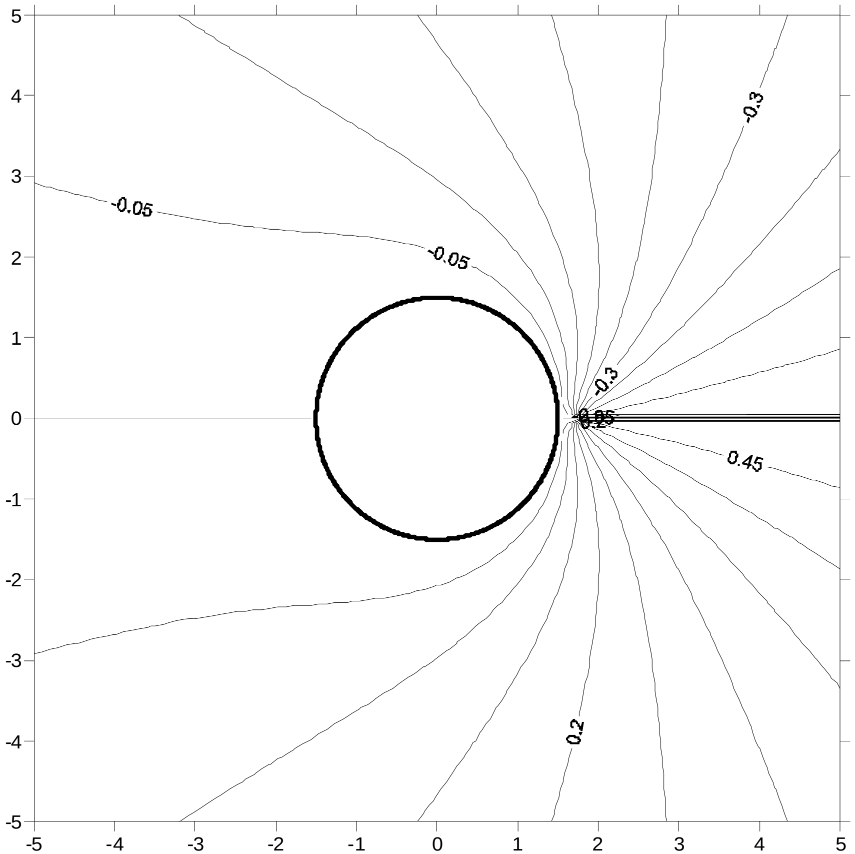
$$w_E(x,y)=\frac{\lambda}{2\pi i}[\arg((x+iy)-(x_0+iy_0))+\arg(\frac{a_0^2}{\rho^2}(x-yi)-(x_0-iy_0))]$$

Image method

$$u(x)=\begin{cases}\theta+\sum_{i=1}^{n}\frac{1}{m}[(\frac{\rho}{R})^m-(\frac{a^2}{\rho R})^m]\sin[m(\theta-\phi)], & R\geq\rho\\ \phi-\pi-\sum_{i=1}^{n}\frac{1}{m}[(\frac{\rho}{R})^m+(\frac{a^2}{\rho R})^m]\sin[m(\theta-\phi)], & R<\rho\end{cases}$$

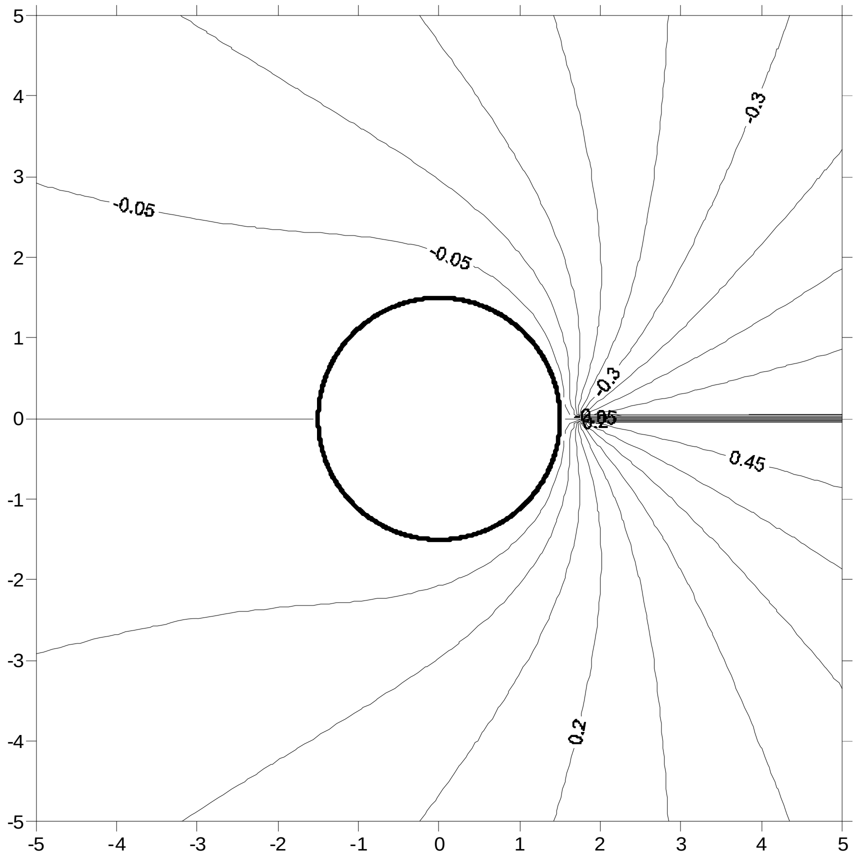
Present method (superposition)

$$u(x)=\begin{cases}\theta-\frac{\ln\rho}{\ln a}\theta+\sum_{m=1}^{\infty}\frac{1}{m}[(\frac{\rho}{R})^m-(\frac{a^2}{R\rho})^m]\sin m(\theta-\phi), & \rho\leq R\\ \phi-\pi-\frac{\ln\rho}{\ln a}\theta-\sum_{m=1}^{\infty}\frac{1}{m}[(\frac{R}{\rho})^m+(\frac{a^2}{R\rho})^m]\sin m(\theta-\phi), & \rho>R\end{cases},\quad x\in D$$



E. Smith close form

$a = 1.5, \quad R = 1.75, \quad \theta = 0$



Present method (M=50)

$a = 1.5, \quad R = 1.75, \quad \theta = 0$

Thanks your kind attentions