A New Numerical Approach to the Calculation of Electromagnetic Scattering Properties of Two-Dimensional Bodies of Arbitrary Cross Section

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Abstract—A new method is introduced for formulating the scattering problem in which the scattered fields (and the interior fields in the case of a dielectric scatterer) are represented in an expansion in terms of free-space modal wave functions in cylindrical coordinates, the coefficients of which are the unknowns. The boundary conditions are satisfied using either an analytic continuation procedure, in which the far-field pattern (in Fourier series form) is continued into the near field and the boundary conditions are applied at the surface of the scatterer; or the completeness of the modal wave functions, to approximately represent the fields in the interior and exterior regions of the scatterer directly. The methods were applied to the scattering of two-dimensional cylindrical scatterers of arbitrary cross section and only the TM polarization of the excitation is considered. The solution for the coefficients of the modal wave functions are obtained by inversion of a matrix which depends only on the shape and material of the scatterer. The methods are illustrated using perfectly conducting square and elliptic cylinders and elliptic dielectric cylinders. A solution to the problem of multiple scattering by two conducting scatterers is also obtained using only the matrices characterizing each of the single scatterers. As an example, the method is illustrated by application to a two-body configuration.

Introduction

Conventional approach to the solution of the electromagnetic scattering problem entails the formulation of an integral equation for the induced surface current density in the case of a perfectly conducting scatterer, or the induced polarization current for a dielectric scatterer. The scattered field, which is often the quantity of primary interest, is then calculated from the knowledge of the induced surface or polarization current.

Here we present two methods for formulating the problem such that the Fourier coefficients of the far-field scattering pattern are the primary unknowns. Thus the pattern may be directly obtained upon solution of the problem. Furthermore, the number of significant unknowns may be determined *a priori* from the size of the body, and then the boundary conditions applied at as many points on the contour of the scatterer as necessary. This enables us to obtain a set of Fourier pattern coefficients which are best in a least-square-error sense. However, in the usual integral equation approach, if one desires to apply the boundary conditions at more points on the contour of the scatterer, the number of unknowns is generally increased.

Although the principle of the methods to be discussed apply to the general case of an arbitrary scatterer, the basic approach will be illustrated by considering the two-dimensional problem of TM scattering from cylindrical scatterers of arbitrary cross section. The incident field is assumed to be a plane wave with the conventional $\exp(j\omega t)$ harmonic time dependence.

Analytic Continuation of Fields

Consider the two-dimensional scatterer S whose cross section is shown in Fig. 1 with a TM electric field E_z^{inc} incident. The vector $\bar{\rho} = (\rho, \phi)$ locates points in cylindrical coordinates on the contour ∂S of the scatterer or in the region outside the scatterer.

Assume that the z component of the scattered electric field is written as an expansion in the cylindrical Hankel function of the second kind so that the radiation condition is satisfied,

$$E_z^S(\bar{\rho}) = \sum_{n=-\infty}^{\infty} a_n j^{-n} H_n^{(2)}(k\rho) \exp(jn\phi). \tag{1}$$

Then in the far field, we use the asymptotic expansion of the Hankel function

$$\lim_{\rho\to\infty} H_n^{(2)}(k\rho) \sim \left(\frac{2j}{\pi k\rho}\right)^{1/2} j^n \exp(-jk\rho)$$

and obtain

$$E_z^S(\bar{\rho}) \underset{\rho\to\infty}{\longrightarrow} \left(\frac{2j}{\pi k\rho}\right)^{1/2} \exp(-jk\rho) \sum_{n=-\infty}^{\infty} a_n \exp(jn\phi).$$

Now the far-field scattering pattern is defined by

$$g(\phi) = \sum_{n=-\infty}^{\infty} a_n \exp(jn\phi) \tag{2}$$

where the a_n are the unknown Fourier expansion coefficients to be determined. While we assume that the scat-

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tering pattern is the primary quantity of interest, once the unknowns a_n are determined even the near field may be calculated according to (1). Equation (1), however, may not be valid for all values of ρ less than the minimum radius $\rho_{\min}$ which encloses the body (Fig. 1). As pointed out by Millar [1], the formal series diverges inside any circle which encloses a singularity of the scattered field, or what might be considered to be the image source of the incident field. The location of the singularities depends on the excitation and the geometry, but since the singularities of the scattered field must be contained inside the scatterer S, we are assured the series will converge outside the radius $\rho_{\min}$.

