Boundary Value Problems of

MATHEMATICAL PHYSICS

Volume I

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This corresponds to heat conduction in a rod whose ends and lateral surface are insulated. Since the sources along the rod generate heat at a rate $f(x)$ per unit length per unit time, a solution is possible only if $\int_0^l f(x)dx = 0$, that is, if the net heat added per unit time is 0. If this condition is violated, it is physically clear that the temperature would rise (or decline) with time and no steady-state solution could exist.

Suppose $\int_0^l f(x)dx = 0$. To be specific, let $f(x) = \sin(2\pi x/l)$. We can solve the differential equation $-d^2y/dx^2 = \sin(2\pi x/l)$ to obtain $y(x) = A + Bx + (l^2/4\pi^2)\sin(2\pi x/l)$. The boundary condition at the left end yields $B = -l/2\pi$; from the boundary condition at the right end, $B + (l^2/4\pi^2)(2\pi/l)\cos 2\pi = 0$, which is satisfied by $B = -l/2\pi$; furthermore, A is arbitrary. Therefore, the general solution of the system is $y(x) = A + (l^2/4\pi^2)\sin(2\pi x/l) - (l/2\pi)x$. It is easy to check that this satisfies all the conditions on the problem.

It is apparent that the alternative theorem does not tell us how to obtain the solution; it merely predicts the existence or nonexistence of solutions.

Example 2. Consider the inhomogeneous system

$$y'' + 2y' + 5y = f(x), \quad 0 < x < \pi; \qquad y(0) = 0, \quad y(\pi) = 0.$$

The corresponding homogeneous system has the nontrivial solution $e^{-x}\sin 2x$. The adjoint homogeneous system is

$$v'' - 2v' + 5v = 0, \quad 0 < x < \pi; \qquad v(0) = 0, \quad v(\pi) = 0,$$

which, by direct calculation, can be shown to have the nontrivial solution $e^x \sin 2x$. The original inhomogeneous system will therefore have solutions if and only if

$$\int_0^\pi f(x)e^x \sin 2x \, dx = 0,$$

and, if this consistency condition is satisfied,

$$y = y_p + Ae^{-x}\sin 2x,$$

where y_p is any particular solution of the inhomogeneous system [which can be obtained explicitly by (1.50) or by the method outlined below].

The Modified Green's Function

We have seen that the self-adjoint system $-d^2u/dx^2 = 0$, $0 \le x \le l$, with $u'(0) = u'(l) = 0$, has the nontrivial solution $u =$ constant.

This means that the system

$$-\frac{d^2y}{dx^2} = f(x); \qquad y'(0) = y'(l) = 0 \tag{1.89}$$

has a solution if and only if $\int_0^l f(x)dx = 0$. It follows that the ordinary Green's function cannot be constructed for this problem. The system

$$-\frac{d^2g}{dx^2} = \delta(x - \xi); \qquad \left.\frac{dg}{dx}\right|_{x=0} = \left.\frac{dg}{dx}\right|_{x=l} = 0 \tag{1.90}$$

has no solution, since $\int_0^l \delta(x - \xi)dx \neq 0$.

There is a simple physical interpretation of this difficulty. We know that (1.90) represents the steady-state temperature in a *completely* insulated rod when a steady heat source is present at the point $x = \xi$ along the rod. These conditions are contradictory, since the temperature would surely rise indefinitely in the interior of such a rod, owing to the presence of the steady source.

If we insist upon constructing something like a Green's function [a function that will help us solve system (1.90) when this system has a solution, that is, when $\int_0^l f(x)dx = 0$], the remedy is rather simple. We introduce an additional source *density* of strength $-1/l$ in the rod. Thus the net heat input per unit time in the rod for this new problem is

$$\int_0^l \left[\delta(x - \xi) - \frac{1}{l}\right] dx = 0.$$

The *modified* Green's function satisfies

$$-\frac{d^2g_M}{dx^2} = \delta(x - \xi) - \frac{1}{l}; \qquad \left.\frac{dg_M}{dx}\right|_0 = \left.\frac{dg_M}{dx}\right|_l = 0. \tag{1.91}$$

Since the consistency condition is satisfied, this system has a solution (determined only up to an additive constant). For $x \neq \xi$, $-g_M'' = -1/l$, so that

$$g_M(x \mid \xi) = \begin{cases} A + Bx + \dfrac{x^2}{2l}, & 0 \le x < \xi; \\[2ex] C + Dx + \dfrac{x^2}{2l}, & \xi < x \le l. \end{cases}$$

