ON THE ROLLING MOTION OF CYLINDERS IN THE SURFACE OF A FLUID†

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SUMMARY

The methods of a previous paper (ref. 1) are extended to permit the evaluation of the fluid motion when a cylinder of arbitrary symmetrical section performs simple harmonic oscillations of small amplitude about a mean position with its axis of symmetry level with the surface. Special attention is paid to the slow motions, for which a method of successive approximation is developed. Applications are made to the virtual mass in slow heaving, to the determination of the roll axis, and to motion in a sea-way. Finally, the waves generated by the forced rolling motion are studied. It is shown that their amplitude, and thus also the damping decrement, depends critically on the details of the section. A section is calculated whose rolling is undamped to the first order. Experiments are needed.

1. Introduction

In a former paper (ref. 1) the motion of a fluid was considered when a long circular cylinder executes vertical heaving oscillations about a mean position in the surface. It will be shown that the method of that paper can be extended to discuss the oscillations of a long cylinder of any section, provided that the cylinder is symmetrical about a vertical plane and the principal axes of the section are of the same order of magnitude. The emphasis will be generally on skew-symmetrical (rolling) motions. In these the length of the waves produced by the rolling is usually much greater than the beam of the ship. Application of a method of successive approximation outlined in ref. 1 is therefore permissible.

2. Formulation of the problem

On the assumption that the motion is two-dimensional and simple harmonic of period $2\pi/\sigma$, a velocity potential ϕ and a stream-function ψ exist satisfying

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = 0, \tag{1}$$

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = 0, \tag{2}$$

where the origin of coordinates is at the mean position of the centre of the cylinder and y increases with depth, the mean free surface being $y = 0$, as in ref. 1. The stream function is prescribed on the cylinder, or, to a first

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approximation, on its mean position. To the same order, the condition for the free surface is

$$K\phi+\frac{\partial\phi}{\partial y}=0 \quad (y=0) \tag{3}$$

where

$$gK=\sigma^2. \tag{4}$$

Introduce dimensionless coordinates ξ, η defined by the isogonal transformation

$$x = c\cosh\xi\sin\eta + c\sum_{r=1}^{\infty} A_{2r+1}e^{-(2r+1)\xi}\sin(2r+1)\eta, \tag{5}$$

$$y = c\sinh\xi\cos\eta - c\sum_{r=1}^{\infty} A_{2r+1}e^{-(2r+1)\xi}\cos(2r+1)\eta. \tag{6}$$

The constants A_{2r+1} are chosen in such a way that the given cylinder becomes the curve $\xi=\xi_0$, and the region occupied by the fluid is mapped isogonally on $-\frac{1}{2}\pi \leqslant \eta \leqslant \frac{1}{2}\pi$, $\xi_0<\xi<\infty$. Equation (1) becomes

$$\frac{\partial^2\phi}{\partial\xi^2}+\frac{\partial^2\phi}{\partial\eta^2}=0 \tag{7}$$

and the surface condition becomes

$$-Kc\Big[\sinh\xi + \sum_{1}^{\infty}(-1)^{r-1}e^{-(2r+1)\xi}A_{2r+1}\Big]\phi \pm \frac{\partial\phi}{\partial\eta}=0 \quad (\eta=\pm\tfrac{1}{2}\pi,\ \xi>\xi_0). \tag{8}$$

3. Skew-symmetric motions

Consider in the first place, motions in which

$$\phi(\xi,\eta) = -\phi(\xi,-\eta), \tag{9}$$

for instance a rolling motion. It is sufficient to consider the region $0\leqslant\eta\leqslant\frac{1}{2}\pi$. Laplace's equation (7) and the surface condition are both satisfied by the skew-symmetric set

$$\begin{aligned}-\phi_{2n+1}(\xi,\eta) &= e^{-(2n+1)\xi}\sin(2n+1)\eta - \\ &\quad -Kc\Big[e^{-2n\xi}\frac{\sin 2n\eta}{4n}+e^{-(2n+2)\xi}\frac{\sin(2n+2)\eta}{4n+4} - \\ &\quad -\sum_{r=1}^{\infty}A_{2r+1}e^{-(2n+2r+2)\xi}\frac{\sin(2n+2r+2)\eta}{2n+2r+2}\Big] \\ &\qquad (n=1,2,3,\ldots).\end{aligned} \tag{10}$$

It will often be convenient to work with the conjugate stream-functions

$$\begin{aligned}\psi_{2n+1}(\xi,\eta) &= e^{-(2n+1)\xi}\cos(2n+1)\eta - \\ &\quad -Kc\Big[e^{-2n\xi}\frac{\cos 2n\eta}{4n}+e^{-(2n+2)\xi}\frac{\cos(2n+2)\eta}{4n+4} - \\ &\quad -\sum_{r=1}^{\infty}A_{2r+1}e^{-(2n+2r+2)\xi}\frac{\cos(2n+2r+2)\eta}{2n+2r+2}\Big] \\ &\qquad (n=1,2,3,\ldots).\end{aligned} \tag{11}$$

It is not to be expected that the general skew-symmetric motion can be expanded in terms of these sets, since at infinity the motion consists of regular wave trains travelling away from the origin, whereas each $\phi_{2n+1}(\xi,\eta)$ vanishes at infinity.

The set is completed by the addition of the potential Φ and stream-function Ψ describing a horizontal dipole at the origin (cf. ref. 2). In particular, the stream-function Ψ is given by

$$\Psi = \Psi_c(\xi,\eta)\cos\sigma t+\Psi_s(\xi,\eta)\sin\sigma t$$

$$= \operatorname{Im}\left[2Kcie^{iKz-i\sigma t}+\frac{2}{\pi}\sin\sigma t\int\limits_0^\infty\frac{kce^{-kz}\,dk}{k+iK}\right]$$

$$= \operatorname{Im}\left[2Kcie^{iKz-i\sigma t}+\frac{2}{\pi}\frac{c}{z}\sin\sigma t+\right.$$

$$\left.+\frac{2}{\pi}\sin\sigma t\left\{iKc(\gamma+\log iKz)\sum_{r=0}^{\infty}\frac{(iKz)^r}{r!}-iKc\sum_{r=1}^{\infty}\frac{(iKz)^r}{r!}\left(\frac{1}{1}+\dots+\frac{1}{r}\right)\right\}\right]. \tag{12}$$

4. Rolling oscillations of small amplitude about the origin

The process which is employed to determine the coefficients of $\psi_{2n+1}(\xi,\eta)$, $n = 1,2,3,\dots$ in the expansion of ψ is best illustrated by an example. Consider a cylinder whose equation is $\xi = \xi_0$ and suppose that it is rolling about the origin, the angular displacement at time t being

$$\theta = \theta_0\cos\sigma t, \tag{13}$$

where θ_0 is small.

