POTENTIAL THEORY

Edited by
Josef Král
Jaroslav Lukeš
Ivan Netuka and
Jiří Veselý

ON A UNIFIED CHARACTERIZATION OF CAPACITY

George C. Hsiao and Ralph E. Kleinman

Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19716 U.S.A.

INTRODUCTION

The notion of capacity as a fundamental quantity associated with a smooth surface was treated exhaustively by Polya and Szego in a series of papers over more than twenty years following earlier work by Fekete (1923) and many others who considered the problem of determining the charge accumulated on an electrostatic condenser maintained at a constant potential. See Polya and Szego (1951) and Payne (1967) for extensive references.

In three dimensions, the capacity, C, of a surface $\partial\Omega$ can be defined in terms of the conductor potential. That is, if Ω_c denotes the exterior of $\partial\Omega$, p is the position vector of a point and u is a function for which $\nabla^2 u = 0$ in Ω_c, $u = 1$ on $\partial\Omega$ and $u = 0(\frac{1}{|p|})$ then $C = -\frac{1}{4\pi}\int_{\partial\Omega}\frac{\partial u}{\partial n}\,ds$. Alternatively capacity may be defined in terms of the solution of the Helmholtz resonator problem, that is, if $B_R(0)$ is a ball of radius R with center at the origin of a coordinate system centered in Ω, $\partial\Omega$ is contained in $B_R(0)$, and u_1 is a function for which $\nabla^2 u_1 = 0$ in $\Omega_c \cap B_R(0)$, $u_1 = 0$ on $\partial\Omega$ and $u_1 = 1$ on $\partial B_R(0)$ then $C = \lim_{R\to\infty}\frac{1}{4\pi}\int_{\partial B_R(0)}\frac{\partial u_1}{\partial n}\,ds$.

These characterizations of capacity are equivalent in $\mathbf{R}^3$ since in general the solutions of the conductor potential and the Helmholtz resonator problems are related by $u = 1 - \lim_{R\to\infty} u_1$. For example if $\partial\Omega$ is a sphere of radius a then the conductor potential is simply

$$u = \frac{a}{r} \quad \text{and} \quad C = -\frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \; \frac{\partial}{\partial r}\left(\frac{a}{r}\right)_{r=a} a^2 \sin\theta = a .$$

In this case the solution of the Helmholtz resonator problem is

$$u_1 = \frac{\frac{1}{r}-\frac{1}{a}}{\frac{1}{R}-\frac{1}{a}} \quad \text{and} \quad \frac{1}{4\pi}\int_{\partial B_R(0)}\frac{\partial u_1}{\partial n}\,ds = \frac{1}{\frac{1}{a}-\frac{1}{R}}. \text{ Hence } C = \lim_{R\to\infty}\frac{1}{\frac{1}{a}-\frac{1}{R}} = a$$

as before.

Unfortunately neither of these problems leads to nontrivial characterizations of capacity in $\mathbf{R}^2$ because of the behavior of regular solutions of Laplace's equation. That is, if we ascribe to regular solutions of Laplace's equation in exterior domains in $\mathbf{R}^2$ the same behavior at infinity as the fundamental solution then the conductor potential problem $\nabla^2 u = 0$ in Ω_c, $u = 1$ on $\partial\Omega$ and $u = 0(\ln |p|)$ does not always have a unique solution ($A \ln |p|$ is a solution for any A when $\partial\Omega$ is a circle of radius 1) and while the Helmholtz resonator problem is uniquely solvable, the quantity comparable to capacity in $\mathbf{R}^3$ always vanishes, i.e. $\lim_{R\to\infty} \int_{\partial B_R(0)} \frac{\partial u_1}{\partial n} ds = 0$. For example if $\partial\Omega$ is a circle of radius a then $u_1 = \frac{\ln r - \ln a}{\ln R - \ln a}$ and $\int_{\partial B_R(0)} \frac{\partial u_1}{\partial n} ds = \frac{2\pi}{\ln R - \ln a}$ which obviously vanishes when $R \to \infty$.

One might try to alter the condition at infinity in an effort to arrive at a definition of capacity consistent with those in $\mathbf{R}^3$. But the obvious changes are not helpful. The problem $\nabla^2 u = 0$ in Ω_c ; $u = 1$ on $\partial\Omega$ and $u = 0(1)$ as $|p| \to \infty$ has the unique solution $u = 1$ in which case $C = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = 0$ while if we change the condition at infinity to $u = o(1)$, then the problem has no solution. In either case one cannot define capacity as in $\mathbf{R}^3$.

Polya and Szego were, of course, aware of this behavior and as early as 1931 defined two dimensional capacity in terms of the solution of the problem $\nabla^2 u = 0$ in Ω_c ; $u = \ln \frac{1}{|p|} + 0(\frac{1}{|p|})$ and u = constant on $\partial\Omega$. The constant boundary value could not be prescribed but was part of the solution and they called this constant $\ln \frac{1}{C}$ with C being the capacity constant. In 1945 they changed the definition slightly by formulating the potential problem $\nabla^2 u = 0$ in Ω_c, $u = 0$ on $\partial\Omega$ and $u = \ln \frac{1}{|p|} - \ln \frac{1}{C} + 0(\frac{1}{|p|})$ where as before C could not be prescribed but appeared as part of the solution. They termed C the outer radius or transfinite diameter (following Fekete) because of its interpretation as the diameter of the circle whose exterior is the image of Ω_c under a conformal transformation that preserves the point at infinity. However they refrained from referring to C as the capacity constant despite their earlier designation. This may have stemmed from the apparent inconsistency with the definition of capacity in $\mathbf{R}^3$. One may infer from their repeated and varied treatments of this capacity question that, in addition to the study of isoperimetric inequalities, they hoped to achieve a uniform characterization of capacity in terms of the solution of a potential problem in all dimensions and remove the annoying necessity of using one definition in two dimensions and another in three.