The boundary conditions yield $B = 0$, $D = -1$. Continuity at $x = \xi$ implies $A + (\xi^2/2l) = C - \xi + (\xi^2/2l)$, or $C = A + \xi$. The jump condition on dg_M/dx at $x = \xi$ gives $1 - (\xi/l) + (\xi/l) = 1$, which is automatically satisfied. Therefore,

$$g_M(x \mid \xi) = \begin{cases} A + \dfrac{x^2}{2l}, & 0 \le x < \xi; \\[2ex] A + \xi - x + \dfrac{x^2}{2l}, & \xi < x \le l. \end{cases} \tag{1.92}$$

As was expected, an arbitrary constant A appears in the solution. It is often convenient to choose a particular modified Green's function which is a symmetric function of x and ξ. To accomplish this, consider $g_M(x\,|\,\xi_1)$ and $g_M(x\,|\,\xi_2)$, which satisfy the systems

$g_M(x|\xi_2)$

$$-\frac{d^2 g_M(x\,|\,\xi_1)}{dx^2} = \delta(x-\xi_1) - \frac{1}{l}; \qquad \left.\frac{dg_M}{dx}\right|_{x=0} = \left.\frac{dg_M}{dx}\right|_{x=l} = 0$$

and

$g_m(x,\xi_1)$

$$-\frac{d^2 g_M(x\,|\,\xi_2)}{dx^2} = \delta(x-\xi_2) - \frac{1}{l}; \qquad \left.\frac{dg_M}{dx}\right|_{x=0} = \left.\frac{dg_M}{dx}\right|_{x=l} = 0,$$

respectively. Combining these equations in the usual way, we obtain

$$g_M(\xi_1\,|\,\xi_2) - g_M(\xi_2\,|\,\xi_1) - \frac{1}{l}\int_0^l g_M(x\,|\,\xi_2)dx + \frac{1}{l}\int_0^l g_M(x\,|\,\xi_1)dx = 0.$$

If we impose the condition $\int_0^l g_M(x\,|\,\xi)dx = 0$ for every ξ, then $g_M(x\,|\,\xi)$ will be symmetric. In our particular case this condition yields

$$\int_0^\xi \left(A + \frac{x^2}{2l}\right)dx + \int_\xi^l \left(A + \xi - x + \frac{x^2}{2l}\right)dx = 0$$

or

$$A = \frac{1}{l}\left(\frac{l^2}{3} + \frac{\xi^2}{2} - \xi l\right).$$

The *symmetric* modified Green's function is given by

$$g_M(x\,|\,\xi) = \begin{cases} \dfrac{l}{3} - \xi + \dfrac{x^2+\xi^2}{2l}, & 0 \le x < \xi; \\[2ex] \dfrac{l}{3} - x + \dfrac{x^2+\xi^2}{2l}, & \xi < x \le l. \end{cases}$$

This result could have been obtained by inspecting (1.92) and making a judicious choice of A.

The modified Green's function can be used to solve the system

$$-\frac{d^2y}{dx^2} = f(x); \qquad \frac{dy}{dx}(0) = \frac{dy}{dx}(l) = 0, \tag{1.93}$$

where $f(x)$ satisfies $\int_0^l f(x)dx = 0$ [no solution exists unless $\int_0^l f(x)dx = 0$].

Multiply (1.93) by $g_M(x\,|\,\xi)$, (1.91) by $y(x)$, subtract, and integrate to obtain

$$-\int_0^l (g_M y'' - y g_M'')dx = \int_0^l g_M(x\,|\,\xi)f(x)dx - y(\xi) + \frac{1}{l}\int_0^l y(x)dx.$$

Therefore, after imposing the boundary conditions,

$$y(\xi) = \text{constant} + \int_0^l g_M(x\,|\,\xi)f(x)dx,$$

where $g_M(x\,|\,\xi)$ is any solution of (1.91). If we choose $g_M(x\,|\,\xi)$ to be the symmetric solution of (1.91) then $y(\xi) = \text{constant} + \int_0^l g_M(\xi\,|\,x)f(x)dx$, or

$$y(x) = \text{constant} + \int_0^l g_M(x\,|\,\xi)f(\xi)d\xi.$$

Consider now the problem of a general self-adjoint system. We assume that all the nontrivial solutions of the homogeneous system $Lu = 0$; $B_1(u) = B_2(u) = 0$, are of the form $Cu_1(x)$, where $u_1(x)$ is a normalized solution; that is,

$$\int_a^b u_1^2\,dx = 1.$$

$U(x,s) = h_1 + - \Sigma$

The *modified Green's function* satisfies

$$Lg_M(x\,|\,\xi) = \delta(x-\xi) - u_1(x)u_1(\xi); \qquad B_1(g) = B_2(g) = 0. \tag{1.94}$$