The boundary condition is (ref. 3, Art. 72)

$$\psi = \frac{1}{2}\frac{d\theta}{dt}(x^2+y^2)+F(t), \tag{14}$$

on the cylinder, where $F(t)$ is a function of t only; for small θ_0 this condition may be replaced by

$$\psi = \frac{1}{2}\frac{d\theta}{dt}(x^2+y^2)+F(t) \quad \text{on} \quad \xi = \xi_0. \tag{15}$$

(For other skew-symmetrical motions x^2+y^2 is replaced by some other even function of η, to which all the following considerations may be applied.) Suppose that ψ is multiplied by a constant so as to make the coefficient of Ψ unity; A is a function of θ_0 and the period. Then,

$$A\psi = \Psi_c(\xi,\eta)\cos\sigma t+\Psi_s(\xi,\eta)\sin\sigma t-$$
$$-\sum_{r=1}^{\infty}p_{2r+1}(Kc,\xi_0)\psi_{2r+1}(\xi,\eta)\cos\sigma t-$$
$$-\sum_{r=1}^{\infty}q_{2r+1}(Kc,\xi_0)\psi_{2r+1}(\xi,\eta)\sin\sigma t, \tag{16}$$

and p_{2r+1}, q_{2r+1} must be determined in such a way that on $\xi = \xi_0$ the stream-function ψ differs from a multiple of x^2+y^2 by a function of t only, that is

$$\Psi_c(\xi_0,\eta)\cos\sigma t+\Psi_s(\xi_0,\eta)\sin\sigma t-$$
$$-\sum_1^\infty p_{2r+1}(Kc,\xi_0)\psi_{2r+1}(\xi_0,\eta)\cos\sigma t-$$
$$-\sum_1^\infty q_{2r+1}(Kc,\xi_0)\psi_{2r+1}(\xi_0,\eta)\sin\sigma t$$
$$= C(t)(x^2+y^2)_0+F(t). \tag{17}$$

There must be no source-singularities on the cylinder, i.e. ψ is continuous in the range $0 \leqslant \eta \leqslant \frac{1}{2}\pi$. It is therefore assumed that the series (16) converges uniformly in η on $\xi = \xi_0$.

Suppose that the beam of the cylinder is $2a$ and that the draught is b. Put $\eta = \frac{1}{2}\pi$ and subtract the new equation from (17), thus eliminating $F(t)$.

$$[\Psi_c(\xi_0,\eta)-\Psi_c(\xi_0,\tfrac{1}{2}\pi)]\cos\sigma t+[\Psi_s(\xi_0,\eta)-\Psi_s(\xi_0,\tfrac{1}{2}\pi)]\sin\sigma t-$$
$$-\sum_1^\infty p_{2r+1}[\psi_{2r+1}(\xi_0,\eta)-\psi_{2r+1}(\xi_0,\tfrac{1}{2}\pi)]\cos\sigma t-$$
$$-\sum_1^\infty q_{2r+1}[\psi_{2r+1}(\xi_0,\eta)-\psi_{2r+1}(\xi_0,\tfrac{1}{2}\pi)]\sin\sigma t$$
$$= C(t)(x^2+y^2-a^2)_0$$
$$= (p_1\cos\sigma t+q_1\sin\sigma t)\left(\frac{x^2+y^2-a^2}{b^2-a^2}\right)_0, \quad \text{say;} \tag{18}$$

that is

$$\Psi_c(\xi_0,\eta)-\Psi_c(\xi_0,\tfrac{1}{2}\pi)$$
$$= p_1\left(\frac{x^2+y^2-a^2}{b^2-a^2}\right)_0+\sum_1^\infty p_{2r+1}[\psi_{2r+1}(\xi_0,\eta)-\psi_{2r+1}(\xi_0,\tfrac{1}{2}\pi)], \tag{19}$$

$$\Psi_s(\xi_0,\eta)-\Psi_s(\xi_0,\tfrac{1}{2}\pi)$$
$$= q_1\left(\frac{x^2+y^2-a^2}{b^2-a^2}\right)_0+\sum_1^\infty q_{2r+1}[\psi_{2r+1}(\xi_0,\eta)-\psi_{2r+1}(\xi_0,\tfrac{1}{2}\pi)]. \tag{20}$$

These equations must be solved for the infinite set of unknowns p_{2r+1}, q_{2r+1}. One approximate method is to fit a finite set of polynomials by least squares, as was done for the case of the heaving circular cylinder in ref. 1. However, in this paper the solution of the set of equations will be considered only under the condition that Kce^{ξ_0} is small and that a and b are of the same order of magnitude.

The expansion in power series of Kc of the functions $\Psi_c(\xi,\eta)$, $\Psi_s(\xi,\eta)$

suggests that

$$p_{2r+1} = \sum_{n=0}^{\infty} P_{n,2r+1}(Kc)^n \tag{21}$$

$$q_{2r+1} = \sum_{n=0}^{\infty} Q^{(1)}_{n,2r+1}(Kc)^n + \log Kc \sum_{n=1}^{\infty} Q^{(2)}_{n,2r+1}(Ks)^n, \tag{22}$$

since the expression in square brackets on the right-hand side of (19) and (20) involves only $(Kc)^0$ and $(Kc)^1$. It is therefore possible to determine the coefficients $P_{n,2r+1}$, $Q^{(1)}_{n,2r+1}$, $Q^{(2)}_{n,2r+1}$ recursively by equating coefficients of $(Kc)^n$ and $(Kc)^n \log Kc$ on the two sides of (19) and (20). It is assumed that the resulting power series defining p_{2r+1}, q_{2r+1} converge for sufficiently small values of Kc. This is known to be true for the heaving motion of the circular cylinder.

