ON THE HEAVING MOTION OF A CIRCULAR CYLINDER ON THE SURFACE OF A FLUID

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SUMMARY

We consider the motion of a fluid of infinite depth which arises when a horizontal cylinder of circular cross-section oscillates with small amplitude about a mean position, in which the axis of the cylinder is assumed to lie in the mean surface. It is further assumed that the resulting motion is two-dimensional; this assumption is justified when the cylinder is long compared with a wave-length, or when the fluid is contained between vertical walls at right angles to the axis of the cylinder. Expressions are obtained for the wave motion at a distance from the cylinder, and for the increase in the inertia of the cylinder due to the presence of the fluid.

Introduction

A CYLINDER of circular section is immersed in a fluid with its axis in the free surface. If the cylinder is given a forced simple harmonic motion of small amplitude about its initial position, a surface disturbance is set up in which waves travel away from the cylinder, and a stationary state is rapidly attained. When the cylinder is very long or when the fluid is contained between vertical walls at right angles to the axis of the cylinder, the velocity component parallel to the axis of the cylinder vanishes and the motion is two-dimensional. It is well known that at a distance of a few wave-lengths from the cylinder the motion on each side is described by a single regular wave-train travelling away from the cylinder, and that the wave-amplitude is proportional to the amplitude of oscillation of the cylinder, provided that the latter is sufficiently small compared with the radius of the cylinder, and that the wave-length is not much smaller than the diameter of the cylinder.

In this paper it will be shown how the fluid motion can be calculated when the cylinder is oscillating vertically. The foregoing assumptions will be made, and viscosity and surface tension will be neglected. Then a velocity potential and a conjugate stream function exist, and it will be assumed that terms involving their squares may be neglected. From the potential or the stream function it is easy to deduce the wave-amplitude at a distance from the cylinder and the added mass of the cylinder due to the fluid motion.

Formulation of the problem

Take the origin of rectangular Cartesian coordinates at the mean position of the axis of the cylinder. The x-axis is horizontal and perpendicular to the axis of the cylinder, the y-axis is vertical, y increasing with depth. Define polar coordinates by the equations

$$x = r\sin\theta, \qquad y = r\cos\theta.$$

Since the motion is symmetrical about the y-axis, it is sufficient to consider the quadrant $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The velocity potential ϕ satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0, \tag{A}$$

the stream function ψ satisfies

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = 0. \tag{B}$$

On the free surface the pressure is constant, whence to the first order

$$K\phi+\frac{\partial\phi}{\partial y} = 0, \qquad \theta = \tfrac{1}{2}\pi, \qquad r > a, \tag{C}$$

where $K = \sigma^2/g$ and $2\pi/\sigma$ is the period (cf. ref. 1). Also, by symmetry,

$$\partial\phi/\partial\theta = 0, \qquad \theta = 0. \tag{D}$$

It remains to express the boundary condition on the cylinder. This is that the velocity component normal to the boundary just inside the fluid is equal to the corresponding component of the velocity of the cylinder. Suppose that the ordinate of the axis of the cylinder is

$$y = l\cos(\sigma t+\epsilon).$$

Then at $(a\sin\alpha,\, a\cos\alpha+l\cos(\sigma t+\epsilon))$, the normal velocity is

$$-\frac{1}{a}\frac{\partial\psi}{\partial\alpha} = \frac{dy}{dt}\cos\alpha,$$

$$\psi = -a\frac{dy}{dt}\sin\alpha,$$

and to the first order this condition holds at $(a\sin\alpha,\, a\cos\alpha)$, whence

$$\psi = l\sigma a\sin(\sigma t+\epsilon)\sin\theta \quad \text{on} \quad r = a. \tag{E}$$

It is required to find a velocity potential and a stream function satisfying the boundary conditions (C), (D), and (E), and representing a diverging wave-train at infinity. To this end a series of non-orthogonal harmonic polynomials will be constructed satisfying (C) and (D); these will be superposed to satisfy (E) for values of Ka less than $3\pi/2$ by a numerical process. It is shown that when Ka is less than 1·5, this is permissible, and it will be supposed that the process is permitted in the wider range.

