A UNIQUENESS THEOREM FOR THE REDUCED WAVE EQUATION GOVERNING THE ACOUSTIC WAVE IN A HETEROGENEOUS MEDIUM

By D. J. N. WALL

(*Department of Mathematics, University of Canterbury, Christchurch, New Zealand*)

[Received 17 April 1986. Revise 12 March 1987]

SUMMARY

A uniqueness theorem is established for the scattering of harmonic small-amplitude longitudinal (acoustic) waves by a body with spatially varying parameters. The theorem in particular incorporates structures suitable for application to problems formulated for computational solution.

1. Introduction

MANY authors (see (**1**, **2**) for bibliography) have considered uniqueness theorems for the scattering of harmonic small-amplitude (acoustic) waves by penetrable bodies. The majority of these papers consider homogeneous, or at least piecewise homogeneous bodies. A few consider bodies with spatially varying parameters with smoothness conditions placed on the parameters for the whole region (**3**, **4**). As explained in (**2**) many of these are unsatisfactory for computational reasons. We follow the approach used by (**2**), but extend the results considerably beyond the piecewise homogeneous bodies they analyse; their uniqueness results are a special case of our result. In particular, we have tried to produce uniqueness theorems that will cover the cases met with, when considering computational solutions of the equations considered here.

We stress that our interest here is for computational purposes, where it is essential that two factors are present; these are often excluded from other uniqueness theorems. These are that

1. the composition of the body must not have a nested restriction (**4**, **5**),
2. the restriction on the composition of the body must allow for the material parameters to be piecewise smooth, but spatially varying.

These factors ensure that any weak formulations of the governing equations may be successfully tackled, via a discretization procedure. In order to include the above two requirements, we find it necessary to restrict the smoothness of the material parameters to be piecewise real-analytic. This we claim, does *not* restrict our results in a computational setting, as then the material parameters of the body must *always* be able to be approximated by piecewise polynomials in order to obtain any results.

Jones (**4**) has considered uniqueness theorems for scattering problems in elastodynamics, with spatially varying parameters, but his proofs make extensive use of the properties of the spherical harmonics. In contrast, our proofs are straightforward and we produce results which allow for the two factors mentioned earlier. As only *finite-degree* piecewise polynomials lie within the piecewise real analytic class, the Stone–Weierstrass theorem is *not* applicable to give uniform approximation to any piecewise continuous function. However, from within this polynomial class, approximation theory shows that we may approximate, with an accuracy bounded away from zero, any continuous function. It is for this reason that we imply that the uniqueness theorem has wide computational application.

In section 2 we introduce the equations we produce the uniqueness theorems for, and describe the geometry of the problem.

Throughout, we subsume the standard uniqueness argument for linear operators, that of assuming the possibility of two solutions satisfying the same partial differential equation and boundary data. It then suffices, by taking the difference of the two assumed solutions, to show that the only possible solution to the homogeneous equation, with zero boundary data, is the zero solution. As Jones (**4**) points out, the proof of uniqueness for the exterior problem revolves about the proof for the interior body. In section 3, we prove uniqueness for the interior problem, by use of the Holmgren uniqueness corollary to the Cauchy–Kovalevskaia theorem. For exterior problems, we must utilise this result, and the Rellich lemma, together with analyticity arguments, to show uniqueness for the exterior problem. This is carried out in section 4. The interior proof in section 3 is the only part which utilizes the piecewise analyticity of the material parameters; if this restriction could be relaxed then the same weakening would be carried through immediately for the exterior problem.

2. The governing equations and the problem geometry

We make the usual assumption; that we are considering a small-amplitude linear acoustic wave propagating in a static elastic fluid and the wave motion is adiabatic and isentropic. Then, if the fluid has static density $\rho(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$, the velocity potential $u(\mathbf{x})$ is related to the velocity of the fluid particles $\mathbf{v}(\mathbf{x})$ by

$$\nabla u = \rho \mathbf{v}. \tag{1}$$

The equation of motion for a driving-force potential F can then be written as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \rho \nabla \cdot (\rho^{-1} \nabla u) - \sigma \nabla \cdot \left(\rho^{-1} \nabla \frac{\partial u}{\partial t}\right) = \frac{\partial F}{\partial t}, \tag{2}$$

where $c(\mathbf{x})$ and $\sigma(\mathbf{x})$ are, respectively, the wave speed and the coefficient of the expansive friction in the fluid. Note that the bulk modulus κ is related to the wave speed through $\kappa = \rho c^2$. We are considering time-harmonic problems so, with a time dependence $\exp(i\omega t)$, (2) can be written as

$$\rho \nabla . (\rho^{-1} \nabla u) + k^2 u = 0, \tag{3}$$

where the wave number is $k = [\omega^2/(c^2 + i\omega\sigma/\rho)]^{\frac{1}{2}}$, and for the reason mentioned in the introduction we are only interested in the homogeneous equation, that is, $F \equiv 0$. We note that u *now* denotes the complex velocity potential.

