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The method of fundamental solutions for eigenproblems in domains with and without interior holes

BY C. C. TSAI[1], D. L. YOUNG[2,*], C. W. CHEN[2] AND C. M. FAN[2]

[1]*Department of Information Technology, Toko University, Chia-Yi County 613, Taiwan, ROC*
[2]*Department of Civil Engineering & Hydrotech Research Institute, National Taiwan University, Taipei 106, Taiwan, ROC*

The main purpose of the present paper is to provide a general method of fundamental solution (MFS) formulation for two- and three-dimensional eigenproblems without spurious eigenvalues. The spurious eigenvalues are avoided by utilizing the mixed potential method. Illustrated problems in the annular and concentric domains are studied analytically and numerically to demonstrate the issue of spurious eigenvalues by the discrete and continuous versions of the MFS with and without the mixed potential method. The proposed numerical method is then verified with the exact solutions of the benchmark problems in circular and spherical domains with and without holes. Further studies are performed in a three-dimensional peanut shaped domain. In the spirit of the MFS, this scheme is free from meshes, singularities and numerical integrations.

Keywords: method of fundamental solutions; eigenproblems; spurious eigenvalues; mixed potential method

1. Introduction

In the applications of computational engineering, the mesh generation of a complicated geometry is time-consuming for traditional numerical methods, such as the finite difference method, the finite element method (FEM), and the finite volume method. In recent years, there has been an increasing interest in the idea of meshless numerical methods for solving partial differential equations. Roughly speaking, such methods can be divided into two categories. The first one is the domain type, such as Kansa's method (or multi-quadrics method; Kansa 1990*a*,*b*); and the second one is the boundary type, such as the method of fundamental solutions (MFS; Johnston & Fairweather 1984; Karageorghis & Fairweather 1987; Golberg 1995; Young *et al.* 2004*a*,*b*, 2005*a*,*b*). The MFS has been successfully applied to the potential flow problems (Johnston & Fairweather 1984), the biharmonic equations (Karageorghis & Fairweather 1987), the Poisson equations (Golberg 1995), the Stokes flow problems (Young *et al.*

* Author for correspondence (dlyoung@ntu.edu.tw).

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2005*a*,*b*) and the diffusion equations (Young *et al.* 2004*a*,*b*). In this paper, we will concentrate on the MFS for acoustical problems, which are governed by the Helmholtz equations.

The boundary integral equations (BIEs) have been used to solve the interior and exterior acoustic problems for a long time. Several approaches, such as complex-valued boundary element method (Yeih *et al.* 1998), multiple reciprocity method (Chen & Wong 1997, 1998; Chen *et al.* 1999), and real-part BEM (Chen *et al.* 1999; Kuo *et al.* 2000) have been developed for acoustic problems. All the above methods have to face the singular and hypersingular integrals. To avoid the singular and hypersingular integrals, DeMey (1977) used the imaginary-part kernel to solve the resulting eigenvalue problems governed by the Helmholtz equations.

To further overcome the difficulties of mesh generation, singularities, and numerical integrations, the MFS has been widely adopted as an alternative to the traditional BIE. Kondapalli *et al.* (1992) were the first to apply the MFS for the Helmholtz equation in the analysis of acoustic scattering in fluids and solids. Recently, Kang *et al.* (Kang *et al.* 1999; Kang & Lee 2000) applied the non-dimensional dynamic influence function (NDIF) method to solve the eigenproblem of an acoustic cavity. Chen *et al.* (2000) commented that the NDIF method is a special case of the MFS with imaginary-part kernel, and later further developed the method in a series of papers (Chen *et al.* 2002*a*–*c*). Karageorghis (2001) applied the complex-valued MFS for the two-dimensional simply connected domain. In this paper, we further develop the complex-valued MFS, which has been utilized in exterior problems by Kondapalli *et al.* (1992) and the two-dimensional simply-connected domain by Karageorghis (2001), to interior Helmholtz problems in domains with and without interior holes of two- and three-dimensional geometry.

To avoid the fictitious frequency of the exterior acoustic problems, Burton & Miller (1971) have successfully applied the combination of the single- and double-layer potentials for direct BIE. On the other hand, it was shown independently by Panich (1965) and by Brakhage & Werner (1965) that the mixed potential method for indirect BIE is also available. In domains without interior holes, there are no spurious eigenvalues if the complex-valued BIE is employed (Tai & Shaw 1974). However, the spurious eigenvalues always appear when the complex-valued BIE is employed to the domain with interior holes (Kitahara 1985; Chen *et al.* 2001, 2003*b*). Chen *et al.* (2001, 2003*b*) applied the mixed potential method to deal with the eigenproblems successfully. In the present work, we combine the mixed potential method with the MFS to solve two- and three-dimensional Helmholtz problems free from spurious eigenvalues in domains with interior holes. Independently, Chen *et al.* (2005) recently presented a similar work for two-dimensional multiply connected domains.

2. MFS formulation for domains without interior holes

For acoustic problems, the governing equation is the Helmholtz equation with boundary conditions:

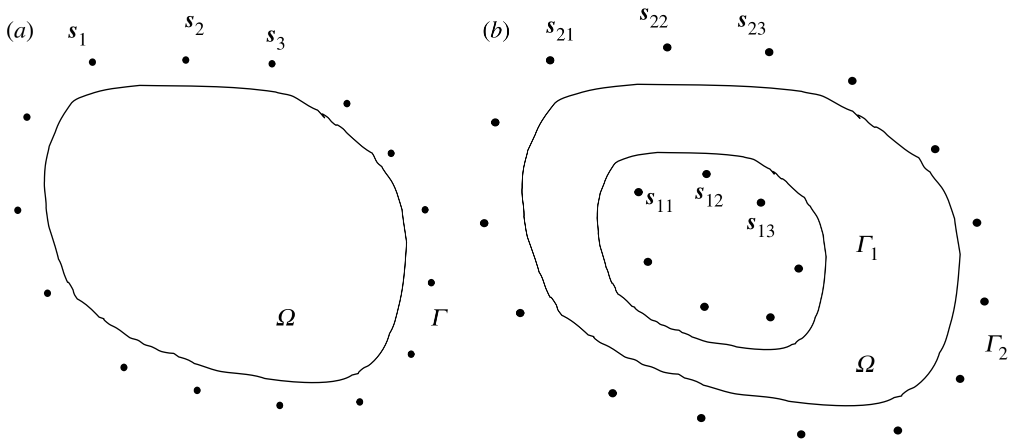


Figure 1. Geometry configuration of domain (*a*) without, (*b*) with interior holes.

$$\begin{cases} (\nabla^2 + k^2)u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Gamma^D, \\ \dfrac{\partial u(\boldsymbol{x})}{\partial n} = 0, & \boldsymbol{x} \in \Gamma^N, \end{cases} \tag{2.1}$$

where ∇^2 is the Laplacian operator, k is the wavenumber, Ω is the domain of interest, and $\Gamma = \Gamma^D + \Gamma^N$ is the boundary of Ω (figure 1*a*). The fundamental solution of the Helmholtz equation (2.1) is defined by

$$-(\nabla^2 + k^2)G_k(\boldsymbol{x}, \boldsymbol{s}) = \delta(\boldsymbol{x}, \boldsymbol{s}), \tag{2.2}$$

where $\boldsymbol{x}$ are the coordinates of field points and $\boldsymbol{s}$ are the coordinates of source points. Then, the fundamental solutions are obtained:

$$G_k(\boldsymbol{x}, \boldsymbol{s}) = \begin{cases} \dfrac{-\mathrm{i}}{4} H_0^{(2)}(k|\boldsymbol{x} - \boldsymbol{s}|), & \text{for two-dimensional,} \\ \dfrac{1}{4\pi|\boldsymbol{x} - \boldsymbol{s}|} \mathrm{e}^{-\mathrm{i}k|\boldsymbol{x}-\boldsymbol{s}|}, & \text{for three-dimensional,} \end{cases} \tag{2.3}$$

where $H_n^{(2)}(\ )$ is the second Hankel function of order n. For simplicity, we define the following notations:

$$\left. \begin{aligned} U_k(\boldsymbol{x}, \boldsymbol{s}) &= G_k(\boldsymbol{x}, \boldsymbol{s}), & L_k(\boldsymbol{x}, \boldsymbol{s}) &= \frac{\partial G_k(\boldsymbol{x}, \boldsymbol{s})}{\partial \boldsymbol{n}_x}, \\ T_k(\boldsymbol{x}, \boldsymbol{s}) &= \frac{\partial G_k(\boldsymbol{x}, \boldsymbol{s})}{\partial \boldsymbol{n}_s}, & M_k(\boldsymbol{x}, \boldsymbol{s}) &= \frac{\partial^2 G_k(\boldsymbol{x}, \boldsymbol{s})}{\partial \boldsymbol{n}_x \partial \boldsymbol{n}_s}. \end{aligned} \right\} \tag{2.4}$$