More recently, Fichera (1961), applied integral equations of the first kind to a variety of elliptic problems. This was further developed by Hsiao and MacCamy (1973) in some specific cases including Laplace's equation where they showed how logarithmic capacity (essentially the logarithm of the transfinite diameter) could be obtained as part of a solution of a system of first kind integral equations in $\mathbf{R}^2$. Symm (1967) and Jaswon and Symm (1977) also used first kind integral equations to characterize capacity in both $\mathbf{R}^2$ and $\mathbf{R}^3$ but not in a uniform manner.

In the present paper we show that the concept of outer radius, C, may be generalized to n dimensions through

$$\omega = - g_n(C)$$

where g_n is the fundamental solution of Laplace's equation (c.f. Eq. (2) below) and ω is a constant which appears as part of the solution of a uniquely solvable system of integral equations which coincide, for $n = 2$, to that studied by Hsiao and MacCamy

(1973). We also formulate a uniquely solvable exterior boundary value problem for the Laplacian in $\mathbf{R}^n$, which generalizes that defined by Polya and Szego (1931) for $n = 2$, and contains as part of the solution the constant ω.

NOTATION, DEFINITIONS AND REPRESENTATION FORMULA

Let Ω be a bounded simply connected domain in $\mathbf{R}^n$ with smooth (Lyapunoff) boundary $\partial\Omega$. Denote the complement of the closure by Ω_c and the unit exterior normal at each point $q \in \partial\Omega$ by $\hat{n}_q$. Let S_n denote the surface area of a unit ball in $\mathbf{R}^n$, i.e.,

$$S_n = \int_{|q|=1} ds_q = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{1}$$

where $\Gamma(\cdot)$ is the usual gamma function and $|\cdot|$ is the Euclidian distance. Denote the fundamental solution of Laplace's equation for p and q in $\mathbf{R}^n$ by

$$g_n(|p-q|) = \begin{cases} -\dfrac{1}{S_2} \ln \dfrac{1}{|p-q|} & , \; n = 2 \\[2ex] \dfrac{-1}{(n-2)S_n} |p-q|^{2-n} & , \; n > 2 \end{cases} \tag{2}$$

The normal derivative of g_n is seen to be

$$\frac{\partial g_n(|p-q|)}{\partial n_q} = \frac{\hat{n}_q \cdot (q-p)}{S_n \; |p-q|^n} . \tag{3}$$

Let $B_\epsilon(p)$ denote a ball of radius ϵ and center p and define

$$\alpha(p) = \lim_{\epsilon \to o} \int_{\Omega \cap \partial B_\epsilon(p)} \frac{\partial}{\partial n_q} g_n(|p-q|) ds_q = \begin{cases} 1 & , \; p \in \Omega \\ 1/2 & , \; p \in \partial\Omega \\ 0 & , \; p \in \Omega_c \end{cases} \tag{4}$$

in terms of which Gauss' integral may be written

$$\int_{\partial\Omega} \frac{\partial g_n(|p-q|)}{\partial n_q} ds_q = \alpha(p) . \tag{5}$$

It is convenient to have the following forms of Green's theorem for interior and exterior domains:

If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\nabla^2 u = 0$ in Ω then

$$\int_{\partial\Omega} \left\{ u(q) \frac{\partial}{\partial n_q} g_n(|p-q|) - g_n(|p-q|) \frac{\partial u}{\partial n_q}(q) \right\} ds_q = \alpha(p) u(p) ; \tag{6}$$

and

if $u \in C^2(\Omega_c) \cap C^1(\overline{\Omega}_c)$, $\nabla^2 u = 0$ in Ω_c , $u = A g_n(r) + \omega + 0(1)$ and $\dfrac{\partial u}{\partial n} = A \dfrac{\partial g_n(r)}{\partial r} + o(\dfrac{1}{r ln r})$ as $r := |p| \to \infty$, where A and ω are constants but otherwise arbitrary, then

$$\int_{\partial\Omega} \left\{ g_n(|p-q|) \frac{\partial u(q)}{\partial n_q} - u(q) \frac{\partial}{\partial n_q} g_n(|p-q|) \right\} ds_q = [1-\alpha(p)] u(p) - \omega . \tag{7}$$

A function $u \in \Omega_c$ is said to be regular at infinity if

$$u = 0\,(\frac{1}{r^{n-2}}) \quad \text{and} \quad \frac{\partial u}{\partial r} = 0\,(\frac{1}{r^{n-1}}) \quad \text{as} \quad r \to \infty \;.$$

Thus when Green's theorem is applied to harmonic functions regular at infinity the constant ω will be absent in (7) for $n > 2$.

PREVIOUS DEFINITIONS OF CAPACITY AND RELATED QUANTITIES

In this section we present some of the existing definitions of capacity and related quantities.

If Ω_o is a bounded simply connected domain in $\mathbf{R}^3$ with smooth (Lyapunoff) boundary $\partial\Omega_o$ which contains $\overline{\Omega}$ in its interior then the capacity of $\partial\Omega$ with respect to $\partial\Omega_o$ is defined to be (see, e.g. Szego (1945) and Protter and Weinberger (1965))

$$C_o = -\frac{1}{4\pi} \int_{\partial\Omega} \frac{\partial u_o(q)}{\partial n_q} ds_q \tag{8}$$

where u_o is the capacitory potential of $\partial\Omega$ with respect to $\partial\Omega_o$ defined through

$$\begin{aligned} \nabla^2 u_o &= 0 \quad \text{in} \quad \Omega_o \setminus \overline{\Omega} \\ u_o &= 1 \quad \text{on} \quad \partial\Omega \;. \\ u_o &= 0 \quad \text{on} \quad \partial\Omega_o \end{aligned} \tag{9}$$