In most practical applications ρ and σ are real and are required to be positive, while c^2 can be complex in some problems, for example visco-elastic media. We assume in the sequel that $\rho, c, \sigma \in \mathbb{R}$. The fluid parameters ρ, c, σ characterize the properties of the fluid for wave propagation; however, when considering equation (3), ρ and k suffice. In most problems these parameters take on constant values as $|\mathbf{x}| \to \infty$; we shall assume that this is the case in the sequel. Regions in which the material parameters differ from these constant values constitute scattering centres for wave-propagation problems, and as such can be considered as wave-scattering bodies.

When solutions of (3) are to be found with $\mathbf{x} \in \mathbb{R}^3$ it is sufficient to specify that ρ and k be piecewise continuous and u satisfies the radiation condition

$$\int_{S_R} \left| \frac{\partial u}{\partial \nu} + iku \right|^2 ds = o(1) \quad \text{as} \quad R \to \infty \tag{4}$$

on the sphere $S_R = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = R\}$ in order to fully specify the problem. In (4), $\partial/\partial\nu$ denotes the directional derivative in the direction of the outward normal to S_R and ds is the surface measure.

When considering problems involving edges or corners, to ensure uniqueness it will be necessary to require that the field satisfy the *edge condition* (**6**, **7**). An edge condition that will give sufficiency conditions for a unique solution is that the acoustic energy density is integrable over any finite region, even if this domain contains singularities of the field. Another equivalent form of this condition, which will be utilized in proving Lemma 2, is that the surface integral of the complex energy-flux density over a small surface S_ε enclosing the edge reduces to zero, as S_ε contracts to the edge in the limit with $\varepsilon \to 0$. Stated mathematically,

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \rho^{-1} u \frac{\partial \bar{u}}{\partial \nu} ds = 0, \tag{5}$$

where the directional derivative is along a normal to S_ε.

The specification of the problem just given requires (3) to be interpreted in the weak sense and leads to the modern weak-solution formulation; see for example (**8**). This is the approach that is used in most computational examinations of the problem, and is our aim here, in obtaining uniqueness theorems for (3) with as weak restrictions on k and ρ as possible.

However, we must also examine how the classical formulation of the problem proceeds; then u is required to be second-partial-derivative continuous, that is, $u \in C^2$. We now require ρ to be piecewise differentiable and k to be piecewise continuous; then equation (3) holds only on open regions Ω, where $\rho \in C^1$ and $k, \sigma, c \in C$. On the surfaces of these open regions, that is, $\partial\Omega$, where these restrictions to ρ and k do not apply, (3) and (4) must be supplemented by the jump conditions

$$\left[\frac{1}{\rho}\frac{\partial u}{\partial v}\right] = 0, \qquad [u] = 0. \tag{6}$$

We now define the notation used in equation (6); if $\hat{\mathbf{v}}$ is a unit normal vector on a surface Σ, the difference between the values taken by the field ϕ on the sides of Σ towards which and away from which $\hat{\mathbf{v}}$ is directed is called the jump of ϕ on Σ; it is denoted by $[\phi]$. The conditions (6) constitute just the usual continuum-mechanical specification of a bonded material, that is, the velocity vector and the traction vector are continuous, respectively.

We must now define more precisely the geometry of the problem which (3) to (6) describe. Let $D_0 \in \mathbb{R}^3$ denote an unbounded, open region, and assume that there is one bounded open region D_1 such that $D_0 \cap D_1 = \varnothing$, $\overline{D_0} \cup D_1 = D_0 \cup \overline{D_1} = \mathbb{R}^3$. *We note that* D_1 connected (respectively non-connected) will allow for one (respectively many) scattering regions. We assume in the sequel that D_1 is connected for simplicity, without restricting our final result. The interface boundary $S_1 = \partial D_1$ is assumed to be a closed, sufficiently smooth surface (regular (**9**, p. 100)), so that an application of Green's first theorem at every finite part of D_0 and D_1 is valid.