Construction of polynomial set

It is easily verified that the set of stream functions

$$a^{2m}\left[\frac{\sin 2m\theta}{r^{2m}}+\frac{K}{2m-1}\,\frac{\sin(2m-1)\theta}{r^{2m-1}}\right]\begin{matrix}\cos\\ \sin\end{matrix}\,\sigma t \quad (m = 1,2,3,...)$$

is such that the conjugate velocity potentials satisfy (C) and (D), while on $r = a$ it takes the values

$$\sin 2m\theta+\frac{Ka}{2m-1}\sin(2m-1)\theta.$$

It is clear on physical grounds that this set is not closed on $r = a$, since the sum of functions of the set tends to zero as r tends to infinity, whereas in fact the stream function for large r must represent a diverging wave-train. It is therefore necessary to add a function satisfying (C) and (D) and representing such a train of waves, e.g. the function describing a source at the origin (cf. ref. 3)

$$\frac{gb}{\pi\sigma}[\Psi_c(Kr;\theta)\cos\sigma t+\Psi_s(Kr;\theta)\sin\sigma t],$$

where b is the amplitude at infinity and

$$\Psi_c(Kr;\theta) = \pi e^{-Kr\cos\theta}\sin(Kr\sin\theta),$$

$$\Psi_s(Kr;\theta) = \int\limits_0^\infty \frac{e^{-kr\sin\theta}}{K^2+k^2}\{k\sin(kr\cos\theta)+K\cos(kr\cos\theta)\}\,dk-$$

$$-\pi e^{-Kr\cos\theta}\cos(Kr\sin\theta).$$

The functions of the closed set must be superposed so as to satisfy (E), and since there are no singularities on the cylinder, ψ must be continuous on $r = a$ when $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

Suppose, then, that the stream function ψ is expressed in the form

$$\frac{\pi\sigma\psi}{gb} = \Psi_c(Kr;\theta)\cos\sigma t+\Psi_s(Kr;\theta)\sin\sigma t+$$

$$+\cos\sigma t\sum_1^\infty p_{2m}(Ka)a^{2m}\left[\frac{\sin 2m\theta}{r^{2m}}+\frac{K}{2m-1}\,\frac{\sin(2m-1)\theta}{r^{2m-1}}\right]+$$

$$+\sin\sigma t\sum_1^\infty q_{2m}(Ka)a^{2m}\left[\frac{\sin 2m\theta}{r^{2m}}+\frac{K}{2m-1}\,\frac{\sin(2m-1)\theta}{r^{2m-1}}\right],$$

where the coefficients $p_{2m}(Ka)$, $q_{2m}(Ka)$ are assumed to be of order $1/m^2$.

This series converges uniformly outside and on $r = a$. On $r = a$ the function ψ is a multiple of $\sin\theta$, by condition (E), i.e. putting $r = a$,

$$\Psi_c(Ka;\theta)\cos\sigma t + \Psi_s(Ka;\theta)\sin\sigma t +$$
$$+\cos\sigma t\sum_1^\infty p_{2m}(Ka)\left[\sin 2m\theta + \frac{Ka}{2m-1}\sin(2m-1)\theta\right] +$$
$$+\sin\sigma t\sum_1^\infty q_{2m}(Ka)\left[\sin 2m\theta + \frac{Ka}{2m-1}\sin(2m-1)\theta\right]$$
$$= C(Ka;t)\sin\theta.$$

To determine $C(Ka;t)$, put $\theta = \frac{1}{2}\pi$ (say); then

$$C(Ka;t) = \Psi_c(Ka;\tfrac{1}{2}\pi)\cos\sigma t + \Psi_s(Ka;\tfrac{1}{2}\pi)\sin\sigma t +$$
$$+\cos\sigma t\sum_1^\infty p_{2m}(Ka)\frac{Ka}{2m-1}\sin\tfrac{1}{2}(2m-1)\pi$$
$$+\sin\sigma t\sum_1^\infty q_{2m}(Ka)\frac{Ka}{2m-1}\sin\tfrac{1}{2}(2m-1)\pi,$$

whence it follows that $p_{2m}(Ka)$, $q_{2m}(Ka)$ are the coefficients in the expansions

$$\Psi_c(Ka;\theta) - \Psi_c(Ka;\tfrac{1}{2}\pi)\sin\theta = \sum_1^\infty p_{2m}(Ka)f_{2m}(Ka;\theta),$$
$$\Psi_s(Ka;\theta) - \Psi_s(Ka;\tfrac{1}{2}\pi)\sin\theta = \sum_1^\infty q_{2m}(Ka)f_{2m}(Ka;\theta),$$

where

$$f_{2m}(Ka;\theta) = -\left[\sin 2m\theta + \frac{Ka}{2m-1}\{\sin(2m-1)\theta - \sin\theta\sin\tfrac{1}{2}(2m-1)\pi\}\right].$$