In the spirit of the MFS, the solution is assumed to be

$$u(\boldsymbol{x}) = \sum_{j=1}^{N} \alpha_j U_k(\boldsymbol{x}, \boldsymbol{s}_j), \tag{2.5}$$

where α_j is the intensity of the source point at $\boldsymbol{s}_j$, and N is the number of source points as depicted in figure 1*a*. Then, we seek to minimize the least squares functional,

$$S = \sum_{i=1}^{M} \left| \sum_{j=1}^{N} \alpha_j \begin{cases} U_k(\boldsymbol{x}_i, \boldsymbol{s}_j), & \text{if } \boldsymbol{x}_i \in \Gamma^D \\ L_k(\boldsymbol{x}_i, \boldsymbol{s}_j), & \text{if } \boldsymbol{x}_i \in \Gamma^N \end{cases} \right|^2, \tag{2.6}$$

where $\boldsymbol{x}_i \in \Gamma$ are the boundary collocated points and M is the number of the boundary collocated points. In general, the minimization involves the α_j, $\boldsymbol{s}_j$, and N. In the present work, we assume $\boldsymbol{s}_j$ to be known *a priori* distributed points and the collocated number is equal to the number of source points for simplicity. Therefore, it results in a $N \times N$ linear system with the same numbers of unknowns and equations.

$$\begin{bmatrix} A(k, \boldsymbol{x}_1, \boldsymbol{s}_1) & A(k, \boldsymbol{x}_1, \boldsymbol{s}_2) & \cdots & \cdots & A(k, \boldsymbol{x}_1, \boldsymbol{s}_N) \\ A(k, \boldsymbol{x}_2, \boldsymbol{s}_1) & A(k, \boldsymbol{x}_2, \boldsymbol{s}_2) & \cdots & \cdots & A(k, \boldsymbol{x}_2, \boldsymbol{s}_N) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A(k, \boldsymbol{x}_N, \boldsymbol{s}_1) & A(k, \boldsymbol{x}_N, \boldsymbol{s}_2) & \cdots & \cdots & A(k, \boldsymbol{x}_N, \boldsymbol{s}_N) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \tag{2.7}$$

where $A(k, \boldsymbol{x}_i, \boldsymbol{s}_j) = U_k(\boldsymbol{x}_i, \boldsymbol{s}_j)$, if $\boldsymbol{x}_i \in \Gamma^D$ and $A(k, \boldsymbol{x}_i, \boldsymbol{s}_j) = L_k(\boldsymbol{x}_i, \boldsymbol{s}_j)$, if $\boldsymbol{x}_i \in \Gamma^N$. The equation (2.7) is a nonlinear eigenproblem for k that we are searching for eigenvalues $k_1 < k_2 < k_3 < \cdots$ such that the equation (2.7) has non-trivial solutions, which are the eigenvectors $\{\alpha_j\}(k_1), \{\alpha_j\}(k_2), \{\alpha_j\}(k_3), \ldots$. Correspondingly, we are able to find the numerical eigenfunctions of the original Helmholtz equation (2.1) by equation (2.5), $u(\boldsymbol{x})(k_1)$, $u(\boldsymbol{x})(k_2)$, $u(\boldsymbol{x})(k_3)$, …. In the present work, we adopted the direct determinant search method to find the associated eigenvalues (Teukolsky *et al.* 1992; Karageorghis 2001).

3. MFS formulation for domains with interior holes

Next, we consider the problems for which the interior holes exist.

$$\begin{cases} (\nabla^2 + k^2) u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Gamma_1^D, \\ \dfrac{\partial u(\boldsymbol{x})}{\partial n} = 0, & \boldsymbol{x} \in \Gamma_1^N, \\ u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Gamma_2^D, \\ \dfrac{\partial u(\boldsymbol{x})}{\partial n} = 0, & \boldsymbol{x} \in \Gamma_2^N, \end{cases} \tag{3.1}$$

where Ω is a domain with interior hole with inside boundary $\Gamma_1 = \Gamma_1^D + \Gamma_1^N$ and outside boundary $\Gamma_2 = \Gamma_2^D + \Gamma_2^N$ as described in figure 1*b*.

To ensure the solutions without spurious eigenvalues, the mixed potential method is adopted. Therefore, the solution is assumed to be

$$u(\boldsymbol{x}) = \sum_{j=1}^{N} \alpha_j (U_k(\boldsymbol{x}, \boldsymbol{s}_{1j}) + \mathrm{i}kT_k(\boldsymbol{x}, \boldsymbol{s}_{1j})) + \sum_{j=1}^{M} \beta_j U_k(\boldsymbol{x}, \boldsymbol{s}_{2j}), \tag{3.2}$$

where α_j and β_j are the intensities of the inside source point $\boldsymbol{s}_{1j}$ and the outside source points $\boldsymbol{s}_{2j}$, respectively. Moreover, N and M are the corresponding numbers of source points as depicted in figure 1*b*. Also, we seek for minimizing the functional generally,

$$S = \sum_{i=1}^{L} \left| \begin{cases} \displaystyle\sum_{j=1}^{N} \alpha_j (U_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j}) + \mathrm{i}kT_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j})) + \sum_{i=1}^{M} \beta_j U_k(\boldsymbol{x}_i, \boldsymbol{s}_{2j}), & \text{if } \boldsymbol{x}_i \in \Gamma_1^D \cup \Gamma_2^D \\ \displaystyle\sum_{j=1}^{N} \alpha_j (L_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j}) + \mathrm{i}kM_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j})) + \sum_{j=1}^{M} \beta_j L_k(\boldsymbol{x}_i, \boldsymbol{s}_{2j}), & \text{if } \boldsymbol{x}_i \in \Gamma_1^N \cup \Gamma_2^N \end{cases} \right|^2, \tag{3.3}$$

where $\boldsymbol{x}_i \in \Gamma_1 \cup \Gamma_2$ is the boundary collocated point and L is the number of the boundary collocated points. Similarly, we assume $\boldsymbol{s}_{1j}$ and $\boldsymbol{s}_{2j}$ to be known *a priori* distributed points and the collocated number is equal to the number of source points, $L = M + N$, for simplicity. Then, it results in a $L \times L$ linear system with the same unknowns and equations.

$$\begin{bmatrix} A(k, \boldsymbol{x}_1, \boldsymbol{s}_{11}) & \cdots & A(k, \boldsymbol{x}_1, \boldsymbol{s}_{1N}) & B(k, \boldsymbol{x}_1, \boldsymbol{s}_{21}) & \cdots & B(k, \boldsymbol{x}_1, \boldsymbol{s}_{2N}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A(k, \boldsymbol{x}_{N+M}, \boldsymbol{s}_{11}) & \cdots & A(k, \boldsymbol{x}_{N+M}, \boldsymbol{s}_{1N}) & B(k, \boldsymbol{x}_{N+M}, \boldsymbol{s}_{21}) & \cdots & B(k, \boldsymbol{x}_{N+M}, \boldsymbol{s}_{2N}) \end{bmatrix} \times \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ \beta_1 \\ \vdots \\ \beta_M \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.4}$$

where $A(k, \boldsymbol{x}_i, \boldsymbol{s}_{1j}) = U_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j}) + \mathrm{i}kT_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j})$ and $B(k, \boldsymbol{x}_i, \boldsymbol{s}_{2j}) = U_k(\boldsymbol{x}_i, \boldsymbol{s}_{2j})$, if $\boldsymbol{x}_i \in \Gamma_1^D \cup \Gamma_2^D$ and $A(k, \boldsymbol{x}_i, \boldsymbol{s}_{1j}) = L_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j}) + \mathrm{i}kM_k(\boldsymbol{x}_i, \boldsymbol{s}_{1j})$ and $B(k, \boldsymbol{x}_i, \boldsymbol{s}_{2j}) = L_k(\boldsymbol{x}_i, \boldsymbol{s}_{2j})$, if $\boldsymbol{x}_i \in \Gamma_1^N \cup \Gamma_2^N$. Then, the eigenvalues $k_1 < k_2 < k_3 < \cdots$ and the corresponding eigenvectors $\{\alpha_j, \beta_j\}(k_1)$, $\{\alpha_j, \beta_j\}(k_2)$, $\{\alpha_j, \beta_j\}(k_3)$, … can be obtained as described above.