We begin with some definitions on the term '*piecewise real-analytic*', and from this point on, we drop the prefix real and use the term *analytic* as a synonym for real analytic.

Let Ω be an open set in $\mathbb{R}^3$. We recall that a function $f : \Omega \to \mathbb{R}$ is analytic in Ω (see (**8**, p. 24)) if it is infinitely-often differentiable, that is, in $C^\infty(\Omega)$, and for every compact set $K \subset \Omega$ there exists a positive constant r_K for which

$$\sup \frac{1}{\alpha!} r_K^{|\alpha|} |D^\alpha f(\mathbf{x})| < \infty,$$

the supremum being over all points $\mathbf{x} \in K$, and all triples α of non-negative integers. Then $f \in C^\omega(\Omega)$.

A finite family $\{\Omega_j\}_{j=1}^N$ with N members is a disjoint piecewise cover of the closed set $\bar{\Omega}$ if

1. $\bar{\Omega} = \bigcup_{j=1}^{N} \Omega_j$,
2. $\Omega_i \cap \Omega_j = \varnothing$ for $i \neq j$.

Then a function $f : \Omega \to \mathbb{R}$ is piecewise analytic on $\bar{\Omega}$ (relative to the cover $\{\Omega_j\}$) if its restriction to Ω_j is analytic for $\bar{\Omega}_j$, for each j. Notice that we have not excluded the possibility of some Ω_j having a non-smooth surface $\partial\Omega_j = \bar{\Omega}_j \setminus \Omega_j$; we simply make the assumption that the surface is from the regular class (piecewise smooth).

Within S_1 the material parameters are assumed to be piecewise analytic; so that ρ, c, and σ, and hence k, are piecewise analytic on $\bar{D}_1$ relative to separate covers. We denote by $\{\Omega_j\}$, the *smallest* finite cover of $\bar{D}_1$ such that ρ, c and σ are each analytic on $\bar{\Omega}_j$, for each j, that is, $\rho, c, \sigma \in C^{\omega}(\bar{\Omega}_j)$. The cover $\{\Omega_j\}$ has therefore been 'aligned' with the material properties so that they are analytic except possibly across the interfaces $\partial\Omega_j$, where they may change discontinuously. We note that the aforementioned assumptions ensure that regions in which the material parameters are not analytic are of measure zero in $\mathbb{R}^3$. This implies that the solution u of the elliptic partial differential equation (3) is analytic within each $\bar{\Omega}_j$ (**10**, Chapter 5), except at points of $\bar{\Omega}$ on an edge, and those are of measure zero.

It is now convenient to provide further description of the cover of D_1. Within S_1 the material parameters are C^{ω}, except on a finite number of surfaces. We see from the previous discussion that the region D_1 can be further subdivided into open regions, not necessarily simply connected, on which the material parameters are analytic, and these regions themselves permit further subdivision, and so on. Thus the cover of D_1 has a tree-like structure to handle any nested regions, and we divide D_1 further, so as to represent the support of each Ω_j by a particular region. Figure 1 illustrates this structure for a typical problem geometry. In this figure D_1 has been

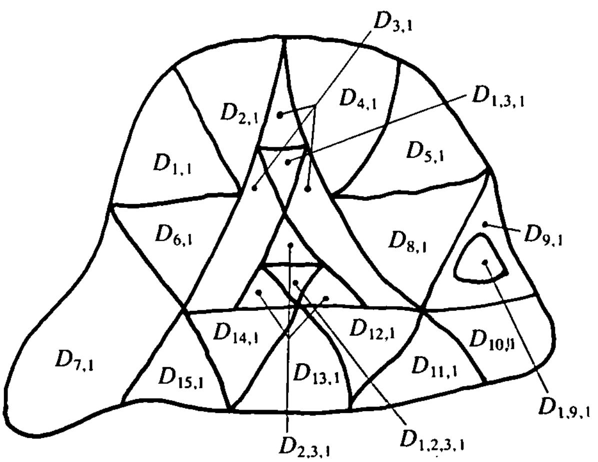

FIG. 1. Cross-section of a typical region, showing the tree-like structure and surfaces across which parameters are not analytic

further sub-divided into regions. These are

at level one, $D_{i,1}$ where $1 \leq i \leq 15$,

at level two, $D_{1,3,1}$, $D_{2,3,1}$, $D_{1,9,1}$,

at level three, $D_{1,2,3,1}$,

within which $\rho, c, \sigma \in C^{\omega}$. The subscript notation is such that the first index denotes the scatterer number in the particular embedding region, and the remaining indices denote in which region the scatterer is embedded. This figure illustrates the type of structure that might occur in a typical computational problem. It incorporates structure inside S_1 which is both nested and non-nested, having the *finite-element* type structure common to modern computational problems.