Write

$$\Psi_c(Ka;\tfrac{1}{2}\pi) + \sum_1^\infty \frac{(-1)^{m-1}Ka}{2m-1}p_{2m}(Ka) = A(Ka),$$
$$\Psi_s(Ka;\tfrac{1}{2}\pi) + \sum_1^\infty \frac{(-1)^{m-1}Ka}{2m-1}q_{2m}(Ka) = B(Ka).$$

Then the stream function on the cylinder is given by

$$\psi = \frac{gb}{\pi\sigma}[A(Ka)\cos\sigma t + B(Ka)\sin\sigma t]\sin\theta,$$

whence by comparison with condition (E) the ratio

$$\frac{\text{wave-amplitude}}{\text{amplitude of forced oscillation}}$$

is

$$\frac{\pi Ka}{\sqrt{(A^2+B^2)}}.$$

Calculation of the coefficients $A(Ka)$ and $B(Ka)$

The coefficients p_{2m}, q_{2m} are the roots of an infinite number of equations in an infinite number of unknowns. For purposes of calculation, the system of equations was replaced by a system involving only a finite number of the polynomials $f_{2m}(Ka;\theta)$. The functions

$$\Psi_c(Ka;\theta)-\Psi_c(Ka;\tfrac{1}{2}\pi)\sin\theta,$$
$$\Psi_s(Ka;\theta)-\Psi_s(Ka;\tfrac{1}{2}\pi)\sin\theta$$

were evaluated at $= 0°(10°)90°$ by quadrature, with

$$Ka/\pi = \tfrac{1}{8},\ \tfrac{1}{4},\ \tfrac{1}{2},\ \tfrac{5}{8},\ \tfrac{3}{4},\ 1,\ \tfrac{5}{4},\ \tfrac{3}{2}$$

and the polynomials $f_2(Ka;\theta),\ldots,f_{2N}(Ka;\theta)$ were fitted at these values by least squares, the corresponding coefficients being

$$\begin{matrix} p_{2m} \\ q_{2m} \end{matrix}(Ka;N) \quad (m = 1,\ldots,N).$$

The least squares condition provides a set of N simultaneous linear equations for $p_{2m}(Ka;N)$ and similarly for $q_{2m}(Ka;N)$. The matrix of the equations is symmetrical and the terms on the principal diagonal are larger than any other terms in the same row (except for the first row). The system could therefore be solved conveniently by relaxation methods. Trial calculations were carried out and it was found that an expansion in terms of six polynomials was adequate, giving a close fit at the chosen values of Ka and θ. Table 1 shows various functions of Ka to three significant figures.

TABLE 1

Ka/π	$A(Ka)$	$B(Ka)$	$\sqrt{(A^2+B^2)}$	$\tan^{-1}(B/A)$	$\dfrac{\pi Ka}{\sqrt{(A^2+B^2)}}$	$m(Ka)$
0	0	−1·57	1·57	−90°	0	∞
1/6	1·75	−2·23	2·83	−52°	0·58	0·78
1/4	2·75	−2·06	3·44	−37°	0·72	0·73
1/2	5·64	+0·65	5·68	+7°	0·87	0·83
2/3	6·17	4·39	7·57	35°	0·87	0·91
3/4	5·55	6·62	8·64	50°	0·86	0·94
1	−0·88	12·33	12·4	94°	0·80	1·01
5/4	−12·6	11·2	16·9	138°	0·73	1·06
3/2	−22·0	−1·17	22·0	183°	0·67	1·09

These calculations are based on six polynomials. The values of $\sqrt{(A^2+B^2)}$ and $\tan^{-1}(B/A)$ are given for convenience in interpolation, each of these functions being monotone increasing in the range; in fact the angle varies nearly linearly except near $Ka = 0$; when $Ka = 0$

$$\frac{d}{d(Ka)}\tan^{-1}(B/A) = 2 \text{ radians},$$

as can be shown by expanding in a power series in Ka.