Let us assume for illustration that we are concerned with the scattering by a perfectly conducting cylinder. In Fig. 1 it is obvious that the region of convergence of the scattered field expression depends on the location of the coordinate origin. If the origin is translated to a new location at $\bar{\rho}_0 = (\rho_0, \phi_0)$ with respect to the original system as shown, we may again write the field in terms of a convergent series of outgoing wave functions in the new (primed) coordinate system, viz.,

$$E_z^S(\bar{\rho}_0 + \bar{\rho}') = \sum_{m=-\infty}^{\infty} a_m' j^{-m} H_m^{(2)}(k\rho') \exp(jm\phi'). \quad (3)$$

In order to express (1) in this form and still retain the original expansion coefficients as unknowns, the addition theorem for Hankel functions is invoked (Stratton [2]),

$$H_n^{(2)}(k\rho) \exp(jn\phi) = \sum_{m=-\infty}^{\infty} \begin{Bmatrix} H_{n-m}^{(2)}(k\rho_0) \\ J_{n-m}(k\rho_0) \end{Bmatrix} \cdot \exp[j(n-m)\phi_0] \begin{Bmatrix} J_m(k\rho') \\ H_m^{(2)}(k\rho') \end{Bmatrix} \cdot \exp(jm\phi'), \quad \text{for} \begin{Bmatrix} \rho_0 > \rho' \\ \rho_0 < \rho' \end{Bmatrix}. \quad (4)$$

If the original coordinate origin lies inside the body, then ρ_0 is always less than ρ' and we may use (4) in (1) to write

$$E_z^S(\bar{\rho}_0 + \bar{\rho}') = \sum_{n=-\infty}^{\infty} a_n j^{-n} \Big[\sum_{m=-\infty}^{\infty} J_{n-m}(k\rho_0) \cdot \exp[j(n-m)\phi_0] H_m^{(2)}(k\rho') \cdot \exp(jm\phi') \Big]. \quad (5)$$

Now in the region where both the conditions $|\bar{\rho}| = |\bar{\rho}_0 + \bar{\rho}'| > \rho_{\min}$ and $|\bar{\rho}'| > |\bar{\rho}_0|$ are satisfied the series in the brackets is absolutely convergent so that we may interchange the order of summation to get

$$E_z^S(\bar{\rho}_0 + \bar{\rho}') = \sum_{m=-\infty}^{\infty} \Big[\sum_{n=-\infty}^{\infty} a_n j^{n-m} J_{m-n}(k\rho_0) \cdot \exp[j(n-m)\phi_0] j^{-m} H_m^{(2)}(k\rho') \cdot \exp[jm\phi'] \Big]. \quad (6)$$

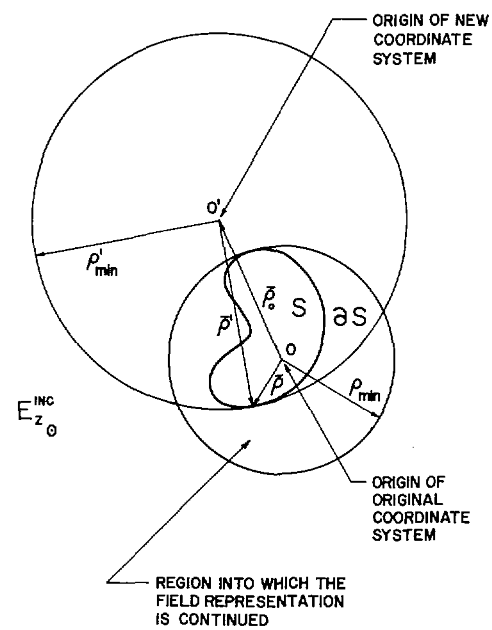

Fig. 1. Geometry for two-dimensional scatterer with TM wave incident.

In regions where *both* (6) and (3) converge we may identify (by the orthogonality of $\exp(jm\phi')$ on a circle of large enough radius) the coefficients of (3) as

$$a_m' = \sum_{n=-\infty}^{\infty} a_n j^{n-m} J_{m-n}(k\rho_0) \exp[j(n-m)\phi_0]. \quad (7)$$

Since (6) is convergent for all $\rho' > \rho_{\min}'$ we may use it to apply the boundary condition $E_z^S + E_z^{\text{inc}} = 0$ at the point on the boundary where the circle $\rho_{\min}'$ is tangent. This process is essentially analytic continuation of the field representation.