A consistent notation is used for the scattering surfaces across which the material parameters change discontinuously or are not C^{ω}. Thus S_1 is the interior closure of D_0 and all other surfaces are the exterior closure of a particular region, for example, $S_{1,1} = \bar{D}_{1,1} \backslash D_{1,1}$. The level of the scatterer can be readily determined as

$$\mathscr{L} = \text{number of tuples } -1,$$

with level 0 being the embedding region D_0. The aforementioned subscript notation soon becomes very cumbersome, so we abbreviate it by using a multi-index to represent the subscript tuples.

We choose to develop uniqueness theorems for the reduced wave equation (3) rather than the Helmholtz equation, because thereby our results are more general. The results we obtain can be applied directly to the Helmholtz equation with spatially varying wave number by taking $\rho \equiv 1$. We point out that (3) can be converted to a Helmholtz equation, if ρ is smooth enough, via the dependent variable transformation

$$w = (\rho)^{-\frac{1}{2}} u.$$

But then, spatial rate restrictions must be placed on the variation of ρ, in order to prove uniqueness results in the resulting Helmholtz equation.

For use in the sequel, we modify the jump conditions (6) to read

$$\left[\gamma \frac{\partial u}{\partial \nu}\right] = 0, \qquad [\mu u] = 0, \tag{7}$$

where $\mu(\mathbf{x}), \gamma(\mathbf{x}) \in \mathbb{C}$ can be spatially varying along the interface. This is a slightly more general jump condition than is necessary for equation (3), wherein it suffices to take $\mu = 1$. When this is the case, it is not possible by renormalization of (3) to reduce one of the jump conditions to have either μ or γ identically unity (as carried out in (**2**) for the piecewise homogeneous problem). We specifically exclude a non-penetrable boundary condition, of Dirichlet or Neumann type, from consideration here, although our results can be modified to handle these.

In section 4 the jump conditions appear in the form

$$u_{j,k,l} \frac{\partial \bar{u}_{j,k,l}}{\partial \nu} = \beta_{i,j,k,l} u_{i,j,k,l} \frac{\partial \bar{u}_{i,j,k,l}}{\partial \nu}, \tag{8}$$

where

$$\beta_{i,j,k,l} = \frac{\mu_{i,j,k,l}\bar{\gamma}_{i,j,k,l}}{\mu_{j,k,l}\bar{\gamma}_{j,k,l}} \tag{9}$$

for all points $\mathbf{x} \in S_{i,j,k,l}$. Although equation (8) is a weaker statement than (7), the latter are required for the interior result, so we shall assume that (7) holds in the sequel. Here we are using the aforementioned scatterer-embedding notation, so for example $\bar{u}_{i,j,k,l}$ denotes the complex conjugate of the field at scatterer level 3, in the ith scatterer embedded in the region j, k, l with the associated jump parameters as given in (7). With the use of the multi-index notation, m can take the value of i, j, k, l and by defining m_{-1} to mean the multi-index m with the first tuple removed, that is in this case $m_{-1} = j, k, l$, equation (8) can be much more simply written as

$$u_{m_{-1}} \frac{\partial \bar{u}_{m_{-1}}}{\partial v} = \beta_m u_m \frac{\partial \bar{u}_m}{\partial v}. \tag{10}$$

As a further example of the use of the multi-index notation, with m denoting the same tuple as previously used, $D_{m_{-1}}$ denotes the embedding region $D_{j,k,l}$ at level 2 and $D_{m_{-3}}$ denotes D_0 or the overall embedding medium. In a similar manner we define m_{+1} to mean multi-index m with an extra first tuple added, that is, it denotes a level one deeper. This notation is used extensively in the next two sections. Note that each D_m is a member of the cover $\{\Omega_j\}$; we have just introduced a hierarchical labelling. The boundary S_m is the exterior boundary of the open set D_m.

3. The interior problem for piecewise analytic inhomogeneity

In this section we examine what can be said about the field inside S_1 when $u = 0$, $\partial u/\partial v = 0$ on S_1, that is, when the normal component of velocity and the pressure vanish on S_1.