Now, if $(Kc)^2 \log Kc$ is negligible,

$$\Psi_c(\xi, \eta) = 2Kc, \tag{23}$$

$$\Psi_s(\xi, \eta) = \operatorname{Im}\left[\frac{2c}{\pi z} + \frac{2}{\pi} iKc(\gamma + \log iKz)\right], \tag{24}$$

$$\Psi_c(\xi_0, \eta) - \Psi_c(\xi_0, \tfrac{1}{2}\pi) = 0, \tag{25}$$

$$\Psi_s(\xi_0, \eta) - \Psi_s(\xi_0, \tfrac{1}{2}\pi) = \operatorname{Im}\left(\frac{2c}{\pi z_0}\right) + \frac{2}{\pi} Kc \log\left|\frac{z_0}{a}\right|$$

$$= \operatorname{Im}\left(\frac{2c}{\pi z_0}\right) + O(Kc), \tag{26}$$

and $$\psi_{2r+1}(\xi_0, \eta) - \psi_{2r+1}(\xi_0, \tfrac{1}{2}\pi) = e^{-(2r+1)\xi_0} \cos(2r+1)\eta + O(Kc). \tag{27}$$

It follows that $$P_{0,2r+1} = P_{1,2r+1} = Q^{(2)}_{1,2r+1} = 0 \tag{28}$$

and $Q^{(1)}_{0,2r+1}$ is given by

$$\operatorname{Im}\left(\frac{2c}{\pi z_0}\right) = Q^{(1)}_{0,1}\left(\frac{x^2+y^2-a^2}{b^2-a^2}\right)_0 + \sum_{r=1}^{\infty} Q^{(1)}_{0,2r+1} e^{-(2r+1)\xi_0} \cos(2r+1)\eta. \tag{29}$$

The amplitude of the motion of the cylinder is determined to the first order by $Q^{(1)}_{0,1}$, which may be found by multiplying (29) by $\cos\eta$ and integrating from 0 to $\frac{1}{2}\pi$. But

$$\operatorname{Im}\left(\frac{2c}{\pi z_0}\right) = \sum_{n=0}^{\infty} a_{2n+1} e^{-(2n+1)\xi} \cos(2n+1)\eta \quad (\xi \geqslant \xi_0), \tag{30}$$

from the theory of harmonic functions, where only a_1 is required in the calculation of $Q^{(1)}_{0,1}$.

Multiply both sides of (30) by e^{ξ} and let ξ tend to infinity. For large ξ,

$$x \sim \tfrac{1}{2}ce^{\xi} \sin\eta, \tag{31}$$

$$y \sim \tfrac{1}{2}ce^{\xi} \cos\eta. \tag{32}$$

In the limit the equation (30) reduces to

$$-(4/\pi)\cos\eta = a_1\cos\eta,$$

or

$$a_1 = -4/\pi. \tag{33}$$

On carrying out the multiplication and integration,

$$-\frac{4}{\pi}e^{-\xi_0} = Q_{0,1}^{(1)}\frac{4}{\pi}\int\limits_0^{\frac{1}{2}\pi}\left(\frac{x^2+y^2-a^2}{b^2-a^2}\right)_0\cos\eta\,d\eta. \tag{34}$$

On the cylinder, the stream-function is given approximately by

$$A\psi = Q_{0,1}^{(1)}\sin\sigma t\left(\frac{x^2+y^2-a^2}{b^2-a^2}\right)_0 = -\tfrac{1}{2}A\theta_0\,\sigma\sin\sigma t(x^2+y^2-a^2)_0, \tag{35}$$

where A is a normalizing constant, and at positive infinity

$$A\psi = 2Kce^{-Ky}\cos(Kx-\sigma t) = \frac{gAB}{\sigma}e^{-Ky}\cos(Kx-\sigma t), \tag{36}$$

where $|B|$ is the amplitude at infinity. Hence

$$B = \frac{K^2c\theta_0(b^2-a^2)}{Q_{0,1}^{(1)}} = K^2ce^{\xi_0}\theta_0\int\limits_0^{\frac{1}{2}\pi}(a^2-x^2-y^2)_0\cos\eta\,d\eta. \tag{37}$$

As an example consider the elliptic cylinder

$$x = c\cosh\xi_0\sin\eta, \qquad y = c\sinh\xi_0\cos\eta$$
$$a = c\cosh\xi_0, \qquad b = c\sinh\xi_0. \tag{38}$$

Here

$$(a^2-x^2-y^2)_0 = (a^2-b^2)\cos^2\eta \tag{39}$$

and the amplitude $|B|$ is

$$K^2ce^{\xi_0}\theta_0\int\limits_0^{\frac{1}{2}\pi}|a^2-b^2|\cos^3\eta\,d\eta = \tfrac{2}{3}K^2\theta_0(a+b)^2|a-b|. \tag{40}$$

It can be shown that this formula is valid whether a is greater or less than b. The calculation shows that the error is $O(K^3a^4)$, which is negligible if Ka is small. When $Q_{0,1}^{(1)}$ has been found, $Q_{0,2r+1}^{(1)}$ is obtained by multiplying (29) by $\cos(2r+1)\eta$ and integrating from 0 to $\frac{1}{2}\pi$.

5. The boundary condition $\partial\phi/\partial y = 0$ $(y = 0)$

There is another way of approaching the problem, which is suggested by the boundary condition

$$K\phi+\frac{\partial\phi}{\partial y} = 0 \quad (y = 0). \tag{41}$$

As K tends to zero, the boundary condition tends formally to

$$\frac{\partial\phi}{\partial y} = 0 \quad (y = 0). \tag{42}$$

It is not unreasonable to expect a close relationship between the first approximation to the wave problem and the complete solution ψ_R of the problem with boundary condition (42). (Cf. ref. **10**.) One way of solving the latter is to expand the stream-function in terms of the set

$$\psi_{2n+1,R} = e^{-(2n+1)\xi}\cos(2n+1)\eta, \tag{43}$$

to which is added the dipole function

$$\Psi_R = \mathrm{Im}\left(\frac{2c}{\pi z}\right). \tag{44}$$

Then the determination of the coefficients in this expansion is clearly the same as the determination of $Q^{(1)}_{0,2r+1}$. Expansion of $\mathrm{Im}(2c/\pi z)$ leads to the following

THEOREM 1. *Suppose that in the skew-symmetric wave-problem ψ is prescribed (except for an additive constant) on the cylinder $\xi = \xi_0$, and that at the mean free surface $y = 0$*

$$K\phi + \frac{\partial\phi}{\partial y} = 0.$$

To obtain the wave amplitude at infinity when Ka is small, solve the simpler problem, in which ψ is prescribed as before on $\xi = \xi_0$, but the free surface is replaced by a rigid plane and

$$\frac{\partial\phi}{\partial y} = 0 \quad (y = 0).$$

If the solution to this latter problem is written in the form

$$\psi_R = \sum_{r=0}^{\infty} B_{2r+1}\, e^{-(2r+1)\xi}\cos(2r+1)\eta \sin\sigma t,$$

where

$$B_{2r+1}\, e^{-(2r+1)\xi_0} = O\left(\frac{1}{r^2}\right). \tag{45}$$

Then for the original problem the waves at infinity are described by

$$\psi = -\tfrac{1}{2}\pi B_1 Kc e^{-Ky}\cos(K|x| - \sigma t). \tag{46}$$

The error in ψ is small of order $K^2a^2B_1$ when $B_1 \neq 0$.