4. Analytic treatments

In general, solutions of the Helmholtz problems obtained by the MFS cannot be solved analytically except for some special shapes. Chen *et al.* (2005) have derived the finite eigenvalues of the annular problem by using the discrete version of the MFS and the circulant theory. Herein, we use the continuous version of the MFS, which is equivalent to the indirect BEM (Golberg & Chen 1998; Chen *et al.* 2004), to derive the eigenequations for the following Helmholtz problems: circular, annular, spherical, and concentric spherical problems with the Dirichlet and Neumann boundary conditions. In the following derivation, we assume the source points $\boldsymbol{s}$ are dense enough so that the equations of the MFS equation (2.5) can be transformed into integral forms (Golberg & Chen 1998; Chen *et al.* 2004),

$$u(\boldsymbol{x}) = \int \alpha(\boldsymbol{s})\, U_k(\boldsymbol{x}, \boldsymbol{s})\mathrm{d}\boldsymbol{s}, \tag{4.1}$$

and similarly for normal derivatives

$$\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_x} = \int \alpha(\boldsymbol{s}) L_k(\boldsymbol{x}, \boldsymbol{s})\mathrm{d}\boldsymbol{s}, \tag{4.2}$$

and also for the MFS equation (3.2) in domains with interior holes,

$$u(\boldsymbol{x}) = \int \alpha(\boldsymbol{s}_1)(U_k(\boldsymbol{x}, \boldsymbol{s}_1) + \mathrm{i}kT_k(\boldsymbol{x}, \boldsymbol{s}_1))\mathrm{d}\boldsymbol{s}_1 + \int \beta(\boldsymbol{s}_2)\, U_k(\boldsymbol{x}, \boldsymbol{s}_2)\mathrm{d}\boldsymbol{s}_2, \tag{4.3}$$

$$\frac{\partial u(\boldsymbol{x})}{\partial \boldsymbol{n}_x} = \int \alpha(\boldsymbol{s}_1)(L_k(\boldsymbol{x}, \boldsymbol{s}_1) + \mathrm{i}kM_k(\boldsymbol{x}, \boldsymbol{s}_1))\mathrm{d}\boldsymbol{s}_1 + \int \beta(\boldsymbol{s}_2) L_k(\boldsymbol{x}, \boldsymbol{s}_2)\mathrm{d}\boldsymbol{s}_2. \tag{4.4}$$

(*a*) *Case I: circular problem with the Dirichlet boundary condition*

Consider the Dirichlet Helmholtz problems in the circular domain described in figure 2*a*, where $\boldsymbol{s}(R, \theta)$ and $\boldsymbol{x}(\rho, \phi)$ are the source and boundary field positions in polar coordinate system, respectively. To describe the MFS numerical solutions in the circular domain, it is convenient to introduce the two-dimensional degenerate kernels:

$$U_k(\boldsymbol{x}, \boldsymbol{s}) = \begin{cases} \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}}{4} J_m(k\rho) H_m^{(2)}(kR)\cos(m(\theta-\phi)), & \mathrm{R} > \rho, \\ \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}}{4} J_m(kR) H_m^{(2)}(k\rho)\cos(m(\theta-\phi)), & \mathrm{R} < \rho, \end{cases} \tag{4.5a}$$

$$T_k(\boldsymbol{x}, \boldsymbol{s}) = \begin{cases} \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}k}{4} J_m(k\rho) H_m^{(2)'}(kR)\cos(m(\theta-\phi)), & \mathrm{R} > \rho, \\ \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}k}{4} J_m'(kR) H_m^{(2)}(k\rho)\cos(m(\theta-\phi)), & \mathrm{R} < \rho, \end{cases} \tag{4.5b}$$

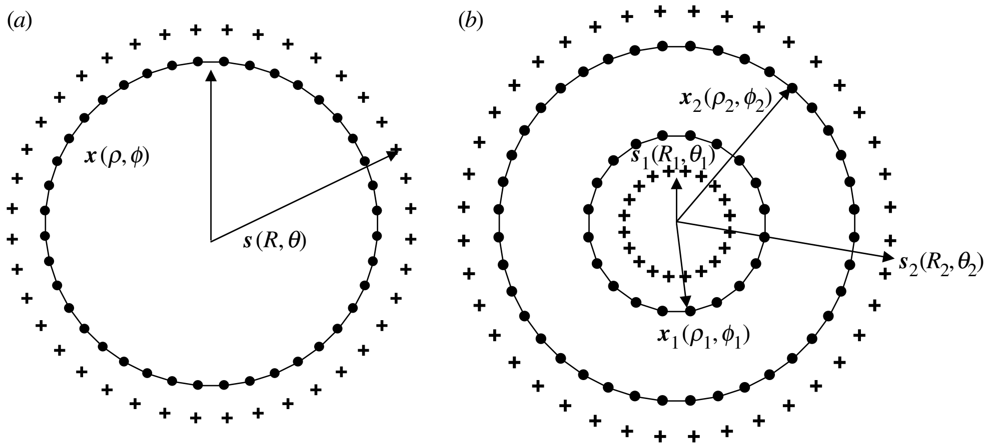


Figure 2. The distributions of source and field points in the (*a*) circular, (*b*) annular problem.

$$L_k(\boldsymbol{x},\boldsymbol{s}) = \begin{cases} \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}k}{4} J'_m(k\rho) H_m^{(2)}(kR)\cos(m(\theta-\phi)), & \mathrm{R} > \rho, \\ \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}k}{4} J_m(kR) H_m^{(2)'}(k\rho)\cos(m(\theta-\phi)), & \mathrm{R} < \rho, \end{cases} \tag{4.5c}$$

$$M_k(\boldsymbol{x},\boldsymbol{s}) = \begin{cases} \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}k^2}{4} J'_m(k\rho) H_m^{(2)'}(kR)\cos(m(\theta-\phi)), & \mathrm{R} > \rho, \\ \displaystyle\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}k^2}{4} J'_m(kR) H_m^{(2)'}(k\rho)\cos(m(\theta-\phi)), & \mathrm{R} < \rho, \end{cases} \tag{4.5d}$$

where $J_n(\ )$ is the Bessel function of the first kind of order n. Then, we decompose the source intensities (in equation (2.5)) in circular harmonics,

$$\alpha(\boldsymbol{s}) = \sum_{n=-\infty}^{\infty} A_n \cos n\theta + B_n \sin n\theta, \tag{4.6}$$

and substitute into equation (4.1) with the boundary condition

$$0 = \int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} -\frac{\mathrm{i}}{4} J_m(k\rho) H_m^{(2)}(kR)\cos(m(\theta-\phi)) \sum_{n=-\infty}^{\infty} A_n \cos n\theta + B_n \sin n\theta \right) R\,\mathrm{d}\theta. \tag{4.7}$$

We then apply the orthogonal relations

$$\begin{cases} \int_0^{2\pi} \sin[n\theta]\cos[m\theta]\mathrm{d}\theta = 0, \\ \int_0^{2\pi} \cos[n\theta]\cos[m\theta]\mathrm{d}\theta = \pi\delta_{mn}, \\ \int_0^{2\pi} \sin[n\theta]\sin[m\theta]\mathrm{d}\theta = \pi\delta_{mn}, \end{cases} \tag{4.8}$$

where δ_{mn} is the Kronecker delta symbol. It will result in

$$0 = \sum_{n=-\infty}^{n=\infty} A_n J_n(k\rho) H_n^{(2)}(kR) \cos m\phi + \sum_{n=-\infty}^{n=\infty} B_n J_n(k\rho) H_n^{(2)}(kR) \sin m\phi. \qquad (4.9)$$

Since the source intensities are arbitrary when k is an eigenvalue, therefore, we can obtain the following relation:

$$J_n(k\rho) = 0, \quad n = 0, 1, 2, \ldots. \qquad (4.10)$$

It is noticed that there are no spurious eigenvalues in equation (4.10).

(b) *Case II: annular problem with the Dirichlet boundary condition*

Next, we consider the Dirichlet Helmholtz problem in the annular domain as depicted in figure 2*b*, where $\boldsymbol{s}_1(R_1, \theta_1)$, and $\boldsymbol{s}_2(R_1, \theta_1)$ and $\boldsymbol{x}_1(\rho_1, \phi_1)$, and $\boldsymbol{x}_2(\rho_2, \phi_2)$ are the source and boundary field positions of the inner and outer circles in the polar coordinate system, respectively. Similarly, we decompose the source intensities in circular harmonics,

$$\begin{cases} \alpha(\boldsymbol{s}_1) = \displaystyle\sum_{n=-\infty}^{\infty} A_n \cos n\theta_1 + B_n \sin n\theta_1, \\ \beta(\boldsymbol{s}_2) = \displaystyle\sum_{n=-\infty}^{\infty} C_n \cos n\theta_2 + D_n \sin n\theta_2, \end{cases} \qquad (4.11)$$

where $\alpha(\boldsymbol{s}_1)$ and $\beta(\boldsymbol{s}_2)$ are source intensities of the outer and inner circles, respectively. Then, we impose boundary condition and substitute to the conventional MFS equation (4.1):

$$\begin{aligned} 0 = &\int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} J_m(kR_1) H_m^{(2)}(k\rho_1) \cos(m(\theta_1 - \phi_1)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} A_n \cos n\theta_1 + B_n \sin n\theta_1 \right) R_1 \, \mathrm{d}\theta_1 \\ &+ \int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} J_m(k\rho_1) H_m^{(2)}(kR_2) \cos(m(\theta_2 - \phi_1)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} C_n \cos n\theta_2 + D_n \sin n\theta_2 \right) R_2 \, \mathrm{d}\theta_2, \end{aligned} \qquad (4.12a)$$

$$\begin{aligned} 0 = &\int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} J_m(kR_1) H_m^{(2)}(k\rho_2) \cos(m(\theta_1 - \phi_2)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} A_n \cos n\theta_1 + B_n \sin n\theta_1 R_1 \, \mathrm{d}\theta_1 \right) \\ &+ \int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} J_m(k\rho_2) H_m^{(2)}(kR_2) \cos(m(\theta_2 - \phi_2)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} C_n \cos n\theta_2 + D_n \sin n\theta_2 \right) R_2 \, \mathrm{d}\theta_2. \end{aligned} \qquad (4.12b)$$

Similarly, the arbitrariness of the source intensities for eigenvalues k ensure the following relation:

$$J_n(kR_1)(J_n(k\rho_1)\,Y_n(k\rho_2) - J_n(k\rho_2)\,Y_n(k\rho_1)) = 0, \quad n = 0, 1, 2, \ldots, \tag{4.13}$$

where $Y_n(\ )$ is the Bessel function of the second kind of order n. It is noticed that spurious eigenvalues occurred at $J_n(kR_1)=0$ when conventional MFS is applied.