If $\Omega_o = B_\rho(0)$ then as $\rho \to \infty$, u_o becomes the solution of the conductor potential problem,

$$\begin{aligned} \nabla^2 u_1 &= 0 \quad \text{in} \quad \Omega_c \\ u_1 &= 1 \quad \text{on} \quad \partial\Omega \\ u_1 &= o(1) \quad \text{as} \quad r \to \infty \end{aligned} \tag{10}$$

and the capacity of $\partial\Omega$ is

$$C = \lim_{\rho \to \infty} C_o = -\frac{1}{4\pi} \int_{\partial\Omega} \frac{\partial u_1(q)}{\partial n_q} ds_q \tag{11}$$

In $\mathbf{R}^2$ one may define the capacity of $\partial\Omega$ with respect to $\partial\Omega_o$ analogously as

$$C_o = -\frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial u_o(q)}{\partial n_q} ds_q \tag{12}$$

where u_o is as before with Ω and Ω_o in $\mathbf{R}^2$ rather than $\mathbf{R}^3$. However, in contrast with the three dimensional case it is not possible to define the capacity of $\partial\Omega$ simply by letting $\Omega_o \to \mathbf{R}^2$ as was pointed out in the introduction.

Alternatively the capacity of $\partial\Omega$ with respect to $\partial\Omega_o$ has been defined as Szego (1945), Polya and Szego (1945)

$$C_o = \frac{1}{S_n} \int_{\Omega_o \setminus \overline{\Omega}} |\nabla u_o|^2 \, dq \quad , \; n = 2, 3 \;. \tag{13}$$

Moreover, it has been shown Kellogg (1953), Stakgold (1968), that u_o minimizes the Dirichlet integral over the class of admissable functions

$$U_{ad} := \left\{ u \;\middle|\; \begin{array}{l} u \in C^1(\Omega_o \setminus \overline{\Omega}) , \\ u = 1 \quad \text{on} \quad \partial\Omega , \\ u = 0 \quad \text{on} \quad \partial\Omega_o \end{array} \right\} \tag{14}$$

and C_o is the minimum value of the Dirichlet integral (13). As before the capacity of $\partial\Omega$ may be obtained as a limiting value of C_o as Ω_o grows to fill the entire space in $\mathbf{R}^3$ but not in $\mathbf{R}^2$.

Another characterization of capacity in $\mathbf{R}^3$ was given by Szego (1945), Riemann-Weber (1930), Stakgold (1968) where by C, also called the outer radius of $\partial\Omega$, is given by

$$\frac{1}{C} = \min_{\sigma \in V_{ad}} \int_{\partial\Omega} \int_{\partial\Omega} \frac{\sigma(p)\sigma(q)}{|p-q|} \, ds_p \, ds_q \tag{15}$$

where

$$V_{ad} = \left\{ \sigma \;\middle|\; \begin{array}{l} \sigma \in C^{\alpha}(\partial\Omega) \,,\; 0 < \alpha < 1 \\ \int_{\partial\Omega} \sigma(q) \, ds_q = 1 \end{array} \right\} . \tag{16}$$

Or, more appropriately in view of modern boundary element analyses Hsiao and Wendland (1977) and Nedelec (1977), the Holder continuous function space $C^{\alpha}(\partial\Omega)$ can be replaced by the Sobolev space $H^{-1/2}(\partial\Omega)$. We will subsequently show that this definition may be extended to $\mathbf{R}^n$ for any $n \geq 2$.

In $\mathbf{R}^2$, the outer radius or transfinite diameter has been characterized as the constant C occurring in the solution of the following problem (see Szego (1945))

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad \Omega_c \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ u &= \log \frac{1}{r} - \log \frac{1}{C} + o(1) \quad \text{as} \quad r \to \infty \,. \end{aligned} \tag{17}$$

Alternatively, Symm (1967) introduced the Robin constant γ as a part of the solution of the coupled set of boundary integral equations

$$\begin{aligned} &\int_{\partial\Omega} \log |p-q| \, \sigma_1(q) \, ds_q \; + \; \gamma = - \log |p| \\ &\int_{\partial\Omega} \sigma_1(q) \, ds_q = 0 \,. \end{aligned} \tag{18}$$

The outer radius of transfinite diameter is given in terms of γ as

$$C = e^{-\gamma} \,. \tag{19}$$

The outer radius is also related Szego (1945), Symm (1967) the conformal mapping, f, of the domain Ω_c to exterior of the unit circle which preserves the point at infinity by

$$C = \lim_{z \to \infty} \frac{1}{|f'(z)|} \,. \tag{20}$$

Other characterizations of transfinite diameter have been given by Fekete (1923) and Polya and Szego (1931). Symm has extended the definition implicitly in (18) to $\mathbf{R}^3$ by introducing capacity through the solution of the integral equation

$$\int_{\partial\Omega} \frac{\sigma_1(q)}{|p-q|}\, ds_q = 1 \tag{21}$$

in which case

$$C = \int_{\partial\Omega} \sigma_1(q)\, ds_q \ . \tag{22}$$

In the next sections we will demonstrate how the characterization of outer radius or capacity in terms of the solution of a minimization problem in $\mathbf{R}^3$ (15), or a boundary value problem in $\mathbf{R}^2$ (17), or an integral equation in $\mathbf{R}^2$ or $\mathbf{R}^3$ (21) or (18) may be extended to $\mathbf{R}^n$ so that a fundamental constant associated with the surface emerges as the solution of problems which are formulated in exactly the same way for all dimensions.

A UNIFORM CHARACTERIZATION

We will generalize the characterization of outer radius in terms of the solution of a boundary value problem given by Szego (1945) in $\mathbf{R}^2$ to $\mathbf{R}^n$, then show that there is an equivalent characterization in terms of solutions of a system of boundary integral equations slightly different from those given by Symm (1967) and Jaswon and Symm (1977). In the next section we will show that the weak formulations of the boundary value problem and the integral equations give rise to the generalization to $\mathbf{R}^n$ of the characterization of outer radius as the minimum of a functional given by Szego (1945) and Riemann-Weber (1930).