THEOREM 1. *If the velocity potential satisfying* (3) *and its normal derivative vanish on the closed bounded surface S_1 of a bonded body D_1, in which the material parameters are piecewise analytic, then the velocity potential is identically zero in D_1.*

Proof. The proof proceeds by use of the Cauchy–Kovalevskaia theorem, which asserts the existence of a unique analytic solution to the non-characteristic Cauchy problem, for an analytic partial differential equation in the neighbourhood of the smooth surface, on which the analytic Cauchy data are prescribed. Also necessary is the Holmgren extension to this theorem, which assures us that when the equation is linear this is the *only* solution (even among non-analytic solutions).

By virtue of these results, specification of zero Cauchy data on S_1 means that only the solution $u \equiv 0$ holds throughout any of the Ω_j such that $\bar{\Omega}_j \cap S_1 \neq \varnothing$, and the solution u belongs to $C^{\omega}(\Omega_j)$. The zero solution can then be continued, via the jump conditions, to prescribe zero Cauchy data on the Ω_j which border the aforementioned ones. Proceeding in this manner D_1 is covered. Even though the $\partial\Omega_j$ may not be smooth the requirement that they be regular ensures that we can use analytic continuation to ensure that $u \equiv 0$ is the only solution.

We remark, again, that it is not required that the cover $\{\Omega_j\}$ be nested as in the previous proofs for piecewise homogeneous material domains (**4**; **5**, p. 67) which utilize integral representations. This is an important extension, as many practical problems do not fit into the nested class. However the cost of this extension is the requirement of piecewise analyticity of the material parameters.

4. The exterior problem for piecewise analytic inhomogeneity

We now pose the problem of uniqueness when the region D_1 is embedded in the region D_0 by bonding the material across S_1. Now, instead of specifying u and its normal derivative on S_1, the behaviour at infinity through the radiation condition (4) is prescribed. It will be supposed that R is the radius of the circle which circumscribes the surface S_1 from an origin assumed for convenience to lie within D_1. Then, as discussed in section 1 we can always assume for the scattering problem under consideration that k and ρ have constant values outside D_1. We call these constant values k_0 and ρ_0 respectively, where $k_0 \in \mathbb{C}$, $\rho_0 \in \mathbb{R}$.

We cannot expect the solutions of (3) to (7) to be unique for all values of the material parameters and associated jump conditions—see (**2**) for an example of non-uniqueness in the piecewise homogeneous case. We must, therefore, impose some restrictions to obtain a uniqueness theorem; these are as follows.

1. $\rho_m \in \mathbb{R}$, $\rho_m > 0$ for each m.
2. $k_m(\mathbf{x}) = |k_m(\mathbf{x})| \exp(i\phi_m)$, where $\phi_m \in (-\pi, 0]$ is constant in each D_m for each m. This assumption means that the wave number has a constant phase angle in each region, and is made to ensure that assumption 4 is possible.
3. If $\mu, \gamma \in \mathbb{C}$ and we define the ratio

$$\beta_m \frac{\rho_m}{\rho_{m-1}} = \frac{\mu_m \bar{\gamma}_m \rho_m}{\mu_{m-1} \bar{\gamma}_{m-1} \rho_{m-1}},$$

we then require that ξ_m, defined by

$$\xi_m = \beta_m \frac{\rho_m}{\rho_{m-1}} \tag{11}$$

on each interface S_m, is a complex constant. We note that this is *not* the same as stating that the material parameters are constant on an interface. This assumption is necessary for the technical reason of being able to take the ratio ξ outside the integral sign in Lemma 2.

4. We define a spatially varying function $\chi_m(\mathbf{x})$ as

$$\chi_m(\mathbf{x}) = \frac{\bar{k}_m^2(\mathbf{x})}{\bar{k}_0^2} \prod_{j=0}^{\mathscr{L}-1} \xi_{m_{-j}}, \qquad x \in D_m, \quad \mathscr{L} \geqslant 1, \tag{12}$$

where m is a multi-index at level $\mathscr{L}$, and the multi-index m_{-j} moves up the tree from the interface S_m to the tree-top interface $S_{m_{-\mathscr{L}+1}} = S_1$. Given assumption 3, the requirement that $\chi_m \in \mathbb{R}$ is seen to be possible only with assumption 2.

5. $\chi_m \geqslant 0$ (respectively <0) if $\operatorname{Re}(k_0)$, $\operatorname{Re}(k_m) \geqslant 0$ (respectively <0).

We should note that the results of (**2**) will be included as a special case of our result here simply by allowing only a one-level scattering body and restricting k and ρ not to be spatially varying within D_1.