Then, we apply the MFS equation for domains with interior holes (3.2) and check if it does not have spurious eigenvalues. Similarly, we substitute the equation (4.11) and the homogeneous Dirichlet boundary condition to the MFS modified by the mixed potential method of equation (4.3), it results in

$$\begin{aligned} 0 = &\int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} (J_m(kR_1) + \mathrm{i}k^2 J_m'(kR_1)) H_m^{(2)}(k\rho_1) \cos(m(\theta_1 - \phi_1)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} A_n \cos n\theta_1 + B_n \sin n\theta_1 \right) \\ &+ \int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} J_m(k\rho_1)(H_m^{(2)}(kR_2)) \cos(m(\theta_2 - \phi_1)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} C_n \cos n\theta_2 + D_n \sin n\theta_2 \right) R_2 \,\mathrm{d}\theta_2, \end{aligned} \tag{4.14a}$$

$$\begin{aligned} 0 = &\int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} (J_m(kR_1) + \mathrm{i}k^2 J_m'(kR_1)) H_m^{(2)}(k\rho_2) \cos(m(\theta_1 - \phi_2)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} A_n \cos n\theta_1 + B_n \sin n\theta_1 \right) R_1 \,\mathrm{d}\theta_1 \\ &+ \int_0^{2\pi} \left(\sum_{m=-\infty}^{m=\infty} \frac{-\mathrm{i}}{4} J_m(k\rho_2)(H_m^{(2)}(kR_2)) \cos(m(\theta_2 - \phi_2)) \right. \\ &\left. \times \sum_{n=-\infty}^{\infty} C_n \cos n\theta_2 + D_n \sin n\theta_2 \right) R_2 \,\mathrm{d}\theta_2. \end{aligned} \tag{4.14b}$$

Also, the arbitrariness of the source intensities ensure the following relation:

$$J_n(k\rho_1)\,Y_n(k\rho_2) - J_n(k\rho_2)\,Y_n(k\rho_1) = 0, \quad n = 0, 1, 2, \ldots. \tag{4.15}$$

Wherein, it is noticed that only true and non-spurious eigenvalues are contained. This is the major declaration of the present paper.

(*c*) *Case III: annular problem with the Neumann boundary condition*

A similar procedure can be applied to the Neumann Helmholtz problem in the annular domain. The resulting relation is

$$J_n'(k\rho_1)\,Y_n'(k\rho_2) - J_n'(k\rho_2)\,Y_n'(k\rho_1) = 0, \quad n = 0, 1, 2, \ldots. \tag{4.16}$$

(*d*) *Case IV: spherical problem with the Dirichlet boundary condition*

The present scheme can be applied not only for two-dimensional Helmholtz problems but also for three-dimensional Helmholtz problems. Consider the

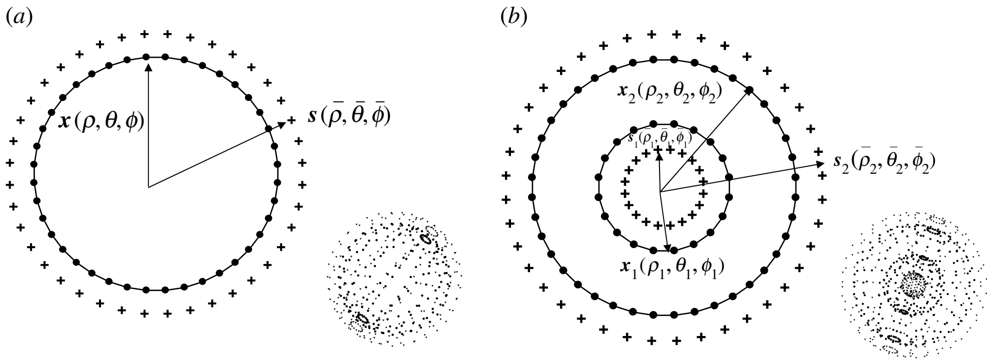


Figure 3. The distributions of source and field points in the (*a*) spherical, (*b*) concentric spherical problem.

Dirichlet Helmholtz problems in the spherical domain described in figure 3*a*, where $\boldsymbol{s}(\bar\rho, \bar\theta, \bar\phi)$ and $\boldsymbol{x}(\rho, \theta, \phi)$ are the source and boundary field positions in the spherical coordinate system, respectively. We first decompose the three-dimensional kernels into circular harmonics, and it will result in the following degenerate kernels:

$$U_k(\boldsymbol{x}, \boldsymbol{s}) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar\phi)] \\ \quad \times P_n^m(\cos\theta)P_n^m(\cos\bar\theta)h_n^{(2)}(k\bar\rho)j_n(k\rho), \quad \bar\rho > \rho, \\ \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar\phi)] \\ \quad \times P_n^m(\cos\theta)P_n^m(\cos\bar\theta)h_n^{(2)}(k\rho)j_n(k\bar\rho), \quad \bar\rho < \rho, \end{cases} \tag{4.17a}$$

$$T_k(\boldsymbol{x}, \boldsymbol{s}) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k^2}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar\phi)] \\ \quad \times P_n^m(\cos\theta)P_n^m(\cos\bar\theta)h_n^{(2)\prime}(k\bar\rho)j_n(k\rho), \quad \bar\rho > \rho, \\ \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k^2}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar\phi)] \\ \quad \times P_n^m(\cos\theta)P_n^m(\cos\bar\theta)h_n^{(2)}(k\rho)j_n'(k\bar\rho), \quad \bar\rho < \rho, \end{cases} \tag{4.17b}$$

$$L_k(\boldsymbol{x}, \boldsymbol{s}) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k^2}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar\phi)] \\ \quad \times P_n^m(\cos\theta)P_n^m(\cos\bar\theta)h_n^{(2)}(k\bar\rho)j_n'(k\rho), \quad \bar\rho > \rho, \\ \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k^2}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar\phi)] \\ \quad \times P_n^m(\cos\theta)P_n^m(\cos\bar\theta)h_n'(k\rho)j_n(k\bar\rho), \quad \bar\rho < \rho, \end{cases} \tag{4.17c}$$

$$M_k(\boldsymbol{x},\boldsymbol{s}) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k^3}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})] \\ \qquad \times P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})h_n^{(2)\prime}(k\bar{\rho})j_n'(k\rho), \quad \bar{\rho}>\rho, \\ \displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{-\mathrm{i}k^3}{4\pi}(2n+1)\varepsilon_m \frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})] \\ \qquad \times P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})h_n^{(2)\prime}(k\rho)j_n'(k\bar{\rho}), \quad \bar{\rho}<\rho, \end{cases} \tag{4.17d}$$

where $P_n^m(\,)$ is the associated Legendre polynomial function, ε_m is the Neumann factor ($\varepsilon_m=1$, when $m=0$ or $\varepsilon_m=2$, when $m>0$), $j_n(\,)$ is the spherical Bessel function of the first kind of order n defined by $j_n(x)=\sqrt{\pi/2x}J_{n+1/2}(x)$, and $h_n^{(2)}(x)=j_n(x)-\mathrm{i}y_n(x)$ is the second spherical Hankel function of order n with the definition $y_n(x)=\sqrt{\pi/2x}\,Y_{n+1/2}(x)$. Then, we decompose the source intensities (in equation (2.5)) in spherical harmonics,

$$\alpha(\boldsymbol{s}) = \sum_{v=0}^{\infty}\sum_{w=0}^{v} A_{vw}P_v^w(\cos\bar{\theta})\cos(w\bar{\phi}), \tag{4.18}$$

and substitute the result into the integral form of the MFS equation (4.1) with the homogenous Dirichlet boundary condition

$$0=\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{-\mathrm{i}k}{4\pi}(2n+1)\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})] \\ \times P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})h_n^{(2)}(k\bar{\rho})j_n(k\rho)\sum_{v=0}^{\infty}\sum_{w=0}^{v}A_{vw}P_v^w(\cos\bar{\theta})\cos(w\bar{\phi})\bar{\rho}^2\sin\bar{\theta}\,\mathrm{d}\bar{\theta}\,\mathrm{d}\bar{\phi}. \tag{4.19}$$

Applying the orthogonal relations,

$$\begin{cases} \int_0^{\pi} P_n^m(\cos\bar{\theta})P_v^w(\cos\bar{\theta})\sin\bar{\theta}\,\mathrm{d}\bar{\theta} = \dfrac{2}{2n+1}\dfrac{(n+m)!}{(n-m)!}\delta_{nv}\delta_{mw}, \\ \int_0^{2\pi}\cos[m(\phi-\bar{\phi})]\cos(w\bar{\phi})\,\mathrm{d}\bar{\phi} = \dfrac{2\pi}{\varepsilon_w}\delta_{mw}\cos(m\phi), \end{cases} \tag{4.20}$$

will result in

$$0=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A_{mn}h_n^{(2)}(k\bar{\rho})j_n(k\rho). \tag{4.21}$$

Since the source intensities are arbitrary where k is an eigenvalue, therefore, we can obtain the following relation:

$$j_n(k\rho)=0, \quad n=0,1,2,\ldots. \tag{4.22}$$

It is noticed that there are no spurious eigenvalues in equation (4.22).