We define the following classical boundary value problem for $u \in C^2(\Omega_c) \cap C^{1+\alpha}(\overline{\Omega}_c)$ and $\omega \in \mathbf{R}$

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad \Omega_c \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ u &= g_n(r) + \omega + 0(r^{1-n}) \quad \text{as} \quad r \to \infty \end{aligned} \tag{23}$$

where g_n is defined in (2). The constant ω which appears as part of the solution pair (u,ω) is related to the outer radius C through

$$\omega = -\, g_n(C) \tag{24}$$

which constitutes an implicit definition of C, the capacity of $\partial\Omega$ for $n \geq 2$ (when $n = 2$ this is also called logarithmic capacity or transfinite diameter. The quantity ω is called the Robin constant in $\mathbf{R}^2$.) Explicitly for $n = 2$,

$$\omega = -\frac{1}{2\pi} \log C \ \ => \ \ C = e^{-2\pi\omega}$$

and for $n \geq 3$,

$$\omega = \frac{1}{(n-2) S_n\, C^{n-2}} \ \ => \ \ C = \{1/[(n-2)S_n\ \omega]\}^{1/n-2} \ .$$

The boundary value problem (23) will be shown to be equivalent to the system of boundary integral equations for $(\sigma,\omega) \in C^\alpha(\partial\Omega) \times \mathbf{R}$

$$\int_{\partial\Omega} \sigma(q) g_n(|p-q|\, ds_q = -\omega$$

$$\int_{\partial\Omega} \sigma(q)\, ds_q = 1 \quad . \tag{25}$$

The unique solvability of (23) and (25) and the relation between the two formulations are established in the following theorems.

Theorem 1 – *The boundary value problem (23) has a unique solution pair* (u,ω) *for all* $n \geq 2$. *If* $n > 2$ *then* $\omega \neq 0$.

Theorem 2 – *The system of integral equations (25) has a unique solution pair* (σ,ω) *for all* $n \geq 2$. *If* $n > 2$ *then* $\omega \neq 0$.

Theorem 3 – *If* (u,ω_1) *is the solution of (23) and* (σ,ω_2) *is the solution of (25) then*

$$\sigma = \frac{\partial u}{\partial n} \quad \text{on} \quad \partial\Omega \, , \ \omega_1 = \omega_2 : = \omega$$

$$\text{and} \quad u(p) = \int_{\partial\Omega} \sigma(q) g_n(|p-q|)\, ds_q + \omega \quad \text{in} \quad \overline{\Omega}_c \quad .$$

Proof of Theorem 1

If $n > 2$ and $\omega = 0$ then the boundary value problem (23) admits only the trivial solution since the homogeneous exterior Dirichlet problem for functions regular at infinity has no nontrivial solution from, for example, the maximum principle. Hence we assume $\omega \neq 0$ and define

$$v = 1 - \frac{u}{\omega} \tag{26}$$

in which case v solves the boundary value problem

$$\begin{aligned} \nabla^2 v &= 0 \quad \text{in} \quad \Omega_c \\ v &= 1 \quad \text{on} \quad \partial\Omega \\ v &= -\frac{1}{\omega} g_n(r) + 0(\frac{1}{r^{n-1}}) = 0(\frac{1}{r^{n-2}}) \end{aligned} \tag{27}$$

But this has a unique solution $v \in C^2(\Omega_c) \cap C^{1+\alpha}(\overline{\Omega}_c)$, since the boundary data is $C^{1+\alpha}(\partial\Omega)$. Now define

$$\omega : = \lim_{r \to \infty} - \frac{g_n(r)}{v(p)} \quad \text{and} \quad u(p) : = \omega - \omega v(p) \tag{28}$$

which establishes existence of at least one solution pair of (23) for $n > 2$.

For $n = 2$, we define

$$v = u - g_2(|p|) \tag{29}$$

in which case v solves the boundary value problem

$$\begin{aligned} \nabla^2 v &= 0 \quad \text{in} \quad \Omega_c \\ v &= -g_2(|p|) \quad \text{on} \quad \partial\Omega \\ v &= \omega + 0(r^{-1}) = 0(1) \quad \text{as} \quad r \to \infty \,. \end{aligned} \tag{30}$$

This problem has a unique solution $v \in C^2(\Omega_c) \cap C^{1+\alpha}(\overline{\Omega}_c)$. In fact, following Mikhlin (1970), the solution can be constructed by taking it to be of the form

$$v(p) = \int_{\partial\Omega} \phi(q) \frac{\partial}{\partial n_q} g_2(|p-q|) ds_q + \int_{\partial\Omega} \phi(q) ds_q \quad , \; p \in \Omega_c \,. \tag{31}$$

Here the density function ϕ satisfies the integral equation

$$-\frac{1}{2}\phi(p) + \int_{\partial\Omega} [\frac{\partial}{\partial n_q} g_2(|p-q|) + 1] \, \phi(q) ds_q = -g_2(|p|) \quad , \; p \in \partial\Omega \tag{32}$$

which is always solvable (see Mikhlin (1970), p. 387).

Now define

$$\omega := \lim_{r \to \infty} v(p) = \int_{\partial\Omega} \phi(q) ds_q \,. \tag{33}$$

This together with (29) establishes existence of at least one solution pair of (23) for $n = 2$.

Now for the uniqueness, assume there exist two solution pairs (u_1, ω_1) and (u_2, ω_2) of (23). As usual, let $u = u_1 - u_2$ and $\omega = \omega_1 - \omega_2$. Then (u, ω) solves the boundary value problem

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad \Omega_c \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ u &= \omega + 0(r^{1-n}) \quad \text{as} \quad r \to \infty \,. \end{aligned} \tag{34}$$

Since the Dirichlet integral

$$\int_{\Omega_c} |\nabla u|^2 dq = \lim_{R \to \infty} \int_{\partial B_R(0)} u \frac{\partial u}{\partial r} ds = 0 \,,$$

it follows that $\nabla u = 0$ in Ω_c and hence u = constant in Ω_c. From the boundary condition $u = 0$ on $\partial\Omega$, this implies $u \equiv 0$ in Ω_c and consequently $\omega = 0$. This establishes uniqueness and completes the proof of Theorem 1.