This system has a solution because $\int_a^b [\delta(x-\xi) - u_1(x)u_1(\xi)]u_1(x)dx = 0$. The construction is entirely similar to that for the ordinary Green's function described in Section 1.5, but the modified Green's function is not uniquely determined. We can add $Cu_1(x)$ to a Green's function without violating any of the requirements on it. The reader can verify that the modified Green's function will by symmetric if

$$\int_a^b g_M(x\,|\,\xi)u_1(x)dx = 0.$$

The modified Green's function enables us to solve the inhomogeneous system

$$Ly = f(x); \qquad B_1(y) = B_2(y) = 0.$$

We require $\int_a^b f(x)u_1(x)dx = 0$, for otherwise no solution exists. By the usual procedure we obtain

$$\int_a^b (g_M Ly - yLg_M)dx = \int_a^b g_M(x\,|\,\xi)f(x)dx - y(\xi) + \int_a^b y(x)u_1(x)u_1(\xi)dx.$$

The left side vanishes by the boundary conditions and, therefore,

$$y(\xi) = \int_a^b g_M(x\,|\,\xi)f(x)dx + Cu_1(\xi).$$

If we use the symmetric modified Green's function, the above result can be written

$$y(x) = \int_a^b g_M(x \mid \xi) f(\xi) d\xi + C u_1(x).$$

The above arguments can be modified in a suitable manner when there exist two linearly independent solutions of the homogeneous equation (see Exercise 1.55).

EXERCISES

1.54 Show that a sufficient condition for the modified Green's function to be symmetric is

$$\int_a^b g_M(x \mid \xi) u_1(x) dx = 0.$$

Show that there are problems where this condition is not necessary.

1.55 Develop the theory of modified Green's functions in the case of a self-adjoint system where the completely homogeneous system has two linearly independent solutions $u_1(x)$ and $u_2(x)$ (hence every solution of the homogeneous equation satisfies the boundary conditions).

1.56 Consider steady-state heat conduction in an insulated thin ring of constant cross section. Let x be a coordinate along the center line of the ring, where x ranges from 0 to l. The equation governing the temperature distribution $u(x)$ in the ring is $-kA(d^2u/dx^2) = f(x)$; $0 < x < l$.

Although, at first glance, there appear to be no boundary conditions associated with the system, further thought shows that the temperature and its derivative must have the same value at $x = 0$ and $x = l$. Therefore, $u(0) = u(l)$, $(du/dx)(0) = (du/dx)(l)$. The completely homogeneous system has the nontrivial solution $u = \text{constant}$. Find the modified Green's function for this problem. Use this Green's function to solve

$$-\frac{d^2y}{dx^2} = f(x); \qquad y(0) = y(l), \quad y'(0) = y'(l),$$

when $\int_a^b f(x)dx = 0$.

1.57 Find the modified Green's function for the system $L = d^2/dx^2$ with $y(0) = -y(1)$, $y'(0) = y'(1)$. This system is not self-adjoint.

1.58 Apply the result of Exercise 1.54 to find the modified Green's function for the system $L = (d^2/dx^2) + 1$ with $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$.

1.59 (a) Find the consistency condition for the system

$$-\frac{d^2y}{dx^2} = f(x); \qquad y'(0) = \alpha, \quad y'(1) = \beta.$$

(b) Find the consistency conditions for the system

$$\frac{d^2y}{dx^2} + y = f(x); \qquad y(0) - y(2\pi) = \alpha, \quad y'(0) - y'(2\pi) = \beta.$$

SUGGESTED READINGS FOR CHAPTER 1

The letters E, I, and A indicate elementary, intermediate, and advanced books, respectively.

Ordinary Differential Equations

(E) Agnew, R. P., *Differential Equations*, McGraw-Hill, New York, 1960.

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(A) Coddington, E. A., and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

(I) Indritz, J., *Methods in Analysis*, Macmillan, New York, 1963.

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(I) Friedman, B., *Principles and Techniques of Applied Mathematics*, Wiley, New York, 1956.

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Theory of Distributions

(A) Bremermann, H. J., *Distributions, Complex Variables, and Fourier Transforms*, Addison-Wesley, Reading, 1965.

(I) Erdélyi, A., *Operational Calculus and Generalized Functions*, Holt, Rinehart and Winston, New York, 1962.

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(I) Schwartz, L., *Théorie des Distributions*, Vol. I, Hermann, Paris, 1950.

(I) Zemanian, A. H., *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, 1965.