Generally the largest number $|n| = N$ of the spectrum of coefficients a_n which is significant is on the order of about $k\rho_{\min}$. For $|n| > k\rho_{\min}'$ the coefficients decay exponentially faster than the exponential growth of the Hankel functions for $\rho > \rho_{\min}$ so that the scattered field representation converges. Furthermore the largest number of significant terms of $j^{-n}J_n(k\rho_0) \exp(-jn\phi_0)$ is about $|n| = k\rho_0$ because of the so-called spatial filtering of the Bessel wavefunctions. Owing to the convolution form of these two spectra in (7), we conclude that the largest number $|m| = M$ of the coefficients a_m' which are significant is about $k\rho_{\min} + k\rho_0$. Thus we may determine *a priori* the number of terms needed in (6). We simply choose an M somewhat larger than $k\rho_0 + k\rho_{\min}$ and an N somewhat larger than $k\rho_{\min}$, truncate the series and reverse the order of summation, writing

$$E_z^S(\bar{\rho}_0 + \bar{\rho}') = \sum_{n=-N}^{N} a_n \Big\{ \sum_{m=-M}^{M} j^{n-m} J_{m-n}(k\rho_0) \cdot \exp[j(n-m)\phi_0] j^{-m} H_m^{(2)}(k\rho') \cdot \exp(jm\phi') \Big\}. \quad (8)$$

The process described, of changing coordinate origins and analytically continuing the fields, may be used to find the value of the scattered field, at any point on a convex body. Thus if we designate the coordinates of $\bar{\rho}$, $\bar{\rho}_0$, and $\bar{\rho}'$ with subscripts i to denote their values for each different point on the body where the scattered field is matched to the negative incident field, we have a system of equations which may be written in the matrix form

$$[\beta_{in}][a_n] = -[E_i^{\text{inc}}] \tag{9}$$

where

$$\beta_{in} = \sum_{m=-M}^{M} j^{n-m} J_{m-n}(k\rho_{i0}) \cdot \exp[j(n-m)\phi_{i0}] j^{-m} H_m^{(2)}(k\rho_i') \exp(jm\phi_i') \tag{10}$$

$$E_i^{\text{inc}} = E_z^{\text{inc}}(\rho_i,\phi_i), \qquad i=1,2,\cdots,P. \tag{11}$$

If the number of points matched P is equal to the number of coefficients $2N+1$, then the solution may be found by inverting the matrix $[\beta_{in}]$ which must be nonsingular due to the uniqueness condition for scattering (Jones [3]). Thus we write

$$[a_n] = -[\beta_{in}]^{-1}[E_i^{\text{inc}}]. \tag{12}$$

In order to make the solution less dependent on the particular choice of points where the field is matched, one may attempt to satisfy (9) at many points such that there are more equations than unknowns. A set of coefficients a_n can then be found which satisfies (9) in a least-square-error sense. The solution is

$$[a_n] = -\{[\beta_{in}]^\dagger[\beta_{in}]\}^{-1}[\beta_{in}]^\dagger[E_i^{\text{inc}}] \tag{13}$$

where the dagger denotes the transpose conjugate of the matrix. The quantity $\{[\beta_{in}]^\dagger[\beta_{in}]\}^{-1}[\beta_{in}]^\dagger$ is called the "pseudoinverse" (Penrose [4], [5]) of the matrix $[\beta_{in}]$. Of course once the coefficients a_n have been found by either (12) or (13) the scattering pattern is calculated by (2).

The possibility of representing the scattered field in terms of outgoing cylindrical wave functions (which we will designate as an "outside expansion") at every point on a body requires that the body must be convex and that the largest radius of curvature of the body not be so large that ρ_0 must be chosen large. If ρ_0 is too large, then the number of terms needed to calculate the matrix element may become prohibitively large.

A simple modification of the preceeding method is available which will always work and is generally preferable. Instead of expanding the field in the new coordinate system in terms of outgoing wave functions, we expand in terms of the cylindrical wave functions which are valid inside the largest circle of radius $\rho_{\max}'$ which excludes the scatterer (Fig. 2). This field expansion involves the ordinary Bessel functions of the first kind,

$$E_z^S(\bar{\rho}_0 + \bar{\rho}') = \sum_{m=-\infty}^{\infty} a_m' j^{-m} J_m(k\rho') \exp(jm\phi'). \tag{14}$$

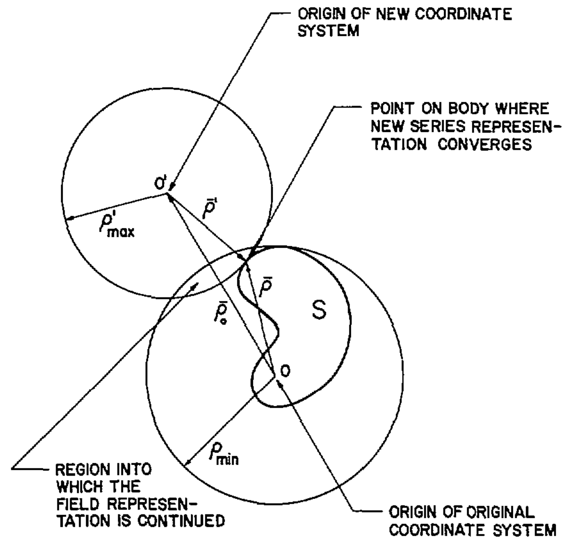

Fig. 2. Geometry for inside expansion using analytic continuation.