Proof of Theorem 2

For $n > 2$ the exterior problem

$$\begin{aligned} \nabla^2 u^e &= 0 \quad \text{in} \quad \Omega_c \\ u^e &= 1 \quad \text{on} \quad \partial\Omega \\ u^e & \quad \text{regular at } \infty \end{aligned} \tag{35}$$

is uniquely solvable. From Green's Theorem (7) the boundary conditions and Gauss' integral (5), it follows that

$$\int_{\partial\Omega} g_n(|p-q|) \frac{\partial u^e}{\partial n_q} ds_q = 1 \quad \text{for} \quad p \in \partial\Omega \,. \tag{36}$$

Now there exists a solution $(\tilde{\sigma} = \frac{\partial u^e}{\partial n})$ of

$$\int_{\partial\Omega} \tilde{\sigma}(q) g_n(|p-q|)\, ds_q = 1 \ . \tag{37}$$

To show that (37) is uniquely solvable, assume two solutions $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ and let $\tilde{\sigma} = \tilde{\sigma}_1 - \tilde{\sigma}_2$ in which case

$$\int_{\partial\Omega} \tilde{\sigma}(q) g_n(|p-q|)\, ds_q = 0 \quad \text{on} \quad \partial\Omega \ . \tag{38}$$

Now define

$$\left.\begin{array}{l} v^e \\ v^i \end{array}\right\} := \int_{\partial\Omega} \tilde{\sigma}(q) g_n(|p-q|) ds_q \quad \text{for} \quad \begin{cases} p \in \Omega_c \\ p \in \Omega \end{cases} \tag{39}$$

Then v^e is a solution of the homogeneous Dirichlet problem regular at infinity and v^i is a solution of the homogeneous interior Dirichlet problem. Hence both v^e and v^i vanish identically as do their normal derivatives in which case the usual jump conditions for the single layer imply that

$$\tilde{\sigma}(q) \equiv 0 \quad \text{on} \quad \partial\Omega$$

and therefore (37) has a unique solution. Next we show that if $\tilde{\sigma}$ solves (37) then

$$\int_{\partial\Omega} \tilde{\sigma}(q)\, ds_q \neq 0 \ . \tag{40}$$

Again define

$$v^e(p) = \int_{\partial\Omega} \tilde{\sigma}(q) g_n(|p-q|)\, ds_q \ , p \in \overline{\Omega}_c \tag{41}$$

Then $v^e(p)$ is regular at infinity and Green's theorem implies that

$$\begin{aligned} \int_{\partial\Omega} g_n(|p-q|) \frac{\partial v^e}{\partial n_q} ds_q &= \frac{1}{2} v^e(p) + \int_{\partial\Omega} v^e(q) \frac{\partial g_n(|p-q|)}{\partial n_q} \quad \text{for } p \in \partial\Omega \\ &= 1 \end{aligned} \tag{42}$$

where the fact that $\tilde{\sigma}$ satisfies (37) and Gauss' integral (5) have been employed.

But since we have established the unique solvability of (37) it follows that

$$\tilde{\sigma}(p) = \frac{\partial v^e}{\partial n_p} \tag{43}$$

Hence

$$\begin{aligned} \int_{\partial\Omega} \tilde{\sigma}(q)\, ds_q &= \int_{\partial\Omega} \frac{\partial v^e}{\partial n_q}\, ds_q = \int_{\partial\Omega} v^e(q) \frac{\partial v^e}{\partial n_q}\, ds_q \\ &= -\int_{\Omega_c} |\nabla v^e(q)|^2\, dq \ . \end{aligned} \tag{44}$$

Thus if $\int_{\partial\Omega} \tilde{\sigma}(q)\, ds_q = 0$ it follows that $\nabla v^e \equiv 0$ in Ω_c which, with the boundary condition implies that $v^e(p) \equiv 1$. However this contradicts the fact that v^e is regular at ∞. Therefore (40) is satisfied. We therefore define, for the uniquely defined solution of (37) satisfying (40)

$$\omega := -\frac{1}{\int_{\partial\Omega} \tilde{\sigma}(q)\, ds_q} \neq 0 \tag{45}$$

$$\sigma = \frac{\tilde{\sigma}(p)}{\int_{\partial\Omega} \tilde{\sigma}(q)\, ds_q} . \tag{46}$$

Clearly (σ , ω) satisfies the system (25). To establish uniqueness assume there exist two solution pairs (σ_1,ω_1) and (σ_2 , ω_2) and let

$$\sigma = \sigma_1 - \sigma_2$$

$$\omega = \omega_1 - \omega_2 .$$

Then,

$$\int_{\partial\Omega} \sigma(q) g_n(|p-q|) ds_q = -\omega \tag{47}$$

$$\int_{\partial\Omega} \sigma(q) ds_q = 0 . \tag{48}$$

If $\omega \neq 0$ then $\tilde{\sigma} := -\frac{\sigma}{\omega}$ satisfies (37) and since (40) is satisfied it follows that

$$\int_{\partial\Omega} \frac{\sigma(q)}{\omega}\, ds_q \neq 0$$

which contradicts (48). Thus we conclude that $\omega = 0$ or $\omega_1 = \omega_2$. Hence

$$\int_{\partial\Omega} \sigma(q) g_n(|p-q|) ds_q = 0 \ , \ p \in \partial\Omega \ , \tag{49}$$

and using the same argument following (38) we find that $\sigma = 0$ or $\sigma_1 = \sigma_2$. This concludes the proof of Theorem 2 for $n > 2$.

For $n = 2$ the above analysis is not directly applicable because of the logarithmic growth of the fundamental solution. To establish the theorem in this case we follow the work of Hsiao and MacCamy (1973) based on results of Muskhelishvili (1953) and Fichera (1961). The desired result is contained in Theorem 3 of Hsiao and MacCamy (1973) and the main points of the proof specialized to the present case are included here for completeness.