A similar investigation for symmetrical motions, using a source function and a set of polynomials $\psi_{2r}(\xi,\eta)$ (cf. ref. **1**), leads to

THEOREM 2. *If the solution of the problem modified as in Theorem 1 is written in the form*

$$\psi_R = \left[C_0\,\eta + \sum_{r=1}^{\infty} C_{2r}\, e^{-2r\xi}\sin 2r\eta\right]\sin\sigma t, \tag{47}$$

where C_0 is adjusted to make

$$C_{2r}\, e^{-2r\xi_0} = O\left(\frac{1}{r^2}\right),$$

then for the original problem the waves at infinity are described by

$$\psi = \pi C_0 e^{-Ky}\sin(K|x|-\sigma t), \tag{48}$$

the error in ψ *being of order* $K^2a^2C_0\log Ka$, *when* $C_0 \neq 0$.

6. The pressure on the cylinder

The pressure at any point in the fluid is connected with the velocity potential by the relation

$$p = g\rho y - \rho\frac{\partial\phi}{\partial t} + F(t),$$

where $F(t)$ is a function of time to be determined from the condition that at the free surface the pressure is zero. In the skew-symmetric motion ϕ is considered as the superposition of the non-orthogonal polynomials, defined by equation (10), and the dipole potential taken in the form

$$\Phi = \text{Real part of}\left[2Kcie^{iKz-i\sigma t}+\frac{2}{\pi}\sin\sigma t\int\limits_0^\infty \frac{kce^{-kz}\,dk}{k+iK}\right]. \tag{49}$$

The non-orthogonal set tends to zero as $|z|$ tends to infinity, (49) reduces to the potential describing a regular wave train for which the pressure is given by

$$p = g\rho y - \rho\frac{\partial\Phi}{\partial t} = 0 \quad \text{on the surface.} \tag{50}$$

Hence without loss of generality $F(t)$ may be taken as zero. A simple calculation gives

THEOREM 3. *If in the modified skew-symmetrical motion*

$$[\partial\phi_R/\partial y = 0, \quad y = 0]$$

$$\psi_R = \sum_{r=0}^{\infty} B_{2r+1}e^{-(2r+1)\xi}\cos(2r+1)\eta\sin\sigma t,$$

where

$$B_{2r+1}e^{-(2r+1)\xi_0} = O\left(\frac{1}{r^2}\right), \tag{51}$$

then in the original motion the hydrodynamic pressure $-\rho\frac{\partial\phi}{\partial t}$ *near the cylinder can be obtained approximately from*

$$\phi_R = -\sum_{r=0}^{\infty} B_{2r+1}e^{-(2r+1)\xi}\sin(2r+1)\eta\sin\sigma t. \tag{52}$$

The error in the pressure is of order $Kc\rho\,\partial\phi_R/\partial t$. *A rather more complicated calculation shows that if the modified symmetrical motion is given by*

$$\psi_R = \left[C_0\eta + \sum_{r=1}^{\infty} C_{2r}e^{-2r\xi}\sin 2r\eta\right]\cos\sigma t, \quad \textit{where} \quad C_{2r}e^{-2r\xi_0} = O\left(\frac{1}{r^2}\right) \tag{53}$$

then the hydrodynamic pressure $-\rho\dfrac{\partial\phi}{\partial t}$ *is obtained from*

$$\phi_R = \Big[-C_0(\gamma+\log\tfrac{1}{2}Kc+\xi)+\sum_{r=1}^{\infty} C_{2r}\,e^{-2r\xi}\cos 2r\eta\Big]\cos\sigma t+\pi C_0\sin\sigma t. \quad (54)$$

7. Applications

The theory which has been given in the previous sections may be applied to a variety of problems concerning the motion of cylinders, and under suitable conditions to the discussion of the motion of ships of long parallel middle body. If numerical computation is to be avoided, the parameter Ka must be so small that the first approximation is valid. This condition is usually fulfilled when a long ship is given a slow forced rolling motion, but not for a heaving motion. Also, the error in the calculation of a symmetrical motion is of order $Ka\log Ka$, and is larger than for a skew-symmetrical one. For these reasons most of the calculations will be made for cases of rolling motions. The section of the cylinder will first be taken as elliptical, so that direct comparison with experiments on ships will not be possible. The extension to other sections involves rather tiresome calculations which, though simple in principle, would obscure the ideas underlying the work. However, the calculation of the amplitude of waves generated by the rolling of a cylinder of nearly rectangular section will be given in detail.

For convenience the modified motion in which the free surface is supposed to be replaced by a rigid plate $\left(\dfrac{\partial\phi}{\partial y}=0,\ y=0\right)$ will be described as the R-motion set up by the motion of the cylinder.

(A) *The roll axis of a cylinder of elliptic section*

When a ship executes a slow rolling motion in still water, there is a point of least movement, called the tranquil point, about which the rolling motion may be supposed to take place. The position of the roll axis in the corresponding two-dimensional motion will now be investigated.

Consider a ship the immersed part of which is a cylinder of semi-elliptic section with its centre in the mean free surface, and consider the skew-symmetrical forced rolling motion about any axis parallel to the axis of the cylinder and lying in the plane of symmetry. For slow rolling the first approximation is adequate; also, the damping is so small that during a roll period there is little difference between free and forced rolling. To maintain a motion at constant amplitude a small amount of energy must be supplied per unit time, most conveniently by a couple. In general there will be an oscillatory reaction on the roll axis, but when this axis coincides with the axis of free rolling, the reaction will be small. For small angles of roll, the rolling motion about a fixed axis can be considered as made up of a rolling motion about the centre, together with an oscillatory horizontal displacement of the centre in phase with the rolling motion. As the cylinder rolls, the fluid exerts a varying pressure on the cylinder, given by

$$p = g\rho y - \rho\frac{\partial\phi}{\partial t}, \tag{55}$$

the first term representing the hydrostatic, the second the hydrodynamic pressure. The potential is adjusted in accordance with Theorem 3. By the principle of Archimedes the total hydrostatic force is vertical and depends only on the displacement, which is constant (to the first order in θ_0) during the rolling motion and equal to the weight of the ship.

The horizontal forces acting on the cylinder are:

(1) the resultant force due to the hydrodynamic pressure;

(2) the reaction of the roll axis.

The resultant of these two forces is equal to the product of the ship's mass and the horizontal acceleration of the centre of gravity. When the roll axis is the axis of the free rolling motion, the second force will be neglected. An equation is thus obtained to determine the roll axis.