Before proving our main theorem we need the following four lemmas.

LEMMA 1. *Let the field u satisfy* (3) *in the region $D_{0R} = D_0 \cap \{|\mathbf{x}| < R\}$ with $k = k_0$. Then*

$$\operatorname{Im}\left(\frac{1}{\bar{k}_0^2}\int_{S_1} \rho^{-1} u \frac{\partial \bar{u}}{\partial \nu} ds\right) = \operatorname{Im}\left(\frac{1}{\bar{k}_0^2}\int_{S_R} \rho^{-1} u \frac{\partial \bar{u}}{\partial \nu} ds\right) - \operatorname{Im}\left(\frac{1}{\bar{k}_0^2}\int_{D_{0R}} \rho^{-1} |\nabla u|^2 d\mathbf{x}\right). \tag{13}$$

Proof. We first note that the subscript zero has been left off all occurrences of the symbols u and ρ in (13) for simplicity. Apply the divergence theorem to $\nabla . (u\rho^{-1}\nabla\bar{u})$ on the region D_{0R} to yield

$$\int_{D_{0R}} u\nabla . (\rho^{-1}\nabla\bar{u})\, d\mathbf{x} = -\int_{S_1} \rho^{-1} u \frac{\partial \bar{u}}{\partial \nu} ds + \int_{S_R} \rho^{-1} u \frac{\partial \bar{u}}{\partial \nu} ds - \int_{D_{0R}} \rho^{-1} |\nabla u|^2 d\mathbf{x}.$$

The use of (3), and dividing through by $\bar{k}_0^2$ yields the desired result.

LEMMA 2. *Let the field u satisfy equation* (3) *in a region D_m, with outer surface S_m, having regions $D_{m_{+1}}$ with surfaces $S_{m_{+1}}$ embedded in it, together with jump conditions*

$$u_{m_{-1}} \frac{\partial \bar{u}_{m_{-1}}}{\partial \nu} = \beta_m u_m \frac{\partial \bar{u}_m}{\partial \nu} \tag{14}$$

on the surface S_m. Then

$$\operatorname{Im}\left(\frac{1}{\bar{k}_0^2}\int_{S_m} \rho_{m_{-1}}^{-1} u_{m_{-1}} \frac{\partial \bar{u}_{m_{-1}}}{\partial \nu} ds \prod_{j=0}^{\mathscr{L}-1} \xi_{m_{-j-1}}\right) = \operatorname{Im}\left(\frac{\chi_m(\mathbf{x})}{\bar{k}_m^2(\mathbf{x})}\int_{D_m} \rho_m^{-1} |\nabla u_m|^2 d\mathbf{x}\right) + $$
$$+ \operatorname{Im}\left(\frac{1}{\bar{k}_0^2}\int_{S_{m_{+1}}} \rho_m^{-1} u_m \frac{\partial \bar{u}_m}{\partial \nu} ds \prod_{j=0}^{\mathscr{L}-1} \xi_{m_{-j}}\right), \qquad \mathscr{L} \geqslant 1. \tag{15}$$

REMARK. Here, $\xi_{m_{-1}}$ with m corresponding to level 1 is replaced by unity. We note the multi-index m_{-j} traces the jump parameter $\xi_{m_{-j}}$ up the tree from surface S_m to S_1 as j varies from 0 to $\mathscr{L}-1$. We should also point out that $\chi_m(\mathbf{x})/\bar{k}_m^2(\mathbf{x})$ is a complex constant, as is seen from examination of assumption 4.

Proof. With assumption 3 and (14) we can write

$$\int_{S_m} \rho_{m_{-1}}^{-1} u_{m_{-1}} \frac{\partial \bar{u}_{m_{-1}}}{\partial \nu} ds = \xi^m \int_{S_m} \rho_m^{-1} u_m \frac{\partial \bar{u}_m}{\partial \nu} ds$$

$$= \xi_m \left[\int_{D_m} \rho_m^{-1} |\nabla u_m|^2 \, d\mathbf{x} - \int_{D_m} \bar{k}_m^2(\mathbf{x}) \, |u_m|^2 \, d\mathbf{x} + \int_{S_{m+1}} \rho_m^{-1} u_m \frac{\partial \bar{u}_m}{\partial \nu} ds \right],$$

where the divergence theorem has been used on the region D_m. If both sides are the multiplied by the complex constant $(\bar{k}_0)^{-2} \prod_{j=0}^{\mathscr{L}-2} \xi_{m_{-j-1}}$ and the imaginary part is taken, then by

$$\xi_m \prod_{j=0}^{\mathscr{L}-2} \xi_{m_{-j-1}} = \prod_{j=0}^{\mathscr{L}-1} \xi_{m_{-j}}$$

the lemma is proved.