The system which we wish to show uniquely solvable takes the form

$$\begin{aligned} &\int_{\partial\Omega} \sigma(q) \log |p-q|\, ds_q = -2\pi\omega \\ &\int_{\partial\Omega} \sigma(q)\, ds_q = 1 . \end{aligned} \tag{50}$$

Every σ which is part of a solution pair (σ , ω) of (46) will satisfy

$$\frac{d}{ds_p} \int_{\partial\Omega} \sigma(q) \log |p-q|\, ds_q = 0 \tag{51}$$

where S_p denotes arc length in $\partial\Omega$. As shown in Muskhelishvili (1953), differentiation and integration may be interchanged yielding a singular integral equation with Cauchy kernel. Moreover the singular integral operator has index zero hence the operator and its adjoint have null spaces of the same dimension. Since it is easy to see that constants are solutions of the homogeneous adjoint equation, the dimension of the null space is at least one. To show that the dimension is exactly one, that is, that (50) has exactly one linearly independent solution we proceed as follows.

Assume that there exists two non-trivial solutions σ_i , $i = 1,2$, of (51). Then, integrating (51), we define two constants ω_i , $i = 1,2$, by

$$\omega_i := -\frac{1}{2\pi} \int_{\partial\Omega} \sigma_i(q) \log|p-q| ds_q \quad , \quad p \in \partial\Omega \quad , \quad i = 1,2 \quad . \tag{52}$$

Furthermore define

$$A_i := \int_{\partial\Omega} \sigma_i(q)\, ds_q \quad , \quad i = 1,2 \tag{53}$$

and choose constants α_i, not both zero, such that

$$\alpha_1 A_1 + \alpha_2 A_2 = 0 \tag{54}$$

Next define

$$\sigma := \alpha_1 \sigma_1 + \alpha_2 \sigma_2 \tag{55}$$

and the single layer potential

$$u(p) = \int_{\partial\Omega} \sigma(q) \log|p-q|\, ds_q \quad , \quad p \in \mathbf{R}^2 \tag{56}$$

It is easy to see with (52)-(54) that u is a solution of the exterior Dirichlet problem

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad \Omega_c \\ u &= -2\pi(\alpha_1 \omega_1 + \alpha_2 \omega_2) \quad \text{on} \quad \partial\Omega \\ u &= 0(\frac{1}{r}) \quad \text{as} \quad r \to \infty \quad . \end{aligned} \tag{57}$$

But this problem is solvable only if

$$\alpha_1 \omega_1 + \alpha_2 \omega_2 = 0$$

in which case u defined by (56) vanishes identically in $\overline{\Omega}_c$. In addition u vanishes in Ω since it is a solution of the homogeneous interior Dirichlet problem. Using standard arguments of potential theory, the vanishing of the normal derivatives and the jump conditions for derivatives of single layer potentials allow us to conclude that

$$\sigma = 0$$

in which case, with (55), σ_1 and σ_2 are seen to be linearly dependent. Moreover it has been shown Muskhelsihvili (1953) that every non-trivial solution σ_o of (51) has the property that

$$\int_{\partial\Omega} \sigma_o(q)\, ds_q \neq 0 \quad .$$

Let σ be the unique solution of (51) such that

$$\int_{\partial\Omega} \sigma(q)\, ds_q = 1 ,$$

and define

$$\omega := -\frac{1}{2\pi} \int_{\partial\Omega} \sigma(q) \log|p-q|\, ds_q \ . \tag{58}$$

Then (σ, ω) is a solution pair of the system (50).

To prove uniqueness, assume the existence of two solution pairs, (σ_i, ω_i) $i = 1,2$. Set $\sigma = \sigma_1 - \sigma_2$ and $\omega = \omega_1 - \omega_2$ in which case

$$\int_{\partial\Omega} \sigma(q) \log|p-q|\, ds_q = -2\pi\omega$$

$$\int_{\partial\Omega} \sigma(q)\, ds_q = 0 \tag{59}$$

Then by the same argument as before (since differentiation of (59) with respect to arc length leads to (51)), we conclude that $\sigma = 0$ and $\omega = 0$ hence the system (50) is uniquely solvable. This completes the proof of Theorem 2.

Proof of Theorem 3

If (u, ω_1) is the solution of (23) then Green's Theorem (7) implies that

$$\int_{\partial\Omega} g_n(|p-q|) \frac{\partial u}{\partial n_q}\, ds_q = [1-\alpha(p)]\, u(p) - \omega_1 \ , \ p \in \overline{\Omega}_c \ . \tag{60}$$

For $p \in \partial\Omega$ we have

$$\int_{\partial\Omega} g_n(|p-q|) \frac{\partial u}{\partial n_q}\, ds_q = -\omega_1 \ . \tag{61}$$

The asymptotic form of g_n together with the behaviour of u as $r \to \infty$ and the representation (60) leads to

$$g_n(r) \int_{\partial\Omega} \frac{\partial u}{\partial n_q}\, ds_q = g_n(r) + 0(\frac{1}{r^{n-1}}) \ . \tag{62}$$

Hence

$$\int_{\partial\Omega} \frac{\partial u}{\partial n_q}\, ds_q = 1 \ . \tag{63}$$

Thus $(\frac{\partial u}{\partial n_q}, \omega_1)$ is a solution of the system (61), (63) which coincides with (25).