Let d be the depth of the centre of gravity and l that of the roll axis below the water-line. It is assumed that the horizontal axis is the major axis. Suppose that the inclination of the ship at time t is

$$\theta = \theta_0 \sin\sigma t, \tag{56}$$

then the horizontal displacement of the centre of the ellipse is

$$-l\theta = -l\theta_0 \sin\sigma t.$$

It follows, as in ref. (3), that on the ellipse $\xi = \xi_0$

$$\psi = -\tfrac{1}{4}\theta_0\sigma c^2(1+\cos 2\eta)\cos\sigma t - l\theta_0\sigma c\sinh\xi_0\cos\eta\cos\sigma t$$

$$= -\frac{4}{\pi}\theta_0\sigma c^2\cos\sigma t\sum_{r=0}^{\infty}\frac{(-1)^{r-1}}{(2r-1)(2r+1)(2r+3)}\cos(2r+1)\eta - -l\theta_0\sigma c\sinh\xi_0\cos\eta\cos\sigma t. \tag{57}$$

The solution for the corresponding R-motion is clearly

$$\psi_R = -\frac{4}{\pi}\theta_0\sigma c^2\cos\sigma t\sum_{r=0}^{\infty}\frac{(-1)^{r-1}}{(2r-1)(2r+1)(2r+3)}e^{-(2r+1)(\xi-\xi_0)}\cos(2r+1)\eta - -l\theta_0\sigma c\sinh\xi_0 e^{-(\xi-\xi_0)}\cos\eta\cos\sigma t. \tag{58}$$

By Theorem 3, the corresponding potential function is

$$\phi_R = \frac{4}{\pi}\theta_0\sigma c^2\cos\sigma t\sum_{r=0}^{\infty}\frac{(-1)^{r-1}}{(2r-1)(2r+1)(2r+3)}e^{-(2r+1)(\xi-\xi_0)}\sin(2r+1)\eta + +l\theta_0\sigma c\sinh\xi_0 e^{-(\xi-\xi_0)}\sin\eta\cos\sigma t \tag{59}$$

near the cylinder.

The horizontal force per unit length opposing the motion is

$$-\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p\frac{\partial y}{\partial \eta}\,d\eta = 2\rho\int_{0}^{\frac{1}{2}\pi}\frac{\partial \phi}{\partial t}\frac{\partial y}{\partial \eta}\,d\eta \quad (\xi = \xi_0)$$

$$= \rho c\sigma \sinh \xi_0[\tfrac{2}{3}\theta_0\,\sigma c^2+\tfrac{1}{2}\pi\theta_0\,\sigma cl \sinh \xi_0]\sin \sigma t. \tag{60}$$

The horizontal displacement of the centre of gravity is

$$(d-l)\theta_0 \sin \sigma t.$$

Now the horizontal force per unit length is equal to the product of the mass M per unit length and the horizontal acceleration of the centre of gravity:

$$M\sigma^2(d-l)\theta_0 \sin \sigma t = \rho c\sigma \sinh \xi_0 \sin \sigma t[\tfrac{2}{3}\theta_0\,\sigma c^2+\tfrac{1}{2}\pi\theta_0\,\sigma cl \sinh \xi_0] \tag{61}$$

i.e. (with $c\cosh \xi_0 = a$, $c\sinh \xi_0 = b$),

$$M(d-l) = \rho b[\tfrac{2}{3}c^2+\tfrac{1}{2}\pi bl],$$

so

$$l = \frac{Md-\frac{2}{3}\rho bc^2}{M+\frac{1}{2}\pi\rho b^2}. \tag{62}$$

But the cylinder is in equilibrium in its mean position, so that

$$M = \tfrac{1}{2}\pi\rho ab,$$

$$l = \frac{ad-\frac{4}{3\pi}(a^2-b^2)}{a+b}. \tag{63}$$

To this approximation the position of the roll axis is independent of the frequency. It follows that (63) also gives the position of the roll axis for the case of free slow rolling, which can be expressed as the superposition of a number of undamped forced oscillations with periods lying in a narrow band near the mean rolling period. Equation (62) was first derived by R. Brard (ref. **10**), who compared it with experiments and found satisfactory agreement.

When the semi-axes a and b are nearly equal,

$$l = \tfrac{1}{2}d. \tag{64}$$

This formula is in agreement with the experimental results mentioned by Sir William White (ref. **8**, p. 169) for rolling without bilge keels. It appears that the position of the roll-axis depends critically on the difference $(a-b)$ of the semi-axes. When higher terms are included in the calculation, it is no longer permissible to assume that the two skew-symmetrical motions are in phase. Under these conditions, i.e. when the diameter of the ellipse is not small compared with the wave-length, there is no roll-axis; the instantaneous centre of rotation is not confined to a small region.

(B) *The added mass in a slow heaving motion*

Suppose that an elliptic cylinder is given a forced heaving motion

$$y = l\sin\sigma t.$$

Then on the cylinder $\xi = \xi_0$

$$\psi = -l\sigma c\cosh\xi_0 \sin\eta\cos\sigma t$$

$$= -l\sigma c\cosh\xi_0\left[\frac{2}{\pi}\eta+\frac{2}{\pi}\sum_1^\infty \frac{(-1)^{r-1}}{r(4r^2-1)}\sin 2r\eta\right]\cos\sigma t. \quad (65)$$

It follows that in the corresponding R-motion

$$\psi_R = -l\sigma c\cosh\xi_0\left[\frac{2}{\pi}\eta+\frac{2}{\pi}\sum_1^\infty \frac{(-1)^{r-1}}{r(4r^2-1)}e^{-2r(\xi-\xi_0)}\sin 2r\eta\right]\cos\sigma t. \quad (66)$$

By Theorem 3, the corresponding potential function ϕ is

$$-l\sigma c\cosh\xi_0\left[-\frac{2}{\pi}(\gamma+\log\tfrac{1}{2}Kc+\xi)+\frac{2}{\pi}\sum_1^\infty \frac{(-1)^{r-1}}{r(4r^2-1)}e^{-2r(\xi-\xi_0)}\cos 2r\eta\right]\cos\sigma t-$$

$$-2l\sigma c\cosh\xi_0\sin\sigma t. \quad (67)$$

The vertical hydrodynamic force opposing the motion of the cylinder is

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} p\frac{\partial x}{\partial\eta}\,d\eta = -2\rho\int_0^{\frac{1}{2}\pi}\frac{\partial\phi}{\partial t}\frac{\partial x}{\partial\eta}\,d\eta \quad (\xi=\xi_0), \quad (68)$$

$$= 2\rho c\cosh\xi_0\, l\sigma^2 c\cosh\xi_0\left[\frac{2}{\pi}(\gamma+\log\tfrac{1}{2}Kc+\xi_0)\int_0^{\frac{1}{2}\pi}\cos\eta\,d\eta\,\sin\sigma t-\right.$$