(*e*) *Case V: concentric spherical problem with the Dirichlet boundary condition*

Next, we consider the Dirichlet Helmholtz problem in the concentric spherical domain as depicted in figure 3*b*, where $\boldsymbol{s}_1(\bar{\rho}_1,\bar{\theta}_1,\bar{\phi}_1)$ and $\boldsymbol{s}_2(\bar{\rho}_2,\bar{\theta}_2,\bar{\phi}_2)$, and

$\boldsymbol{x}_1(\rho_1, \theta_1, \phi_1)$, and $\boldsymbol{x}_2(\rho_2, \theta_2, \phi_2)$ are the source and boundary field positions of the inner and outer spheres in the spherical coordinate system, respectively. Then, we decompose the source intensities (in equation (3.2)) in spherical harmonics,

$$\begin{cases} \alpha(\boldsymbol{s}_1) = \displaystyle\sum_{v=0}^{\infty}\sum_{w=0}^{v} A_{vw} P_v^w(\cos\bar{\theta}_1)\cos(w\bar{\phi}_1), \\ \beta(\boldsymbol{s}_2) = \displaystyle\sum_{v=0}^{\infty}\sum_{w=0}^{v} B_{vw} P_v^w(\cos\bar{\theta}_2)\cos(w\bar{\phi}_2), \end{cases} \tag{4.23}$$

where $\alpha(\boldsymbol{s}_1)$ and $\beta(\boldsymbol{s}_2)$ are source intensities of the inner and outer spheres, respectively. Next, we substitute the equation (4.23) and the homogeneous Dirichlet boundary condition to the MFS equation (4.3) for domains with interior holes in the integral form

$$\begin{aligned} 0 = &\int_0^{2\pi}\int_0^{2\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{-\mathrm{i}k}{4\pi}(2n+1)\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi_1-\bar{\phi}_2)] \\ &\times P_n^m(\cos\theta_1)P_n^m(\cos\bar{\theta}_2)h_n^{(2)}(k\bar{\rho}_2)j_n(k\rho_1) \\ &\times\sum_{v=0}^{\infty}\sum_{w=0}^{v}B_{vw}P_v^w(\cos\bar{\theta}_2)\cos(w\bar{\phi}_2)\bar{\rho}_2^2\sin\bar{\theta}_2\,\mathrm{d}\bar{\theta}_2\,\mathrm{d}\bar{\phi}_2 \\ &+\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{-\mathrm{i}k}{4\pi}(2n+1)\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi_1-\bar{\phi}_1)] \\ &\times P_n^m(\cos\theta_1)P_n^m(\cos\bar{\theta}_1)h_n^{(2)}(k\rho_1)(j_n(k\bar{\rho}_1)+\mathrm{i}k^2j_n'(k\bar{\rho}_1)) \\ &\times\sum_{v=0}^{\infty}\sum_{w=0}^{v}A_{vw}P_v^w(\cos\bar{\theta}_1)\cos(w\bar{\phi}_1)\bar{\rho}_1^2\sin\bar{\theta}_1\,\mathrm{d}\bar{\theta}_1\,\mathrm{d}\bar{\phi}_1, \end{aligned} \tag{4.24a}$$

$$\begin{aligned} 0 = &\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{-\mathrm{i}k}{4\pi}(2n+1)\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi_2-\bar{\phi}_2)] \\ &\times P_n^m(\cos\theta_2)P_n^m(\cos\bar{\theta}_2)h_n^{(2)}(k\bar{\rho}_2)j_n(k\rho_2) \\ &\times\sum_{v=0}^{\infty}\sum_{w=0}^{v}B_{vw}P_v^w(\cos\bar{\theta}_2)\cos(w\bar{\phi}_2)\bar{\rho}_2^2\sin\bar{\theta}_2\,\mathrm{d}\bar{\theta}_2\,\mathrm{d}\bar{\phi}_2 \\ &+\int_0^{2\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{-\mathrm{i}k}{4\pi}(2n+1)\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi_2-\bar{\phi}_1)] \\ &\times P_n^m(\cos\theta_2)P_n^m(\cos\bar{\theta}_1)h_n^{(2)}(k\rho_2)(j_n(k\bar{\rho}_1)+\mathrm{i}k^2j_n'(k\bar{\rho}_1)) \\ &\times\sum_{v=0}^{\infty}\sum_{w=0}^{v}A_{vw}P_v^w(\cos\bar{\theta}_1)\cos(w\bar{\phi}_1)\bar{\rho}_1^2\sin\bar{\theta}_1\,\mathrm{d}\bar{\theta}_1\,\mathrm{d}\bar{\phi}_1. \end{aligned} \tag{4.24b}$$

Similarly, the arbitrariness of the source intensities ensures the following relation:

$$j_n(k\rho_1)y_n(k\rho_2)-j_n(k\rho_2)y_n(k\rho_1) = 0, \quad n = 0, 1, 2, \ldots. \tag{4.25}$$

Also, it is noticed that only true eigenvalues are contained in the even three-dimensional domain with interior hole. In a similar way to equations (4.13) and

Table 1. The first 10 eigenvalues for the circular problem with different nodes compared with the analytical solution.

	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9	k_{10}
MFS (20 nodes)	2.4039	3.8309	5.1340	5.5200	6.3801	7.0180	7.5905	8.4210	8.6590	8.7900
MFS (30 nodes)	2.4049	3.8319	5.1360	5.5200	6.3801	7.0160	7.5880	8.4170	8.6540	8.7720
MFS (40 nodes)	2.4049	3.8319	5.1360	5.5200	6.3801	7.0160	7.5880	8.4170	8.6540	8.7720
MFS (50 nodes)	2.4049	3.8319	5.1360	5.5200	6.3801	7.0160	7.5880	8.4170	8.6540	8.7720
analytical solutions	2.4048	3.8317	5.1356	5.5201	6.3802	7.0156	7.5883	8.4172	8.6537	8.7715

(4.25), we obtain that spurious eigenvalues occurred at $j_n(kR_1)=0$ when conventional MFS is applied.

(*f*) *Case VI: concentric spherical problem with the Neumann boundary condition*

A similar procedure can be applied to the Neumann Helmholtz problem in the concentric spherical domain. The resulted eigenequation is

$$j_n'(k\rho_1)y_n'(k\rho_2) - j_n'(k\rho_2)y_n'(k\rho_1) = 0, \quad n = 0, 1, 2, \ldots. \tag{4.26}$$

5. Numerical results and discussions

In this section, numerical experiments are carried out and results are discussed and compared with the solutions studied analytically in §4. Further studies are performed to extend to the three-dimensional peanut shaped domain.

(*a*) *Case I: circular problem with the Dirichlet boundary condition*

A circular cavity with a unit radius, $\rho=1$, subjected to the Dirichlet boundary condition is considered as depicted in figure 2*a*. Table 1 shows the first 10 eigenvalues obtained by the direct determinant search method depicted in figure 4. In figure 4, it is observed that there are no spurious eigenvalues as we have studied analytically in the last section, as addressed in equation (4.10).

(*b*) *Case II: annular problem with the Dirichlet boundary condition*

Next, the annular problem with the Dirichlet boundary condition is considered, where $\rho_1=0.5$ and $\rho_2=2$ (figure 2*b*). Table 2 describes the first five eigenvalues obtained by the direct determinant search method and the comparisons with the analytical solutions of equation (4.15). Figure 5*a*,*b* addresses the results obtained by the present method (3.2) and the conventional method (2.5). It is observed in the figures that the spurious eigenvalues are exactly predicted by equation (4.13), which is analytically derived in §4.

(*c*) *Case III: annular problem with the Neumann boundary condition*

To demonstrate the capability of the present method for the Neumann problems, an annular Helmholtz problem with the Neumann boundary condition

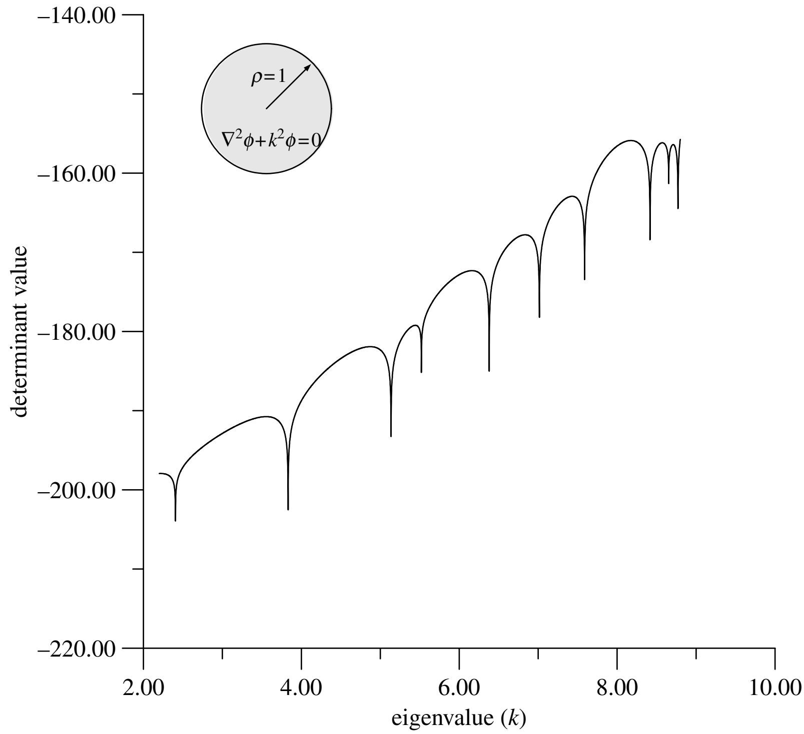


Figure 4. The determinant value versus k for the circular problem (nodes=50, R=1.3).