Conversely if (σ, ω_2) is the unique solution of (25) define

$$u(p) := \int_{\partial\Omega} \sigma(q) g_n(|p-q|)\, ds_q + \omega_2 \quad \text{in} \quad \overline{\Omega}_c \ . \tag{64}$$

Then $(u(p), \omega_2)$ is a solution of the boundary value problem (23) and because of uniqueness, is the only solution. Hence by the first part of the proof $(\frac{\partial u}{\partial n}, \omega_2)$ must be a solution of the system of integral equations (25). Because this system admits only one solution pair, it follows that

$$\sigma = \frac{\partial u}{\partial n} .$$

VARIATIONAL FORMULATION

In the previous section we showed how the Robin constant ω appeared as part of the solution of neither the boundary value problem (23) or the system of integral equations (25). In the case of the boundary value problem one may show that for $n \geq 3$ the weak formulation constitutes the necessary conditions for minimizing the functional

$$J[u] := \int_{\Omega_c} |\nabla u|^2 \, dq - 2\,\omega \tag{65}$$

over the space of functions

$$U_{ad} := \left\{ u \;\middle|\; \begin{array}{l} u \in C_1(\Omega_c) \\ u = g_n(r) + \omega + 0(\frac{1}{r^{n-1}}) \text{ as } r \to \infty \\ u = 0 \text{ on } \partial\Omega \end{array} \right\} . \tag{66}$$

Moreover the functional assumes its minimum when u satisfies (23) and

$$\min_{u \,\varepsilon\, U_{ad}} J[u] = -\,\omega . \tag{67}$$

One may show that this is only a slight change of the standard variational treatment of the Dirichlet integral (see e.g. Kellogg (1929)).

Unfortunately there does not appear to be an analogue of this characterization for $n = 2$ because the Dirichlet integral does not exist for functions with logarithmic growth. However the variational formulation of the system of integral equations (25) is valid for $n = 2$ as well as $n \geq 3$ and these results are presented here.

Consider the following functional of $\sigma \,\varepsilon\, U_{ad}$ where

$$U_{ad} = \left[\sigma \mid \sigma \,\varepsilon\, H^{-1/2}(\partial\Omega) \;,\; \int_{\partial\Omega} \sigma(q) ds_q = 1 \right\} ; \tag{68}$$

$$J[\sigma] := -\int_{\partial\Omega}\int_{\partial\Omega} \sigma(p)\sigma(q) g_n(|p-q|) \, ds_q \, ds_p . \tag{69}$$

We will show that the Robin constant is intimately connected with the minimum of this functional for all $n \geq 2$. Further we show that the weak solution of the system of integral equations (25) minimizes this functional. To this end note that if σ and $\sigma + \varepsilon\eta$ are both in U_{ad} then

$$\int_{\partial\Omega} \eta(q) \, ds_q = 0 \tag{70}$$

and

$$\begin{aligned} J[\sigma + \varepsilon\eta] = J[\sigma] &- 2\varepsilon \int_{\partial\Omega}\int_{\partial\Omega} \eta(p)\, \sigma(q) g_n(|p-q|) ds_q \, ds_p \\ &- \varepsilon^2 \int_{\partial\Omega}\int_{\partial\Omega} \eta(p)\eta(q) g_n(|p-q|) ds_q \, ds_p . \end{aligned} \tag{71}$$

The necessary condition for $J[\sigma]$ to have an extreme value at $\sigma \varepsilon U_{ad}$ is

$$\int_{\partial\Omega}\int_{\partial\Omega} \eta(p)\,\sigma(q)g_n(|p-q|)ds_q\,ds_p = 0 \tag{72}$$

for all $\eta \varepsilon H^{-1/2}(\partial\Omega)$ satisfying (70). Furthermore if (72) is satisfied then $J[\sigma]$ is a minimum since

$$I[\eta] = -\int_{\partial\Omega}\int_{\partial\Omega} \eta(p)\,\eta(q)\,g_n(|p-q|)\,ds_q\,ds_p \ \geq\ 0 \quad \text{for all } \eta \in H^{-1/2}(\partial\Omega) \tag{73}$$

satisfying (70).

To show that (73) holds we follow the argument of Nedelec (1977) for $n \geq 3$ and Hsiao and Wendland (1977) for $n = 2$. For $\eta \varepsilon H^{-1/2}(\partial\Omega)$ satisfying (70) define

$$u(p) := \int_{\partial\Omega} \eta(q)g_n(|p-q|)\,ds_q \ , \ p \ \varepsilon\ \mathbf{R}^n$$

The standard jump conditions for the single layer potential yield

$$\frac{\partial u}{\partial n^+} - \frac{\partial u}{\partial n^-} = \eta(q) \ .$$

Then

$$I(\eta) = -\int_{\partial\Omega} \eta(p)u(p)\,ds_q = -\int_{\partial\Omega}\left[\frac{\partial u}{\partial n^+} - \frac{\partial u}{\partial n^-}\right]u(p)\,ds_q$$

$$= \int_{\Omega^c \cap B_R(0)} |\nabla u|^2\,dp - \int_{\partial B_R(0)} \frac{\partial u}{\partial r}\,u\,ds_q + \int_{\Omega} |\nabla u|^2\,dp \ .$$

The growth of the Green's function is sufficient, for $n \geq 3$, to guarantee that

$$\int_{\partial B_R(0)} \frac{\partial u}{\partial r}\,u\,ds = 0(\frac{1}{R}) \ .$$

However for $n = 2$ it is necessary to employ condition (70) to achieve this same result. Thus for $n \geq 2$, the Dirichlet integral always exists thus

$$I(\eta) = \int_{\mathbf{R}^n} |\nabla u|^2\,dp \geq 0$$

which guarantees that (73) holds.

Observe that (72) constitutes the weak formulation of the system of integral equations (25). Since we have shown that there exists a unique solution pair (σ, ω) of equation (25) it is easy to see that σ will satisfy (72). Moreover the minimum value of J is seen to be, with (25),

$$J[\sigma] = -\int_{\partial\Omega}\int_{\partial\Omega} \sigma(p)\sigma(q)g_n(|p-q|)ds_q\,ds_p$$

$$= \omega\int_{\partial\Omega} \sigma(p)\,ds_q = \omega \ , \tag{74}$$

hence

$$\min_{\sigma \varepsilon U_{ad}} J[\sigma] = \omega \ .$$

This coincides with the characterization given Szego (1945) for $n = 3$.