We should note that if any of the surfaces S_m have edges the divergence theorem cannot be directly applied to the region. Then the standard method of surrounding the edge by a small sphere with surface S_ε is followed. In the limit as this surface contracts to the edge, the edge condition (5) ensures that the contribution from this edge is zero, so that the lemma result follows.

The following lemma is just an extension of the radiation condition and Rellich's lemma (**11**, p. 86).

LEMMA 3. *If u satisfies the homogeneous equation* (3), (4) *and*

$$\operatorname{Im}\left(k_0 \int_{S_R} u \frac{\partial \bar{u}}{\partial \nu} ds\right) \leq 0, \tag{16}$$

then $u \equiv 0$ in D_0.

Proof. From the radiation condition (4), with $k = k_0$ it follows that

$$\int_{S_R} \left[\left| \frac{\partial u}{\partial \nu} \right|^2 + |k_0|^2 \, |u|^2 - 2 \operatorname{Im}\left(k_0 u \frac{\partial \bar{u}}{\partial \nu}\right) \right] ds = o(1).$$

Since all terms on the left-hand side are non-negative it follows that

$$\int_{S_R} |u|^2 \, ds = o(1).$$

Now equation (3) reduces to the Helmholtz equation in the region $\mathbb{R}^3 \backslash D_{0R}$. Therefore by Rellich's lemma and analytic continuation, $u \equiv 0$ in D_0.

LEMMA 4. *If u satisfies the homogeneous equation* (3) *and the radiation condition* (4) *for* $\operatorname{Im}(k) < 0$, *then*

$$\int_{S_R} u \frac{\partial \bar{u}}{\partial \nu} ds = o(R^{-p})$$

as $R \to \infty$ for all p, that is, the left-hand side approaches zero exponentially as $R \to \infty$.

Proof. Equation (3) becomes the Helmholtz equation for $|x| > R$, and u is continuous, so we may utilize Muller's result (**11**, Lemma 35) which asserts that there exists a continuous function $F(\mathbf{x}/r)$, where $r = |\mathbf{x}|$, such that

$$u(\mathbf{x}) = \frac{e^{-ikr}}{r} F\left(\frac{\mathbf{x}}{r}\right) + o\left(\frac{1}{r}\right).$$

The result now follows readily on noting $\operatorname{Im}(k) < 0$.

We can now prove the theorem.

THEOREM 2. *With the previous assumptions* 1 *to* 5, *the only solution of the exterior problem is the trivial solution,* $u \equiv 0$.

Proof. At level 1, that is, $\mathscr{L} = 1$ (see Fig. 1), Lemma 2 gives

$$\operatorname{Im}\left(\frac{1}{\bar{k}_0^2} \int_{S_m} \rho_0^{-1} u_0 \frac{\partial \bar{u}_0}{\partial \nu} ds\right) = \operatorname{Im}\left(\frac{\chi_m}{\bar{k}_m^2} \int_{D_m} \rho_m^{-1} |\nabla u_m|^2 d\mathbf{x}\right) + \\ + \operatorname{Im}\left(\frac{\xi_m}{\bar{k}_0^2} \int_{S_{m+1}} \rho_m^{-1} u_m \frac{\partial \bar{u}_m}{\partial \nu} ds\right), \quad (17)$$

where $m = i$, 0, and where i is at least 1 but no greater than the number of first-level scatterers. Then considering Lemma 1, equation (17) can be used to replace the left-hand side of (13) if the surfaces S_{m+1} are of measure zero. If there *are* further levels, that is, nested domains, then Lemma 2 is used recursively to keep replacing the second term on the right-hand side of (17) until the tree is exhausted; that is, down all $\mathcal{N}$ branches to the bottom levels $\mathscr{L}_b$. Then the first term in the left-hand side of (13) is replaced by the right-hand side of (17), which yields after rearrangement

$$\operatorname{Im}\left(\frac{1}{\bar{k}_0^2} \int_{S_R} \rho_0^{-1} u_0 \frac{\partial \bar{u}_0}{\partial \nu} ds\right) = \operatorname{Im}(k_0^2) \frac{1}{|k_0|^4} \int_{D_{0R}} \rho_0^{-1} |\nabla u_0|^2 d\mathbf{x} + \\ + \sum_{b=1}^{\mathcal{N}} \overline{\sum_{m=0}^{\mathscr{L}_b}} \operatorname{Im}(k_m^2(\mathbf{x})) \frac{\chi_m(\mathbf{x})}{|k_m(\mathbf{x})|^4} \int_{D_m} \rho_m^{-1} |\nabla u_m|^2 d\mathbf{x}, \quad (18)$$

where the overbar on the second summation sign signifies that no term is to be duplicated in the summation.