Table 2. The first five eigenvalues for the annular Dirichlet problem with different methods (nodes=60, R_1=0.4, R_2=2.3).

eigenvalue	numerical methods					analytical solutions
	FEM (Chen *et al.* 2003*b*)	BEM (Chen *et al.* 2003*b*)	BEM+chief (Chen *et al.* 2003*b*)	fictitious BEM (Chen *et al.* 2003*b*)	MFS	
k_1	2.03	2.06	2.04	2.04	2.05	2.05
k_2	2.2	2.23	2.2	2.21	2.22	2.23
k_3	2.62	2.67	2.65	2.66	2.66	2.66
k_4	3.15	3.22	3.22	3.21	3.21	3.21
k_5	3.71	3.81	3.81	3.80	3.80	3.80

is considered, where $\rho_1=0.5$ and $\rho_2=2$ (figure 2*b*). Table 3 addresses the first few eigenvalues obtained by the direct determinant search method. The results in the table also show the excellent performance of the present method as compared with the analytical solutions of equation (4.16).

(*d*) *Case IV: spherical problem with the Dirichlet boundary condition*

To validate the method for the three-dimensional Helmholtz problem, a spherical cavity with a unit radius, $\rho=1$, subjected to the Dirichlet boundary

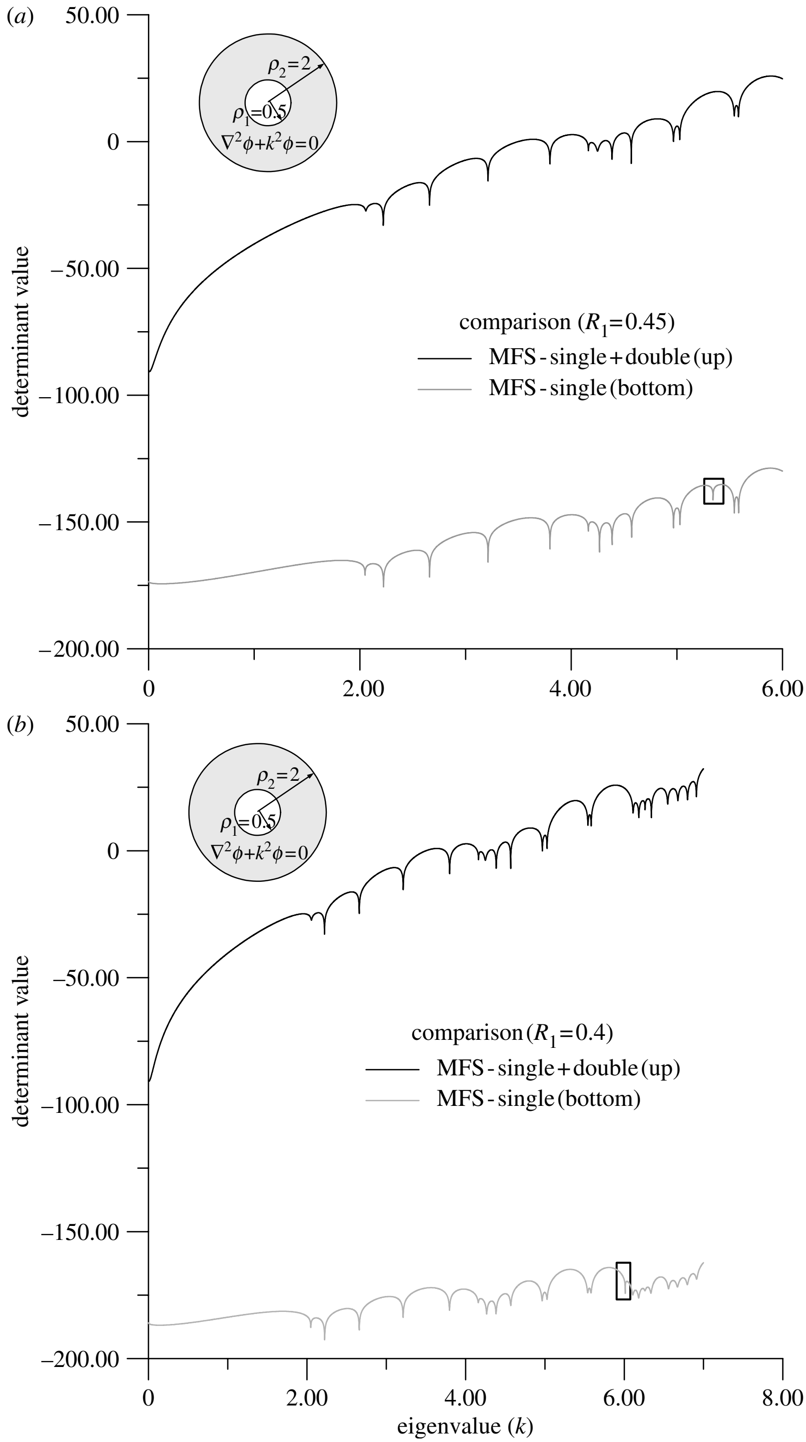


Figure 5. The true and spurious eigenvalues by two methods. (*a*) Nodes $=60$, $R_1=0.45$, $R_2=2.3$; (*b*) nodes $=60$, $R_1=0.4$, $R_2=2.3$.

Table 3. The first few eigenvalues for the annular Neumann problem with analytical solutions (nodes = 80, R_1 = 0.2, R_2 = 2.5).

	eigenvalue						
	k_1	k_2	k_3	k_4	k_5	k_6	k_7
MFS	0.825	1.505	2.095	2.225	2.505	2.655	3.175
analytical solutions	0.823	1.504	2.096	2.223	2.501	2.658	3.178
	k_8	k_9	k_{10}	k_{11}	k_{12}	k_{13}	k_{14}
MFS	3.205	3.755	3.935	4.270	4.405	4.615	4.815
analytical solutions	3.207	3.753	3.932	4.269	4.403	4.619	4.811

Table 4. The first nine eigenvalues for the three-dimensional sphere problem with analytical solution (nodes = 600, ρ = 1, $\bar{\rho}$ = 1.2).

	eigenvalue								
	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9
MFS	3.141	4.493	5.763	6.282	6.987	7.725	8.182	9.095	9.356
analytical solutions	3.141	4.493	5.763	6.283	6.987	7.725	8.183	9.095	9.356

condition is considered (figure 3*a*). Table 4 addresses the first nine eigenvalues obtained by the direct determinant search method sketched in figure 6. The results also show good performance of the present method for the three-dimensional problem compared to the analytical solutions of equation (4.22).

(*e*) *Case V: concentric spherical problem with the Dirichlet boundary condition*

Next, we consider a three-dimensional concentric spherical problem. A concentric spherical domain with Dirichlet boundary condition is studied, in which $\rho_1 = 0.5$ and $\rho_2 = 1$ (figure 3*b*). Table 5*a* shows the first few eigenvalues obtained by the direct determinant search method depicted in figure 7*a*,*b*. In the figure, it is also observed that there is no spurious eigenvalues by utilizing the combination of single- and double-layer potentials.

(*f*) *Case VI: concentric spherical problem with the Neumann boundary condition*

Similarly, we consider a Neumann problem for the three-dimensional domain with interior holes. A concentric spherical domain with the Neumann boundary condition is studied, where $\rho_1 = 0.5$ and $\rho_2 = 1$ (figure 3*b*). The numerical results also show the excellent performance of the present method for the three-dimensional Neumann problem. Table 5*b* describes the first few eigenvalues obtained by the direct determinant search method.