To express (1) in this form, we again use the addition theorem (4) this time noting that $\rho_0 > \rho'$. Using analytic continuation arguments we obtain the inside expansion equivalent of (7),

$$a_m' = \sum_{n=-\infty}^{\infty} a_n j^{n-m} H_{m-n}^{(2)}(k\rho_0) \exp[j(n-m)\phi_0]. \tag{15}$$

It should be noted that the analytic continuation argument will be valid only if $\rho_0 > \rho_{\min}$ so that there exists at least a small circle about the new coordinate origin which lies entirely inside a region where the series representation is valid in both the original and the new coordinate system. Such a condition was not needed for the outside expansion since we needed only to find a large enough circle about the new coordinate system on which the field expressions were valid in either coordinate system.

Upon truncating the series and interchanging the order of summation one obtains the $[\beta_{in}]$ matrix for the inside expansion,

$$\beta_{in} = \sum_{m=-M}^{M} j^{n-m} H_{m-n}^{(2)}(k\rho_{i0}) \cdot \exp[j(n-m)\phi_{i0}] j^{-m} J_m(k\rho_i') \exp(jm\phi_i'). \tag{16}$$

Note that the radius vectors ρ_{i0} and ρ_i' are not the same for matching a given point on the scatterer with the inside expansion as for the outside expansion. However, for points on a body where either expansion is appropriate, either one may be used so that the matrix $[\beta_{in}]$ may contain a mixture of the two types of expansions.

One further problem is that of bodies which are concave to the extent that no coordinate system can be found which would enable one to match the field at some point on the body. An example is the body in Fig. 3 where it is desired to match fields at point P. An auxiliary coordinate system is chosen at $\bar{\rho}_0$ which extends the field part of the way to the boundary as shown. Then a new coordinate system at $\bar{\rho}_0'$ is chosen with respect to the auxiliary system. The fields in this

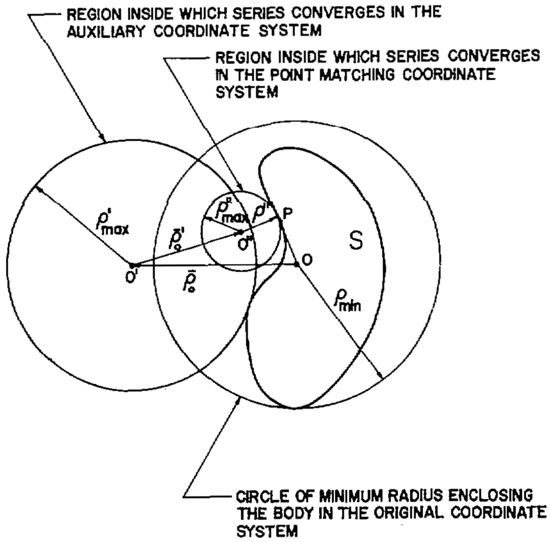

Fig. 3. Use of auxiliary coordinate system to match interior points of concave scatterer.

system may be expanded in terms of the fields in the auxiliary system and the analytic continuation argument used to find the coefficients of expansion. Thus the field in the auxiliary system is written as in (14),

$$E_z^S(\bar{\rho}_0 + \bar{\rho}') = \sum_{m=-\infty}^{\infty} a_m' j^{-m} J_m(k\rho') \exp(jm\phi'). \quad (17)$$

Now we employ the addition theorem for Bessel functions of the first kind (Stratton [2]),

$$J_m(k\rho') \exp(jm\phi') = \sum_{p=-\infty}^{\infty} J_{m-p}(k\rho_0') \cdot \exp[j(m-p)\phi_0'] J_p(k\rho'') \cdot \exp(jp\phi''). \quad (18)$$

This may be substituted into the expression in the auxiliary coordinates and the summations interchanged in the common regions of absolute convergence to get

$$E_z^S(\bar{\rho}_0 + \bar{\rho}_0' + \bar{\rho}'') = \sum_{p=-\infty}^{\infty} \left\{ \sum_{m=-\infty}^{\infty} a_m' j^{p-m} J_{m-p}(k\rho_0') \cdot \exp[j(p-m)\phi_0'] \right\} j^{-p} J_p(k\rho'') \cdot \exp(jp\phi''). \quad (19)$$