Virtual mass due to the fluid

It is well known that when a long cylinder completely immersed in an ideal fluid is moving in any way perpendicular to its axis, the reaction of the fluid may be expressed in the form

$$M'\mathbf{f} \text{ per unit length}$$

where $M' = \pi\rho a^2$ is the mass of fluid displaced by unit length of the cylinder and $\mathbf{f}$ is the acceleration vector (ref. **1**, Art. 68). This expression shows that the motion is unaltered if the fluid is supposed to be removed and a mass M' per unit length is added to the cylinder. The mass M' is called the virtual mass due to the fluid.

When the cylinder is in the free surface, the reaction of the fluid is no longer in phase with the acceleration. There is a component in quadrature which does a positive amount of work in each cycle and is thus simply related to the wave-amplitude. The component in phase with the acceleration does no work: it consists partly of a hydrostatic force which dominates the motion for small values of Ka, and which is easily seen to be

$$\frac{2\rho a^2}{Ka}\frac{d^2y}{dt^2},$$

where y is the displacement of the cylinder. More interesting is the part due to the wave motion which will now be calculated. The velocity potential ϕ is easily derived from the stream function. It is given by

$$\frac{\pi\sigma\phi}{gb} = \Phi_c(Kr;\theta)\cos\sigma t+\Phi_s(Kr;\theta)\sin\sigma t+$$

$$+\cos\sigma t\sum_1^\infty p_{2m}(Ka)a^{2m}\left[\frac{\cos 2m\theta}{r^{2m}}+\frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right]+$$

$$+\sin\sigma t\sum_1^\infty q_{2m}(Ka)a^{2m}\left[\frac{\cos 2m\theta}{r^{2m}}+\frac{K}{2m-1}\frac{\cos(2m-1)\theta}{r^{2m-1}}\right]$$

from which the pressure $-\rho\,\partial\phi/\partial t$ can be derived.

Here $\Phi_c(Kr;\theta)$, $\Phi_s(Kr;\theta)$ are the harmonic conjugates of $\Psi_c'(Kr;\theta)$, $\Psi_s'(Kr;\theta)$:

$$\Phi_c(Kr;\theta) = \pi e^{-Kr\cos\theta}\cos(Kr\sin\theta),$$

$$\Phi_s(Kr;\theta) = -\int_0^\infty \frac{e^{-kr\sin\theta}}{K^2+k^2}\{k\cos(kr\cos\theta)-K\sin(kr\cos\theta)\}\,dk+$$

$$+\pi e^{-Kr\cos\theta}\sin(Kr\sin\theta).$$

It is seen that the pressure on the cylinder is of the form

$$M\cos\sigma t+N\sin\sigma t,$$

while the displacement of the cylinder is of the form

$$P\cos\sigma t+Q\sin\sigma t.$$

The component of the pressure in phase with the displacement is therefore

$$\frac{MP+NQ}{P^2+Q^2}(P\cos\sigma t+Q\sin\sigma t).$$

The force per unit length acting on the cylinder is

$$-\int\limits_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\rho a\frac{\partial\phi}{\partial t}\cos\theta\,d\theta\quad(r=a),\qquad\text{i.e.}\qquad\frac{2\rho abg}{\pi}(M_0\cos\sigma t-N_0\sin\sigma t),$$

where

$$M_0=\int\limits_0^{\frac{1}{2}\pi}\Phi_s(Ka;\theta)\cos\theta\,d\theta+\sum_1^\infty\frac{(-1)^{m-1}q_{2m}(Ka)}{4m^2-1}+\tfrac{1}{4}\pi Kaq_2(Ka),$$

$$N_0=\int\limits_0^{\frac{1}{2}\pi}\Phi_c(Ka;\theta)\cos\theta\,d\theta+\sum_1^\infty\frac{(-1)^{m-1}p_{2m}(Ka)}{4m^2-1}+\tfrac{1}{4}\pi Kap_2(Ka).$$

The displacement is

$$\frac{b}{\pi Ka}(A\sin\sigma t-B\cos\sigma t),$$

where $A(Ka)$, $B(Ka)$ are the functions shown in Table 1.