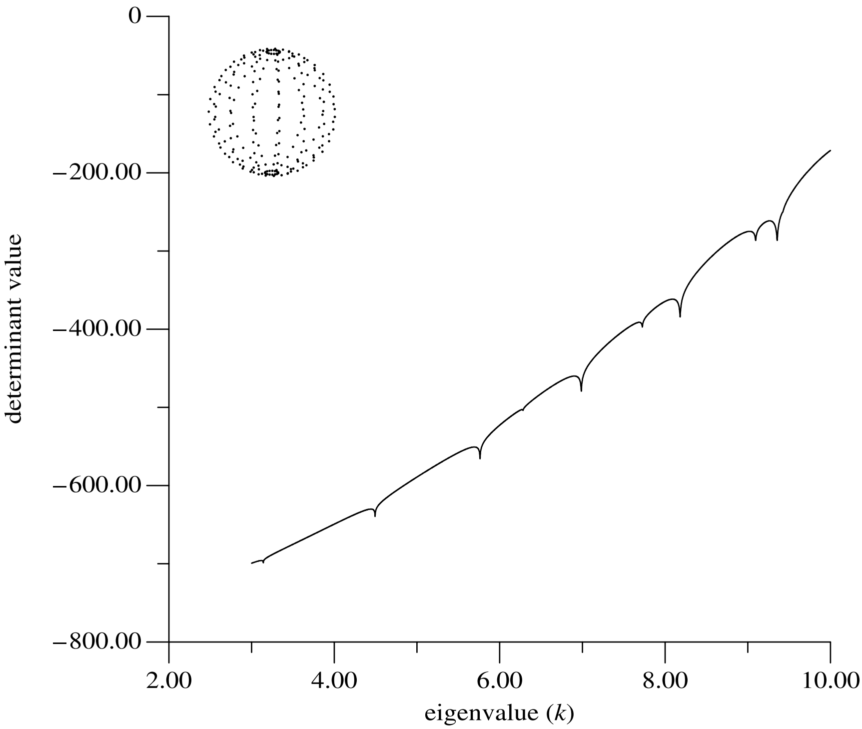


Figure 6. The determinant value versus k for the three-dimensional sphere problem (nodes = 600, $\bar{\rho} = 1.2$, $\rho = 1$).

Table 5. The first few eigenvalues for the three-dimensional (*a*) Dirichlet, and (*b*) Neumann problem in the concentric spherical domain with analytical solution (nodes = 400, $\rho_1 = 0.5$, $\rho_2 = 1$, $\bar{\rho}_1 = 0.35$, $\bar{\rho}_2 = 1.5$).

	eigenvalue						
(*a*)	k_1	k_2	k_3	k_4	k_5	k_6	
MFS	6.284	6.573	7.112	7.846	8.716	9.682	
analytical solutions	6.283	6.572	7.112	7.845	8.711	9.682	
(*b*)	k_1	k_2	k_3	k_4	k_5	k_6	k_7
MFS	1.841	3.151	4.390	5.573	6.575	6.716	6.912
analytical solutions	1.842	3.151	4.392	5.573	6.571	6.717	6.911

(*g*) *Case VII: peanut shaped domain*

In order to demonstrate the flexibility of the proposed numerical method to treat irregular domains, a three-dimensional peanut shaped computational domain (Chen *et al.* 2003*a*) is chosen as the last problem. The first few eigenvalues are found by the direct determinant search method as shown in figure 8. Also, in table 6 the results for different source points by the proposed

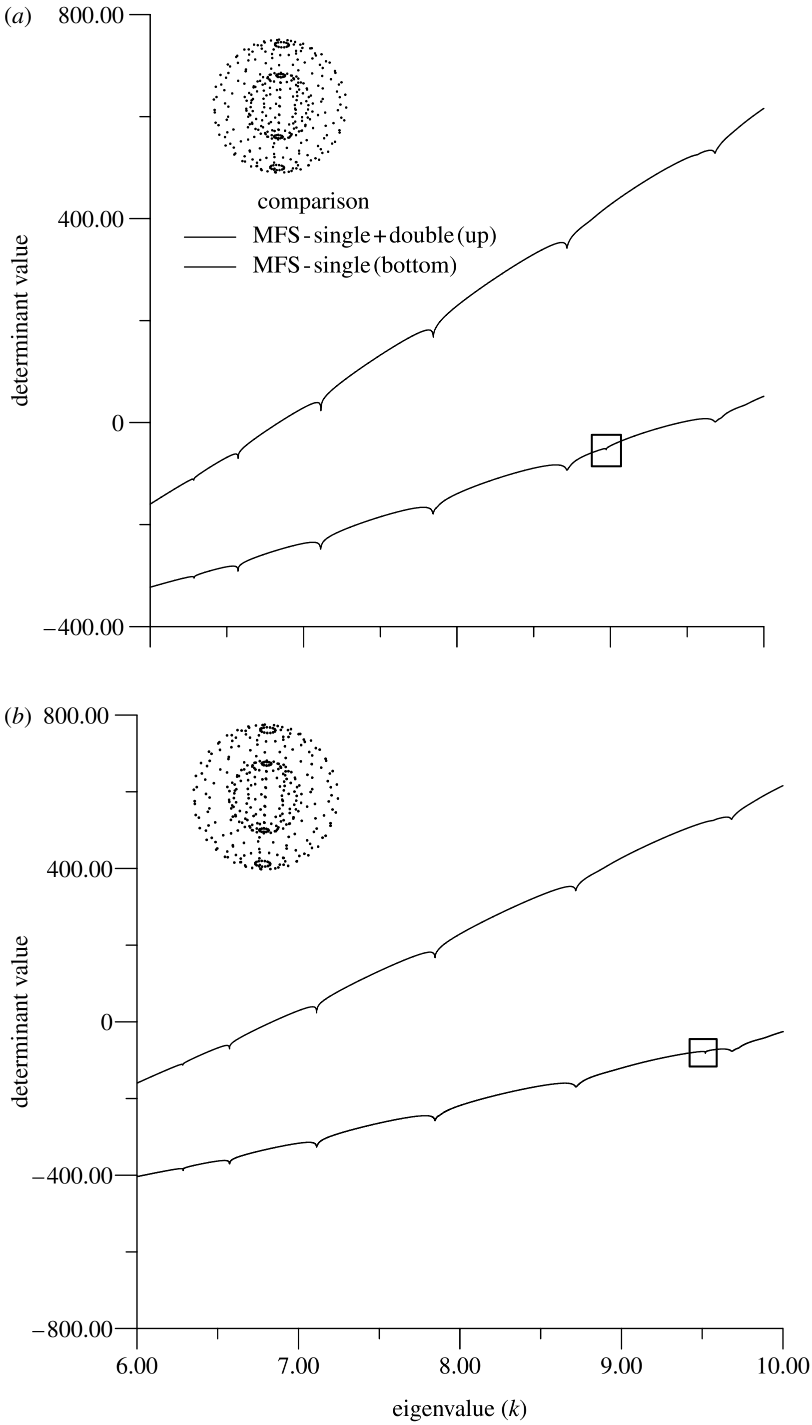


Figure 7. The true and spurious eigenvalues by two methods for the three-dimensional concentric spherical problem with Dirichlet boundary. (*a*) Nodes = 400, $\rho_1 = 0.5$, $\rho_2 = 1$, $\bar{\rho}_1 = 0.35$, $\bar{\rho}_2 = 1.5$; (*b*) nodes = 400, $\rho_1 = 0.5$, $\rho_2 = 1$, $\bar{\rho}_1 = 0.33$, $\bar{\rho}_2 = 1.5$.

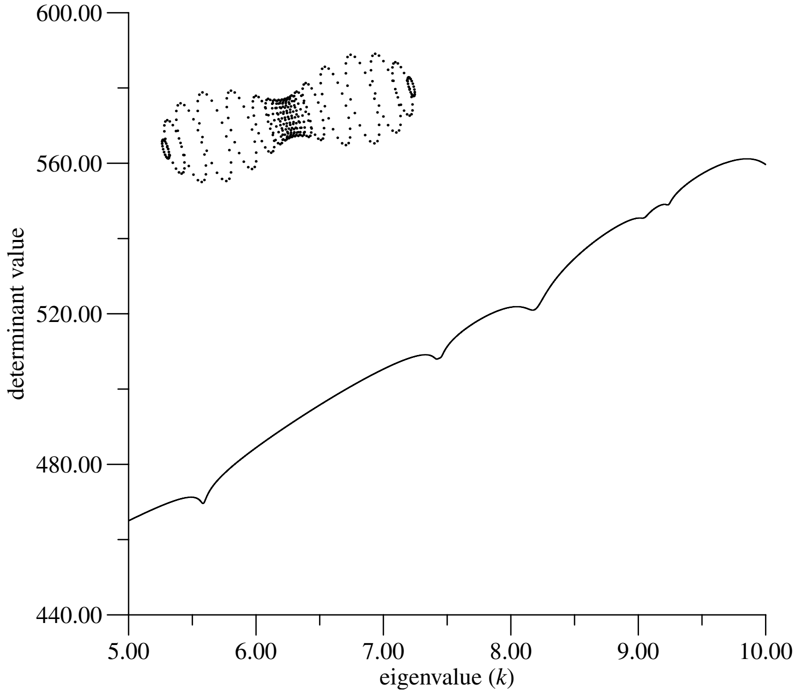


Figure 8. The determinant value versus k for the three-dimensional peanut shaped domain problem with Dirichlet boundary.

Table 6. The first five eigenvalues for the three-dimensional peanut shaped problem for different nodes by MFS and comparison with BEM.

	eigenvalue				
	k_1	k_2	k_3	k_4	k_5
MFS (300 nodes)	5.584	7.419	8.172	9.033	9.232
MFS (450 nodes)	5.646	7.498	8.310	9.168	9.309
MFS (600 nodes)	5.671	7.520	8.340	9.185	9.331
MFS (875 nodes)	5.671	7.519	8.341	9.187	9.337
BEM (2048 elements)	5.693	7.608	8.365	9.281	9.493

method are addressed and compared with the results obtained by using the BEM with 2048 constant elements. By observing these results, it is found that the proposed method is applicable to the problem of the three-dimensional irregular domain.