$$\left.-\frac{2}{\pi}\sum_1^\infty\frac{(-1)^{r-1}}{r(4r^2-1)}\int_0^{\frac{1}{2}\pi}\cos 2r\eta\cos\eta\,d\eta\,\sin\sigma t+2\int_0^{\frac{1}{2}\pi}\cos\eta\,d\eta\,\cos\sigma t\right]$$

$$= 2\rho la^2\sigma^2\left[\frac{2}{\pi}\left(\gamma+\log\frac{Kce^{\xi_0}}{2}\right)\sin\sigma t-\frac{2}{\pi}\sin\sigma t\sum_1^\infty\frac{1}{r(4r^2-1)^2}+2\cos\sigma t\right] \quad (69)$$

$$= 2\rho la^2\sigma^2\left[-\frac{2}{\pi}\left\{\log\frac{1}{K(a+b)}+0\cdot 23\right\}\sin\sigma t+2\cos\sigma t\right], \quad (70)$$

where the numerical constant $0\cdot 23 = 1\cdot 5-\log 2-\gamma$ approximately, and γ is Euler's constant.

Here the first term is 180 degrees out of phase with the acceleration and so represents the effect of added mass. The second term is in quadrature with the acceleration; the work done by it is accounted for by wave damping.

Let M' be the mass per unit length of a semicircular cylinder of radius a, i.e.

$$M' = \tfrac{1}{2}\pi\rho a^2.$$

Then the inertia force per unit length represented by the first term can be written in the form

$$\frac{8}{\pi^2}M'\left[\log\frac{1}{K(a+b)}+0\cdot 23\right]\frac{d^2y}{dt^2}, \tag{71}$$

whence the added mass per unit length is to the first order

$$\frac{8}{\pi^2}M'\left[\log\frac{1}{K(a+b)}+0\cdot 23\right]. \tag{72}$$

It may be noted that the ratio of the inertia force (71) to the hydrostatic restoring force per unit length $2\rho alg$ is of order $Ka\log Ka$, which is assumed to be small.

(C) *Motion of a ship in a sea-way*

Suppose that a ship of the type described in (A) is placed in a progressive train of waves advancing from positive infinity

$$\psi = \frac{gA}{\sigma}e^{-Ky}\sin(Kx+\sigma t) = \frac{gKA}{\sigma}(x\cos\sigma t - y\sin\sigma t)+G(t) \tag{73}$$

(near the origin to the first order); and suppose that as a result the coordinates of the centre of the cylinder at time t are

$$x = L\cos(\sigma t+\alpha), \qquad y = l\sin(\sigma t+\beta),$$

and that the inclination of the minor axis at time t is

$$\theta = \theta_0\cos(\sigma t+\delta).$$

Then, on the ellipse $\xi = \xi_0$,

$$\psi = -l\sigma c\cosh\xi_0\sin\eta\cos(\sigma t+\beta) - L\sigma c\sinh\xi_0\cos\eta\sin(\sigma t+\alpha)+ \\ +\tfrac{1}{4}\theta_0\sigma c^2\sin(\sigma t+\delta)[1+\cos 2\eta]. \tag{74}$$

The stream-function satisfying this condition is

$$\psi = \frac{gA}{\sigma}e^{-Ky}\sin(Kx+\sigma t) - \\ -\frac{gA}{\sigma}Kc\cosh\xi_0\cos\sigma t\left[\frac{2}{\pi}\eta+\frac{2}{\pi}\sum_1^\infty\frac{(-1)^{r-1}}{r(4r^2-1)}e^{-2r(\xi-\xi_0)}\sin 2r\eta\right] - \\ -l\sigma c\cosh\xi_0\cos(\sigma t+\beta)\left[\frac{2}{\pi}\eta+\frac{2}{\pi}\sum_1^\infty\frac{(-1)^{r-1}}{r(4r^2-1)}e^{-2r(\xi-\xi_0)}\sin 2r\eta\right] + \\ +\frac{gA}{\sigma}Kc\sinh\xi_0\,e^{-(\xi-\xi_0)}\cos\eta\sin\sigma t - L\sigma c\sinh\xi_0\,e^{-(\xi-\xi_0)}\cos\eta\sin(\sigma t+\alpha)+ \\ +\frac{4}{\pi}\theta_0\sigma c^2\sin(\sigma t+\delta)\sum_0^\infty\frac{(-1)^{r-1}}{(2r-1)(2r+1)(2r+3)}e^{-(2r+1)(\xi-\xi_0)}\cos(2r+1)\eta. \tag{75}$$

Whence the potential function is to the first order

$$\phi = \frac{gA}{\sigma}e^{-Ky}\cos(Kx+\sigma t)-$$

$$-\left[g\frac{KAa}{\sigma}\cos\sigma t+l\sigma a\cos(\sigma t+\beta)\right]\left[-\frac{2}{\pi}\left(\gamma+\log\frac{Kc}{2}+\xi\right)+\right.$$

$$\left.+\frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^{r-1}}{r(4r^2-1)}e^{-2r(\xi-\xi_0)}\cos 2r\eta\right]-$$

$$-2\left[g\frac{KAa}{\sigma}\sin\sigma t+l\sigma a\sin(\sigma t+\beta)\right]-$$

$$-g\frac{AKb}{\sigma}e^{-(\xi-\xi_0)}\sin\eta\sin\sigma t+L\sigma be^{-(\xi-\xi_0)}\sin\eta\sin(\sigma t+\alpha)-$$

$$-\frac{4}{\pi}\theta_0\sigma c^2\sin(\sigma t+\delta)\sum_{0}^{\infty}\frac{(-1)^{r-1}}{(2r-1)(2r+1)(2r+3)}e^{-(2r+1)(\xi-\xi_0)}\sin(2r+1)\eta. \tag{76}$$

From this velocity potential valid near the cylinder the force and the couple acting on the cylinder can be found, the expression for the pressure being

$$p = g\rho y-\rho\frac{\partial\phi}{\partial t}.$$

When the forces are equated to the product of mass and acceleration in the direction of the force, these equations, with a similar equation for the couple, are sufficient to determine the quantities l, L, θ_0; α, β, δ. For example, consider the heaving motion which is independent of the skew-symmetrical motions.