It follows that the force component in phase with the acceleration is

$$-\frac{2\rho abg}{\pi}\frac{M_0B+N_0A}{A^2+B^2}(A\sin\sigma t-B\cos\sigma t),$$

while the acceleration is

$$-\frac{b\sigma^2}{\pi Ka}(A\sin\sigma t-B\cos\sigma t).$$

The virtual mass is their ratio

$$2\rho a^2\frac{M_0B+N_0A}{A^2+B^2}.$$

The values of the non-dimensional quantity

$$m(Ka)=\frac{M_0B+N_0A}{A^2+B^2},$$

which may be described as an inertia coefficient, are given in the last column of Table 1.

It can be shown that as $Ka \to 0$,

$$m(Ka) - \log\frac{1}{Ka} \to \tfrac{3}{2} - 2\log 2 - \gamma = -0{\cdot}46,$$

where γ is Euler's constant. Hence the hydrodynamic inertia coefficient tends to infinity like $\log\{1/(Ka)\}$, while the hydrostatic coefficient tends to infinity like $\pi/(2Ka)$.

The work done by the cylinder in a cycle must be equal to the energy transmitted by the waves at a distance from the cylinder in the same time. In terms of the various parameters, $M_0 A - N_0 B = \frac{1}{2}\pi^2$. This relation may be used to check the computations.

Discussion of results

The computations show that the amplitude ratio

$$\frac{\pi Ka}{\sqrt{(A^2+B^2)}}$$

is small for small values of Ka/π and increases as Ka/π increases, until Ka/π is approximately 0·6. As Ka/π increases beyond this value, the amplitude ratio decreases steadily throughout the range of computation. This effect may be ascribed to interference between waves originating from different parts of the cylinder surface. Qualitatively similar behaviour is exhibited by cylinders of various other sections. As Ka/π tends to zero, the amplitude ratio tends to $2Ka$. It may be shown that this result, which is in agreement with Holstein's approximate theory (ref. 4), is still valid for cylinders of section other than circular, provided that $2a$ denotes the horizontal diameter in the surface (beam).

The inertia coefficient is of order 1 through the greater part of the range, except near $Ka = 0$, where it tends to infinity. On the other hand, the hydrostatic inertia coefficient tends to infinity even more rapidly, whence it may be concluded that in very slow heaving the deformation of the surface does not cause a significant change in the force on the cylinder. These results may be compared with the measurements made by Holstein (ref. 5) on a cylinder of rectangular section. Holstein measured the amplitude of the waves for various mean depths of immersion of the lower edge. When the depth is equal to about half the beam the amplitude should be comparable to that due to a circular cylinder. It is found that theory and experiment are consistent in the range of measurement ($0{\cdot}5 < Ka/\pi < 0{\cdot}8$). Holstein's experiments on virtual mass (ref. 6) are too rough for comparison with theory.

Note on the magnitude of $p_{2m}(Ka)$, $q_{2m}(Ka)$

It has been shown that the calculation of the wave-motion requires the expansion of a function $G(\theta)$, where

$$0 \leqslant \theta \leqslant \tfrac{1}{2}\pi, \qquad G(0) = G(\tfrac{1}{2}\pi) = 0,$$

in a series of non-orthogonal polynomials

$$f_{2m}(x;\theta) = -\left[\sin 2m\theta + \frac{x}{2m-1}\{\sin(2m-1)\theta - \sin\theta \sin\tfrac{1}{2}(2m-1)\pi\}\right]$$
$$(m = 1, 2, 3, \ldots),$$

where x is a numerical parameter which is usually less than 5.

Let the coefficients be denoted by $p_{2m}(x)$ so that

$$G(\theta) = \sum_{m=1}^{\infty} p_{2m}(x) f_{2m}(x;\theta).$$

This expansion must converge uniformly throughout the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$; it has been assumed in the text that functions $p_{2m}(x)$ exist such that

$$p_{2m}(x) = O(1/m^2) \text{ for fixed } x.$$

A proof that, in fact,

$$p_{2m}(x) = O(1/m^3) \tag{1}$$

will now be given when x is assumed small ($|x| < 1{\cdot}5$).