(*h*) *Investigation of the ill-conditioned problem in MFS*

The ill-conditioned problem always happens in the MFS as the number of collocation point increases. The eigenvalues obtained by different collocation points are depicted in figures 9*a*,*b* and 10*a*,*b*, respectively. In the above studies,

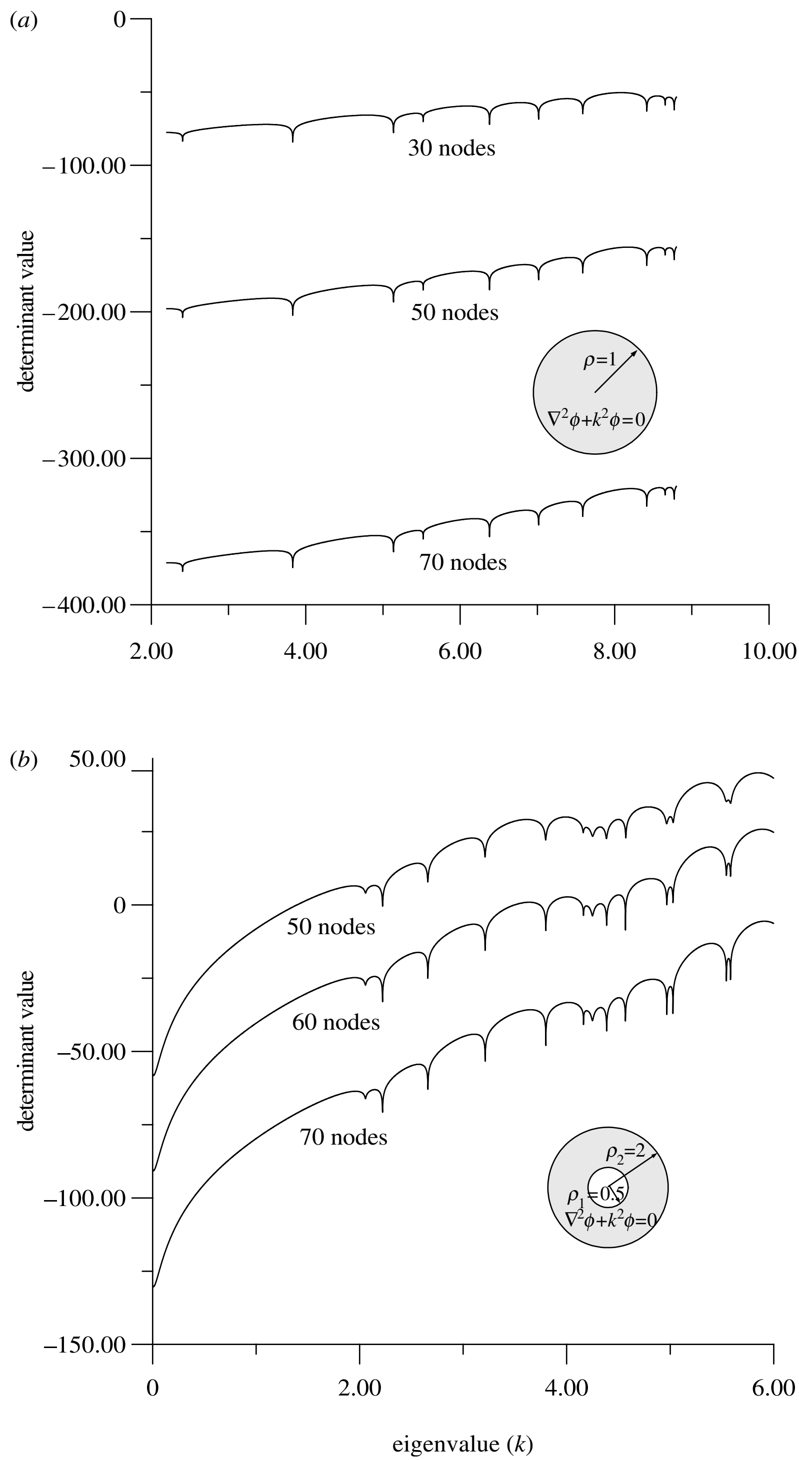
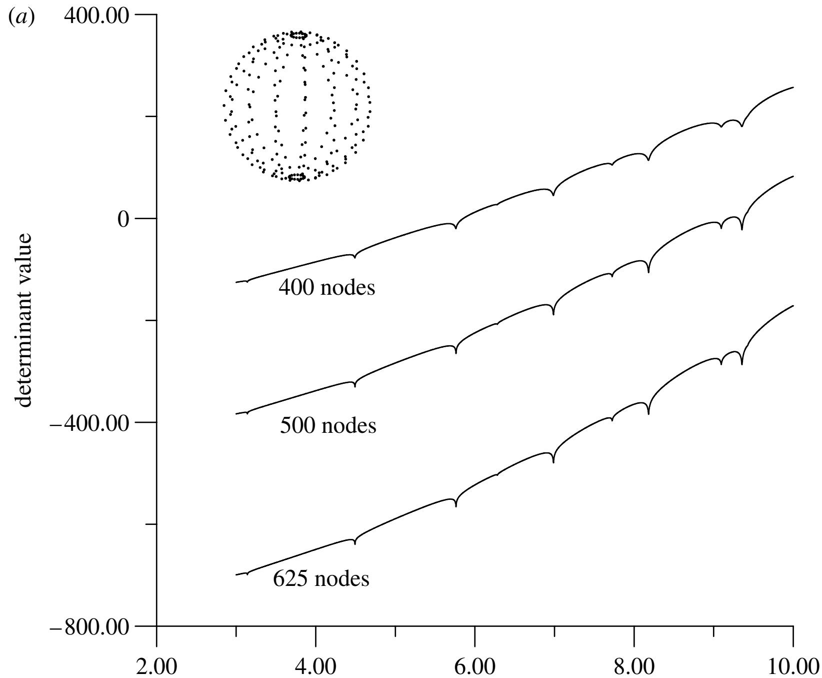


Figure 9. The determinant value versus k by different numbers of the collocation point in (a) a circular domain, and (b) an annular domain.



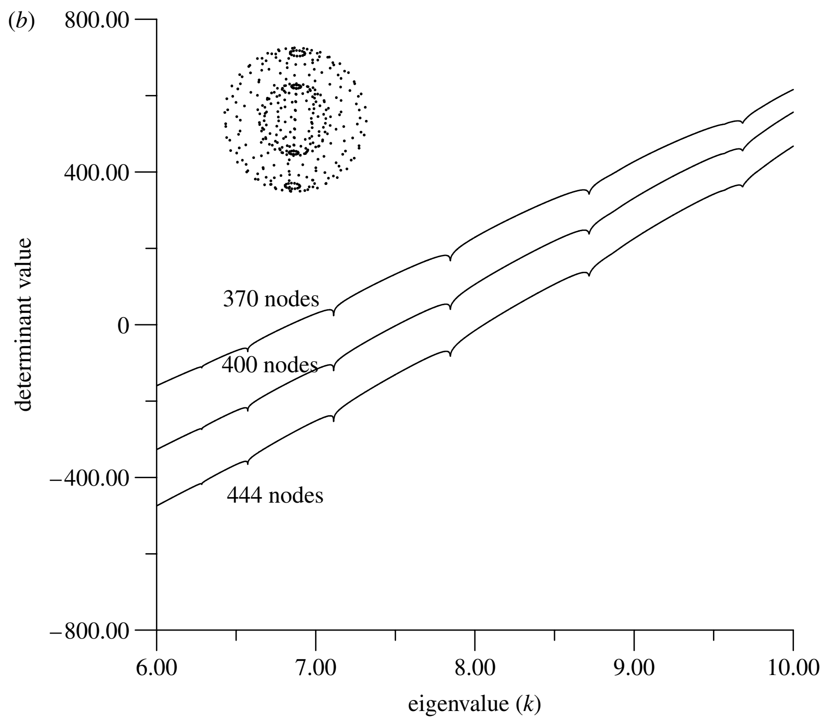


Figure 10. The determinant value versus k by different collocation points in (a) a spherical domain, and (b) a concentric spherical domain.

the ill-conditioned problem will not happen within the range of the collocation point in these cases. The consistency and well-conditioned matrix of the proposed method is demonstrated.

6. Conclusions

We have developed the MFS for eigenproblems in two-dimensional and three-dimensional domains with and without interior holes. We are convinced that there are no spurious eigenvalues if the complex-valued kernels are adopted for domains without interior holes, or the combination of the MFS and the mixed potential method is utilized for domains with interior holes. To demonstrate this fact, the continuous version of the MFS with the degenerate kernels and Fourier series are applied to derive the eigenequations for circular, annular, spherical, and concentric spherical domains. Later, some numerical results also supported the same point. Further studies are also performed in a three-dimensional peanut shaped problem. Moreover, we studied the issue of spurious eigenvalues analytically and numerically for the annular Dirichlet problem by utilizing the conventional MFS and also BEM in a three-dimensional peanut shaped domain. As a meshless numerical method, the scheme is free from meshes, singularities, and numerical integrations.

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