Since the integrals on the right-hand side are clearly real we only need consider the complex nature of k_0^2, k_m^2, and the sign of the real number χ_m. Specifically we distinguish between the two cases in which $\mathrm{Im}\,(k_0) = 0$ and $\mathrm{Im}\,(k_0) < 0$.

1. $\mathrm{Im}\,(k_0) = 0$. In this case, from assumption 2 $\mathrm{Re}\,(k_0) > 0$ and the sign of the second term on the right-hand side of (18) is determined by $\chi_m \,\mathrm{Re}\,(k_m)\,\mathrm{Im}\,(k_m)$. We immediately see that because of assumption 5 and $\mathrm{Im}\,(k_m) \leq 0$ (part of assumption 2),

$$\mathrm{Im}\left(k_0 \int_{S_R} u_0 \frac{\partial \bar{u}_0}{\partial \nu}\, ds\right) \leq 0.$$

Hence, by Lemma 3, $u_0 \equiv 0$ in D_0. It follows that the region inside S_1 corresponds to an interior problem with zero Cauchy data; by Theorem 1 the result follows.

2. $\mathrm{Im}\,(k_0) \leq 0$. Now we work in the opposite direction to 1. Our assumptions, in particular 5, assure us that both terms on the right-hand side of (18) have the same sign, but Lemma 4 indicates that the left-hand side tends to zero as $R \to \infty$. It follows $u_0 \equiv 0$ in D_0 and the proof follows as in the later part of case 1.

When all the wave numbers involved in the scatterer are real the assumptions necessary to prove Theorem 2 can be considerably reduced; this is indicated by the following.

COROLLARY 1. *When the velocity potential satisfies equations* (3) *to* (7), *and* $\rho, c, k, \mu, \gamma \in \mathbb{R}$, $\sigma \equiv 0$ *throughout* $\mathbb{R}^3$ *and* ξ_m *is a constant, with the material parameters in the bonded body* D_1 *piecewise analytic, then the velocity potential is identically zero in* $\mathbb{R}^3$.

5. Conclusions

We have demonstrated a uniqueness theorem which has practical application for computational solution of the reduced wave equation. In a later paper we shall illustrate its usefulness in providing regularity results, for wave-propagation problems. The techniques illustrated here can be readily applied to consider the elastodynamic problem.

We note that assumption 2, which is that the wave number in each region should have a constant-loss phase angle, is a sufficient condition for our proof. We conjecture that it is *not* a necessary one.

Corollary 1 is the important result of this paper as it asserts that the wave-scattering problem, with ρ, k being spatially varying piecewise real-analytic functions, is unique.

Acknowledgement

The author thanks the referees for suggestions which have been incorporated in this revised paper.

REFERENCES

1. G. Kristensson, *SIAM J. Math. Anal.* **11** (1980) 1104–1117.
2. R. Kress and G. F. Roach, *J. math. Phys.*, **19** (1978) 1433–1437.
3. P. Werner, *Arch. ration. Mech. Analysis* **6** (1960) 231–260.
4. D. S. Jones, *Q. Jl Mech. appl. Math.* **37** (1984) 121–142.
5. V. D. Kupradze, *Progress in Solid Mechanics,* Vol 3 (eds I. N. Sneddon and R. Hill, North-Holland, Amsterdam 1963).
6. D. S. Jones, *Prog. Aerospace Sci.* **17** (1977) 149–229.
7. J. Meixner, *IEEE Trans. Antennas Propag.* **20** (1972) 442–446.
8. F. Treves, *Basic Linear Partial Differential Equations* (Academic Press, New York 1975).
9. O. D. Kellog, *Foundations of the Potential Theory* (Dover, New York 1953).
10. P. R. Garabedian, *Partial Differential Equations* (Wiley, New York 1964).
11. C. Müller, *Foundations of the Mathematical Theory of Electromagnetic Waves* (Springer, New York 1969).