THEOREM. *Let $G(\theta)$ be defined in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and let its second differential coefficient be of bounded variation. Then if $|x| < 1{\cdot}5$ there exists an expansion*

$$G(\theta) = \sum_{m=1}^{\infty} p_{2m}(x) f_{2m}(x;\theta), \tag{2}$$

where

$$|p_{2m}(x)| < \frac{A(x)}{(2m-1)^3}. \tag{3}$$

We first prove the following

LEMMA 1. (*Limiting case $x = 0$.*)

Consider the expansion of $G(\theta)$ in terms of the orthogonal polynomials

$$f_{2m}(0;\theta) = -\sin 2m\theta.$$

Then

$$|p_{2m}(0)| < \frac{A(0)}{(2m-1)^3}.$$

Proof. By Fourier's theorem,

$$p_{2m}(0) = -\frac{4}{\pi}\int_0^{\frac{1}{2}\pi} G(\theta)\sin 2m\theta\, d\theta$$

$$= \frac{1}{\pi m^2}\int_0^{\frac{1}{2}\pi} G''(\theta)\sin 2m\theta\, d\theta, \text{ by integration by parts.}$$

Since $G''(\theta)$ is of bounded variation, the last integral is of order $1/m$ (ref. **6**, § 9.41), whence the lemma follows.

Suppose now that each coefficient $p_{2m}(x)$ can be expanded in a power-series in x

$$p_{2m}(x) = \sum_{n=0}^{\infty} a_{2m}^{(n)} x^n \quad (m = 1, 2, 3, \ldots). \tag{4}$$

Substitute in equation (2) and equate coefficients of x^n; then

$$\sum_{m=1}^{\infty} a_{2m}^{(0)} \sin 2m\theta = -G(\theta) \tag{5}$$

$$\sum_{m=1}^{\infty} a_{2m}^{(n)} \sin 2m\theta + \sum_{m=2}^{\infty} \frac{a_{2m}^{(n-1)}}{2m-1}\{\sin(2m-1)\theta - \sin\theta \sin\tfrac{1}{2}(2m-1)\pi\} = 0 \quad (n \geqslant 1). \tag{6}$$

Suppose further that the infinite series in equations (5) and (6) converge uniformly throughout the range. It is then permissible to multiply each equation by $\sin 2r\theta$ and to integrate term by term.

From equation (5),

$$\tfrac{1}{4}\pi a_{2r}^{(0)} = -\int_0^{\frac{1}{2}\pi} G(\theta)\sin 2r\theta \, d\theta. \tag{7}$$

From equation (6),

$$\tfrac{1}{4}\pi a_{2r}^{(n)} = \frac{2r}{4r^2-1} \sum_{m=2}^{\infty} \frac{a_{2m}^{(n-1)}}{2m-1}(-1)^{m+r} \frac{(2m-1)^2-1}{(2m-1)^2-(2r)^2}. \tag{8}$$

It must now be shown that equation (8) defines a double array $a_{2r}^{(n)}$; that the corresponding power-series defined by equation (4) are convergent for small x; that these power-series satisfy (3), and that

$$\sum_{m=1}^{\infty} f_{2m}(x;\theta) \sum_{n=0}^{\infty} a_{2m}^{(n)} x^n = G(\theta).$$

LEMMA 2. *To show that*

$$|a_{2r}^{(0)}| < \frac{A(0)}{(2r-1)^3}. \tag{9}$$

Proof. From equation (7) $a_{2r}^{(0)} = p_{2r}(0)$, so that the result follows from Lemma 1. It also follows that equation (5) holds.