CONCLUDING REMARKS

In summary we have characterized the capacity or outer radius implicitly by

$$\omega = - g_n(c) \tag{24}$$

where g_n is the fundamental solution of the n- dimensional Laplacian, (2), and ω emerges in the unique solution of the boundary value problem (23)

$$\begin{aligned} \nabla^2 u &= 0 \quad \text{in} \quad \Omega_c \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ u &= g_n(r) + \omega + 0(\frac{1}{r^{n-1}}) \quad \text{as } r \to \infty \end{aligned} \tag{23}$$

or the system of boundary integral equations (25)

$$\begin{aligned} \int_{\partial\Omega} \sigma(q) g_n(|p-q|)\, ds_q + \omega = 0 \\ \int_{\partial\Omega} \sigma(q)\, ds_q = 1 \end{aligned} \tag{25}$$

or the optimization problem

$$\omega = \inf_{\sigma \varepsilon U_{ad}} - \int_{\partial\Omega} \int_{\partial\Omega} \sigma(p)\sigma(q) g_n(|p-q|)\, ds_q\, ds_p \tag{74}$$

where U_{ad} is defined in (68).

We conclude by showing how these characterizations encompass existing definitions summarized in section 3.

The definition of capacity (11) in terms of the solution of the conductor potential problem (10) for $n = 3$, u_1, is easily seen to coincide with $\cdot\omega$, through (24), by substituting

$$u_1 = 1 - \frac{u}{\omega}$$

in equation (11).

We already noted at the beginning of section 5 that the characterization of capacity as the minimum of the Dirichlet integral when $n = 3$ is essentially equivalent to the weak form of (23). The generalization of (15) for $n \neq 3$ has been treated in section 5.

The relations between the system of boundary integral equations (25) and those given by Symm are as follows. For $n = 2$, (σ_1, γ) appearing in (18) are related to (σ, ω) of (25) by

$$\gamma = 2\pi\omega$$

$$\sigma_1 = \sigma + \frac{\partial u_o}{\partial n} - \frac{\partial g_2(r)}{\partial n}$$

where u_0 is the solution of the interior Dirichlet problem for the Laplacian with boundary values

$$u_0 = g_2(r) \quad \text{on} \quad \partial\Omega \ .$$

For $n = 3$ the relation between solutions of (21) and (25) is simply

$$\sigma_1 = \frac{\sigma}{4\pi\, \omega} \ .$$

Finally to establish the connection between the conformal mapping (c.f. Eq. (20)) and the boundary value problem (23) for $n = 2$, we let $f(z)$ be the conformal mapping of Ω_c to the exterior of the unit circle such that $|f(z)| = 1$ when $z \in \partial\Omega$ and $\frac{f(z)}{z} = 0(1)$ when $z \to \infty$. We use the complex representation z for the point p. Then

$$f(z) = z\ e^{\phi + i\psi}$$

where ϕ and ψ are harmonic conjugates regular at infinity (i.e. $\phi = 0(1)$). Then it is easy to see that ϕ is related to the solution of (23) by

$$u = \frac{1}{2\pi}\,\phi + \frac{1}{2\pi}\log r \quad , \quad r = |z| \ .$$

Moreover, a simple computation shows that

$$\lim_{z\to\infty} |f'(z)| = \lim_{z\to\infty} e^{\phi} = e^{2\pi\omega}$$

from which equation (20) follows.

Acknowledgment – *This work was partially supported by the National Science Foundation under Grant DMS-8507063.*

References

Fekete, M., 1923, Uber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Mathematische Zeitschrift, Vol. 17, 228-249.

Fichera, G., 1961, Linear Elliptic Equations of Higher Order in Two Independent Variables and Singular Integral Equations *in* Partial Differential Equations and Continuum Mechanics, ed. R.E. Langer, The University of Wisconsin Press, 55-80.

Hsiao, G.C., and MacCamy, R.C., 1973, Solution of Boundary Value Problems by Integral Equations of the First Kind, SIAM Review, *15,* 687-745.

Hsiao, G.C., and Wendland, W., 1977, A Finite Element Method for some Integral Equations of the First Kind, J. Math. Anal. Appl. *58,* 449-481.

Jaswon, M.A., and Symm, G.T., 1977, "Integral Equation Methods *in* Potential Theory and Elastostatics", Academic Press, London,

Kellogg, O.D., 1953, "Foundations of Potential Theory, Dover Publications", New York.

Mikhlin, S.G., 1970, "Mathematical Physics, An Advanced Course", North- Holland Publishing Company, Amsterdam, London.

Muskhelishvili, N., 1953, "Singular Integral Equations", P. Noordhoff, Groningen, The Netherlands.

Nedelec, J.C., 1977, Approximation des Equations Integrals en Mecanique et en Phsique, Centre de Mathematiques Appliques, Ecole Polytechnique - Palaiseau.

Payne, L., 1967, Isoperimetric Inequalities, SIAM Review *9* (3), 453-488.
Polya, G., and Szego, G., 1931, Uber den transfiniten Durchmesser (Kapazitatskonstante) von ebenen und raumlichen Punktmengen, J. Reine Angew. Math. 165, 4-49.
Polya, G., and Szego, G., 1945, Inequalities for the Capacity of a Condenser, Amer. J. Math., 67, 1-32.
Polya, G., and Szego, G., 1951, "Isoperimetric Inequalities *in* Mathematical Physics", Princeton Univ. Press.
Protter, M.H., and Weinberger, H.F., 1965, On the Capacity of Composite Conducters, J. Math. and Phys. 44-45, 375-383.
Riemann-Weber (P. Frank and R.v. Mises), 1930, Die Differential und Integralgleichungen der Mechanik und Physik, Vol. 1, 8 ed. Braunschweig.
Stakgold, I., 1968, "Boundary Value Problems of Mathematical Physics", Vol. 2, Macmillan, New York.
Symm, G.T., 1967, Numerical Mapping of Exterior Domains Nu. Math. 10, 437-445.
Szego, G., 1945, On the Capacity of a Condenser, Bull. Amer. Math. Soc. 51, 325-350.