The force opposing the motion is (correct to the *second* order)

$$2\rho gAa\sin\sigma t[1-\tfrac{1}{4}\pi Kb]-$$

$$-\frac{4}{\pi}\rho a^2\sigma^2\left[\log\frac{1}{K(a+b)}+0{\cdot}23\right][A\sin\sigma t+l\sin(\sigma t+\beta)]+$$

$$+\frac{4}{\pi}\rho a^2\sigma^2[A\cos\sigma t+l\cos(\sigma t+\beta)]+2\rho gla\sin(\sigma t+\beta),$$

which is to be equated to $\quad \frac{1}{2}\pi\rho labo^2\sin(\sigma t+\beta)$ (77)

since the mass of the ship is equal to $\frac{1}{2}\pi\rho ab$ by the principle of Archimedes.

It follows that to the second order

$$\beta = 0, \qquad l+A = 0 \tag{78}$$

(taking account of the equation $gK = \sigma^2$), so that the ship moves with the wave.

The reflection from the ship's side due to heaving can be shown to be zero, to this order.

8. The problem of roll resistance in still water

The damping of the rolling motion is known to be mainly due to three different causes: (1) skin friction, (2) wave-making, (3) eddy-making by keels. It is agreed that skin friction does not account for more than a small fraction of the rolling decrement (ref. **2**). The waves generated in the rolling motion of a ship are of the order of one inch in height and rather difficult to measure. Nevertheless, naval architects have arrived at tentative conclusions about the relative importance of (2) and (3). For instance, G. S. Baker has developed an approximate theory (ref. **4**) from which he deduces that wave-making should account in some cases for more than half the energy lost by the ship. According to Baker, (2) and (3) should make comparable contributions to the damping for any section of rectangular shape with rounded corners. The contribution from eddies is increased when the ship is under way.

The mechanism of wave-damping is independent of the viscosity of the fluid. A reasonable estimate of its magnitude may probably be obtained by assuming that the fluid is frictionless. If the ship has a long parallel middle body of length at least as great as the wave-length set up by the rolling motion, the two-dimensional theory worked out in this paper may be applied. Moreover, the parameter Ka is small; in most cases its value is about 0·1 (ref. **2**). It is therefore sufficient to consider the first approximation. The calculation is unchanged when a uniform flow parallel to the axis of the cylinder is superposed.

It will now be shown that hydrodynamical theory does not support the empirical formulae hitherto put forward. In the next section the waves generated by a cylinder rolling about a point in the mean surface will be calculated. (The tranquil point is usually near the mean surface.) It will be shown that there is at least one nearly rectangular section for which the wave amplitude is zero, so that the damping may be overestimated by Baker's formulae. It is hoped to arrange experiments to check the theory.

9. Determination of the waves made by a conventional section

The formula (37) for the wave amplitude requires that x and y should be of the form (5, 6), with $\xi = \xi_0$.

Transformations are known expressing a rectangle with rounded corners in this form (ref. **5**), but for practical purposes it is necessary to use finite expressions

$$x_N(c,\xi_0) = c\cosh\xi_0\sin\eta + c\sum_{1}^{2N} A_{2r+1,N}\, e^{-(2r+1)\xi_0}\sin(2r+1)\eta, \qquad (79)$$

$$y_N(c,\xi_0) = c\sinh\xi_0\cos\eta - c\sum_{1}^{2N} A_{2r+1,N}\, e^{-(2r+1)\xi}\cos(2r+1)\eta, \qquad (80)$$

for which $x_N^2+y_N^2$ can be found without a prohibitive amount of computation. Equations (79, 80) will represent a roughly rectangular curve if contact with its tangents at $\eta = 0$ and $\eta = \frac{1}{2}\pi$ is of the highest order possible:

$$\left(\frac{\partial}{\partial\eta}\right)^{2s} y_N = 0, \qquad \eta = 0 \quad (s = 1,2,\ldots,N), \tag{81}$$

$$\left(\frac{\partial}{\partial\eta}\right)^{2s} x_N = 0, \qquad \eta = \tfrac{1}{2}\pi \quad (s = 1,2,\ldots,N). \tag{82}$$

The equations for $A_{2r+1,N}$ are

$$\sinh\xi_0 = \sum_{r=1}^{2N} A_{2r+1,N}\, e^{-(2r+1)\xi_0}(2r+1)^{2s} \quad (s = 1,2,\ldots,N), \tag{83}$$

$$\cosh\xi_0 = \sum_{r=1}^{2N} A_{2r+1,N}\, e^{-(2r+1)\xi_0}(-1)^{r-1}(2r+1)^{2s} \quad (s = 1,2,\ldots,N). \tag{84}$$

By addition and subtraction,

$$\tfrac{1}{2}e^{\xi_0} = \sum_{r=1}^{N} A_{4r-1}\, e^{-(4r-1)\xi_0}(4r-1)^{2s} \quad (s = 1,2,\ldots,N), \tag{85}$$

$$-\tfrac{1}{2}e^{-\xi_0} = \sum_{r=1}^{N} A_{4r+1}\, e^{-(4r+1)\xi_0}(4r+1)^{2s} \quad (s = 1,2,\ldots,N). \tag{86}$$

Each of these systems is of the form

$$\sum_{r=1}^{N} B_{r,N} b_r^s = 1 \quad (s = 0,1,\ldots,N-1) \tag{87}$$

the solution of which is

$$B_{r,N} = \prod_{\substack{s=1\\ s\neq r}}^{N} \left(\frac{b_s-1}{b_s-b_r}\right). \tag{88}$$

The equations (83, 84) thus have the solutions

$$A_{4r-1,N}\, e^{-(4r-1)\xi_0}(4r-1)^2 = \tfrac{1}{2}e^{\xi_0} \prod_{\substack{s=1\\ s\neq r}}^{N} \left[\frac{(4s-1)^2-1}{(4s-1)^2-(4r-1)^2}\right], \tag{89}$$

$$A_{4r+1,N}\, e^{-(4r+1)\xi_0}(4r+1)^2 = -\tfrac{1}{2}e^{\xi_0} \prod_{\substack{s=1\\ s\neq r}}^{N} \left[\frac{(4s+1)^2-1}{(4s+1)^2-(4r+1)^2}\right]. \tag{90}$$

As N tends to infinity each coefficient tends to a limit corresponding to the limiting rectangle. Small values of N give satisfactory sections, some of which are shown in Fig. 1. The parameter ξ_0 determines the shape, the parameter c the scale. It is necessary to verify in each case that there is a $(1,1)$ correspondence between points on the boundary and the coordinates ξ or η.