LEMMA 3. *To show that*

$$|a_{2r}^{(n)}| < \frac{A_n}{(2r-1)^3}, \quad \text{where} \quad A_n = (\tfrac{3}{8})^n A(0). \tag{10}$$

Proof by induction on n

From Lemma 2, equation (10) holds when n is zero. Suppose that equation (10) holds for $n = 0, 1, 2,..., N-1$. From equation (8),

$$|\tfrac{1}{4}\pi a_{2r}^{(N)}| = \frac{2r}{4r^2-1}\left|\sum_{m=2}^{\infty}\frac{a_{2m}^{(N-1)}}{2m-1}(-1)^{m+r}\frac{(2m-1)^2-1}{(2m-1)^2-(2r)^2}\right|$$

$$< \frac{2r}{4r^2-1}\sum_{m=2}^{\infty}\frac{|a_{2m}^{(N-1)}|(2m-1)^2}{(2m-1)|(2m-1)^2-(2r)^2|}$$

$$< \frac{2rA_{N-1}}{4r^2-1}\sum_{m=2}^{\infty}\frac{1}{(2m-1)^2|(2m-1)^2-(2r)^2|}$$

$$= \frac{2rA_{N-1}}{4r^2-1}\left[\sum_{m=2}^{\infty}\frac{1}{(2m-1)^2\{(2r)^2-(2m-1)^2\}}+ \right.$$

$$\left. +2\sum_{m=r+1}^{\infty}\frac{1}{(2m-1)^2\{(2m-1)^2-(2r)^2\}}\right].$$

Now

$$\sum_{m=2}^{\infty}\frac{1}{(2m-1)^2\{(2r)^2-(2m-1)^2\}}$$

$$< \frac{1}{4r^2}\left[\sum_{2}^{\infty}\frac{1}{(2m-1)^2}+\sum_{m=1}^{\infty}\frac{1}{(2r)^2-(2m-1)^2}\right]$$

$$< \frac{1}{4r^2}(\tfrac{1}{4}+0) = \frac{1}{16r^2}.$$

The second series may be written

$$\sum_{m=r+1}^{\infty}\frac{1}{(2m-1)^2\{(2m-1)^2-(2r)^2\}} = \frac{1}{(2r+1)^2(4r+1)}+\frac{1}{(2r+3)^2(12r+9)}+$$

$$+\frac{1}{(2r+5)^2(20r+25)}+\sum_{m=r+4}^{\infty}\frac{1}{(2m-1)^2\{(2m-1)^2-(2r)^2\}}.$$

Now

$$\frac{r^2}{(2r+1)^2(4r+1)} \leqslant \frac{1}{45} \qquad (r = 1,2,3,...),$$

$$\frac{r^2}{(2r+3)^2(12r+9)} < \frac{1}{400} \qquad (r = 1,2,3,...),$$

$$\frac{r^2}{(2r+5)^2(20r+5)} < \frac{1}{1000} \qquad (r = 1,2,3,...),$$

$$\sum_{m=r+4}^{\infty} \frac{1}{(2m-1)^2\{(2m-1)^2-(2r)^2\}} < \int_{r+3}^{\infty} \frac{du}{(2u-1)^2\{(2u-1)^2-(2r)^2\}}$$

$$< \frac{1}{2(20r+25)} \int_{r+3}^{\infty} \frac{du}{(2u-1)^3}$$

$$= \frac{1}{4(20r+25)(2r+5)} < \frac{1}{160r^2},$$

$$\sum_{m=r+1}^{\infty} \frac{1}{(2m-1)^2\{(2m-1)^2-(2r)^2\}} < \frac{1}{r^2}\left[\frac{1}{45}+\frac{1}{400}+\frac{1}{1000}+\frac{1}{160}\right] < \frac{1}{31r^2}.$$

Hence

$$|\tfrac{1}{4}\pi a_{2r}^{(N)}| < \frac{2rA_{N-1}}{(4r^2-1)r^2}\left(\frac{1}{16}+\frac{2}{31}\right),$$

$$|a_{2r}^{(N)}| < \frac{\tfrac{2}{3}A_{N-1}}{(2r-1)^3} = \frac{A_N}{(2r-1)^3}.$$

This proves Lemma 3.

It follows immediately that

$$\sum_{m=2}^{\infty} \frac{a_{2m}^{(n-1)}}{2m-1}\{\sin(2m-1)\theta - \sin\theta \sin\tfrac{1}{2}(2m-1)\pi\} \tag{11}$$

is uniformly convergent. Its associated Fourier sine series is

$$-\sum_{m=1}^{\infty} a_{2m}^{(n)} \sin 2m\theta \quad (n \geqslant 1),$$

where $a_{2m}^{(n)}$ is defined by (8). It follows from Lemma 3 that the last series converges throughout the range, and so its sum is equal to (11); that is, equation (6) holds.

It also follows from Lemma 3 that

$$p_{2m}(x) = \sum_{n=0}^{\infty} a_{2m}^{(n)} x^n$$

is defined for $|x| < 1{\cdot}5$; again

$$|p_{2m}(x)| < \frac{A(0)}{(2m-1)^3}[1+\tfrac{2}{3}|x|+(\tfrac{2}{3})^2|x|^2+\ldots]$$

$$= \frac{A(x)}{(2m-1)^3} \quad (|x| < 1{\cdot}5), \tag{12}$$

and since $|f_{2m}(x;\theta)| < 1+2|x|$, $\sum_{m=1}^{\infty} p_{2m}(x) f_{2m}(x;\theta)$ converges.

It only remains to be shown that

$$\sum_{m=1}^{\infty} p_{2m}(x) f_{2m}(x;\theta) = G(\theta).$$

LEMMA 4. *To show that*

$$\lim_{N\to\infty}\sum_{m=1}^{\infty} f_{2m}(x;\theta)\Big(\sum_{n=0}^{N} a_{2m}^{(n)} x^n\Big) = \sum_{m=1}^{\infty} p_{2m}(x) f_{2m}(x;\theta). \tag{13}$$

For
$$\Big|f_{2m}(x;\theta)\sum_{n=0}^{N} a_{2m}^{(n)} x^n\Big| < \frac{A(x)}{(2m-1)^3}(1+2|x|);$$

since $\sum_{m=1}^{\infty}\frac{A(x)(1+2|x|)}{(2m-1)^3}$ converges, the series converges uniformly with respect to N, by Weierstrass's M-test (ref. **2**, § 3.34); hence equation (13) holds.

Proof of the theorem. Consider the expression

$$\sum_{m=1}^{\infty} f_{2m}(x;\theta)\sum_{n=0}^{N} a_{2m}^{(n)} x^n.$$

From (5) and (6)

$$\sum_{m=1}^{\infty} f_{2m}(x;\theta)\Big(\sum_{n=0}^{N} a_{2m}^{(n)} x^n\Big) - G(\theta)$$
$$= -x^{N+1}\sum_{m=2}^{\infty}\frac{a_{2m}^{(N)}}{2m-1}\{\sin(2m-1)\theta - \sin\tfrac{1}{2}(2m-1)\pi\sin\theta\},$$

but

$$\Big|x^{N+1}\sum_{m=2}^{\infty} a_{2m}^{(N)}\{\sin(2m-1)\theta - \sin\tfrac{1}{2}(2m-1)\pi\sin\theta\}\Big|$$
$$< 2|x|^{N+1}(\tfrac{2}{3})^N A(0)\sum_{m=1}^{\infty}\frac{1}{(2m-1)^4} \quad \text{(from Lemma 3)},$$

which tends to zero, as N tends to infinity provided that $|x| < 1{\cdot}5$; so that

$$\lim_{N\to\infty}\sum_{m=1}^{\infty} f_{2m}(x;\theta)\sum_{n=0}^{N} a_{2m}^{(n)} x^n = G(\theta).$$

From Lemma 4, the left-hand side is $\sum_{m=1}^{\infty} p_{2m}(x) f_{2m}(x;\theta)$; also from (12),

$$|p_{2m}(x)| < \frac{A(x)}{(2m-1)^3},$$

which proves the theorem.

Note. The foregoing proof applies only when $|x| < 1{\cdot}5$, but it has been assumed that the theorem is valid in a larger range including $|x| < 1{\cdot}5$.

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REFERENCES

1. H. LAMB, *Hydrodynamics*, 6th ed. (Cambridge, 1932).
2. E. T. WHITTAKER and G. N. WATSON, *Modern Analysis*, 4th ed. (Cambridge, 1927).
3. T. H. HAVELOCK, *Proc. Roy. Soc.* A **113** (1917), 522.
4. M. SCHULER, *Z.A.M.M.* **16** (1936), 65.
5. H. HOLSTEIN, *Ingenieur-Archiv*, **8** (1937), 103.
6. ——, *Werft, Reederei, Hafen*, **23** (1936), 385.