In the subsequent calculation N will be taken as unity. Then the parametric equations of the cylinder are

$$x = \tfrac{9}{16}(a+b)\sin\eta+\tfrac{25}{48}(a-b)\sin\eta+\tfrac{1}{16}(a+b)\sin 3\eta-\tfrac{1}{48}(a-b)\sin 5\eta, \quad (91)$$

$$y = \tfrac{9}{16}(a+b)\cos\eta-\tfrac{25}{48}(a-b)\cos\eta-\tfrac{1}{16}(a+b)\cos 3\eta+\tfrac{1}{48}(a-b)\cos 5\eta, \quad (92)$$

also

$$ce^{\xi_0} = \tfrac{3}{8}(a+b).$$

The amplitude at infinity has been shown to be, by (37),

$$K^2\theta_0\, ce^{\xi_0}\left|\int\limits_0^{\frac{1}{2}\pi}(a^2-x^2-y^2)\cos\eta\, d\eta\right|.$$

In this case

$$x^2+y^2-a^2 = -\tfrac{67}{128}(a^2-b^2)(1+\cos 2\eta)+$$
$$+\tfrac{1}{1152}[25(a-b)^2+81(a+b)^2](1-\cos 4\eta)+\tfrac{3}{128}(a^2-b^2)(1+\cos 6\eta), \quad (93)$$

whence

$$\frac{4}{\pi}\int\limits_0^{\frac{1}{2}\pi}(x^2+y^2-a^2)\cos\eta\, d\eta = -\frac{278}{105\pi}(a^2-b^2)+\frac{1}{270\pi}[25(a-b)^2+81(a+b)^2]$$
$$= -\frac{2{\cdot}25}{\pi}(a-1{\cdot}26b)(a+1{\cdot}05b) \quad (94)$$

and the amplitude at infinity is, to the first order,

$$0{\cdot}63K^2\theta_0(a+b)(a+1{\cdot}05b)|a-1{\cdot}26b|. \quad (95)$$

When a and b are equal the amplitude is

$$\tfrac{27}{40}K^2\theta_0\, a^3. \quad (96)$$

This may be compared with $\tfrac{2}{3}K^2\theta_0\, a^3$ for a thin plate (ref. 9). But when $a = 1{\cdot}26b$, the first order amplitude vanishes. It may be deduced that for this ratio of a to b, the amplitude, and therefore also the wave-damping, is very small. The corresponding section is shown in Fig. 1.

For reference the amplitude is given for the same class of cylinders turning about an axis at height H above the water surface. It is

$$0{\cdot}63K^2\theta_0(a+b)[(1{\cdot}26b-a)(1{\cdot}05b+a)+0{\cdot}015H(a+26b)]. \quad (97)$$

10. Discussion

That wave-making might account for the damping of the rolling motion of ships was first suggested by W. Froude (ref. 7). Assuming that the angular displacement satisfied a differential equation of the second degree with constant coefficients, he was able to give the damping decrement in terms of the characteristics of the ship and the wave height at infinity (see also ref. 2). A considerable advance was made by Havelock (ref. 2), who not only corrected an error in Froude's calculation but also attempted to find the wave height at infinity in terms of the characteristics of the ship. For this purpose he considered the motion as two-dimensional and evaluated the waves made by the rolling about a vertical mean position of a completely submerged thin plate whose width was small compared with the length of surface waves it generated. Comparison with Baker's experiments showed that the calculated amplitude was too small. This was to be expected, since there is a flow between the surface and a submerged cylinder, which tends to reduce the amplitude, and which cannot

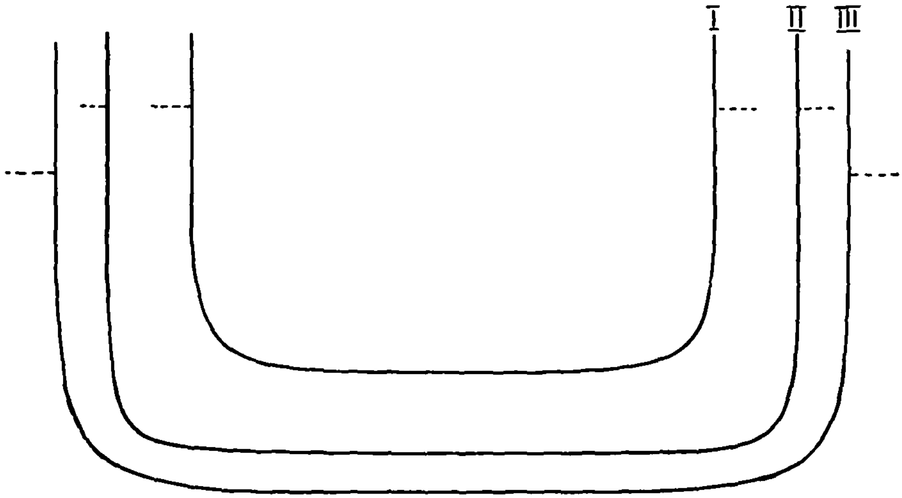

FIG. 1. Some typical sections.

Parametric equations:

I. $x = \frac{a}{8}(9\sin\eta + \sin 3\eta)$,

$y = \frac{a}{8}(9\cos\eta - \cos 3\eta)$. (Minimum radius of curvature $0{\cdot}25a$.)

II. $x = \frac{49a}{640}\left(15\sin\eta + 2\sin 3\eta - \frac{3}{49}\sin 7\eta\right)$,

$y = \frac{49a}{640}\left(15\cos\eta - 2\cos 3\eta + \frac{3}{49}\cos 7\eta\right)$. (Minimum radius of curvature $0{\cdot}16a$.)

III. $x = \frac{9}{16}(a+b)\sin\eta + \frac{25}{48}(a-b)\sin\eta + \frac{1}{16}(a+b)\sin 3\eta - \frac{1}{48}(a-b)\sin 5\eta$,

$y = \frac{9}{16}(a+b)\cos\eta - \frac{25}{48}(a-b)\cos\eta - \frac{1}{16}(a+b)\cos 3\eta + \frac{1}{48}(a-b)\cos 5\eta$,

where $a = 1{\cdot}26b$. When this section is forced to roll about the origin the wave amplitude of the resulting fluid motion vanishes to the first order.

take place when the cylinder is in the surface. This effect was calculated by the present writer, who gave an explicit expression for the waves generated by a thin plate oscillating about the vertical and not completely submerged. Comparison with experiment showed excellent agreement (ref. 9).

In those calculations it was assumed that it would be sufficiently accurate to replace a cylinder by a thin plate of width equal to the draught of the ship. When the finite section of the cylinder is taken into account, formula (95) is obtained. Baker's experimental results, as quoted by Havelock, belong to a ship in which $a = 1{\cdot}31b$ in the centre section (ref. 4), so that the good agreement with the earlier formula cannot be explained on any two-dimensional theory. Dr. Baker has informed the present writer that the model ship on which he made experiments did not possess a long parallel middle body, so that two-dimensional theory could not be expected to apply. No detailed comparison of theory and experiment is possible until experiments are made on suitable models.

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