The bracketed expression may now be recognized as the expansion coefficient in the double-primed coordinate system which is valid at point P. Of course, we have already expressed the a_m' coefficients in terms of the unknown coefficients a_n in (15). When the summations are suitably truncated and the summation over n is interchanged, the result is

$$E_z^S(\bar{\rho}_0 + \bar{\rho}_0' + \bar{\rho}'') \simeq \sum_{n=-N}^{N} a_n \left\{ \sum_{p=-P}^{P} \sum_{m=-M}^{M} (j^{n-m} H_{m-n}^{(2)}(k\rho_0) \cdot \exp[j(n-m)\phi_0] (j^{p-m} J_{m-p}(k\rho_0') \cdot \exp[j(p-m)\phi_0'])(j^{-p} J_p(k\rho'') \exp(jp\phi'')) \right\}. \quad (20)$$

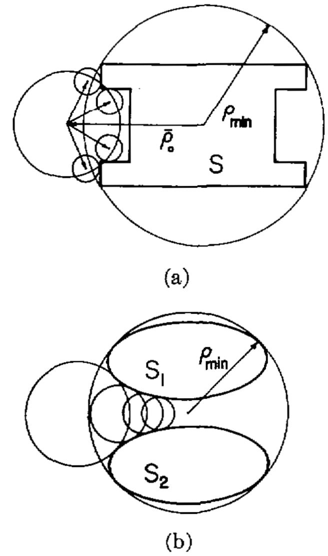

Fig. 4. Use of branches and chains from auxiliary coordinate system. (a) Branching from auxiliary coordinate system. (b) Chains of auxiliary coordinate systems which are also used for point matching.

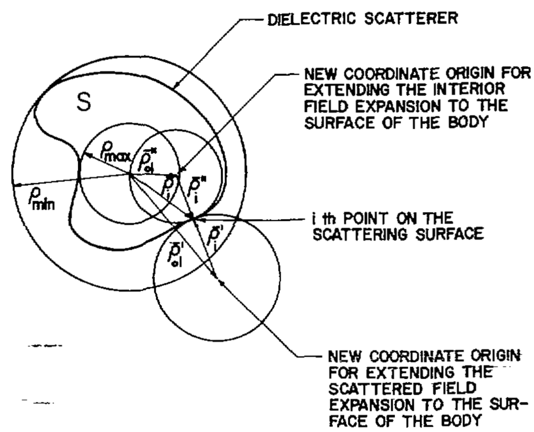

Fig. 5. Geometry of coordinate systems used in dielectric scattering problem.

Obviously, the computational difficulty necessary to match the field at one point is considerable when this method must be used. Fortunately, in many instances the auxiliary coordinate system may be used to calculate the fields at several points on a body by branching out from it as in Fig. 4(a). Particularly difficult points to reach may require a chain of auxiliary coordinate systems as shown for the two elliptic cylinder scatterers in Fig. 4(b).

The analytic continuation method may also be extended to calculate scattering from a dielectric cylinder [6]. The procedure requires only that we begin with an inside expansion for the fields inside the cylinder in addition to the outside expansion for the scattered fields. Thus in Fig. 5 the field in the (ρ,ϕ) coordinate system in the interior of the body is expanded by

$$E_z^I(\bar{\rho}) = \sum_{m=-\infty}^{\infty} b_m j^{-m} J_m(k_d\rho) \exp(jm\phi) \quad (21)$$

where k_d is the wavenumber in the dielectric medium. The internal fields may be continued to the surface using inside expansions with the aid of any auxiliary coordinate transformations which might be needed. The exterior scattered field is also continued to the surface by any of the methods discussed for the perfectly conducting scatterer. The total tangential $\bar{E}$ field must be continuous at the dielectric interface. Thus for the ith point on the boundary the fields satisfy

$$E_z^S(\bar{\rho}_{0i}' + \bar{\rho}_i') - E_z^I(\bar{\rho}_{0i}'' + \bar{\rho}_i'') = -E^{\text{inc}}(\bar{\rho}_i) \quad (22)$$

where $\bar{\rho}_i'$, $\bar{\rho}_i''$, and $\bar{\rho}_i$ are defined as in Fig. 5. At the dielectric interface, the tangential component of the total $\bar{H}$ field must also be continuous. In the two dimensional TM case, the $\bar{H}$ field is given by

$$\bar{H} = \frac{\hat{z}}{j\omega\mu} \times \text{grad } E_z \quad (23)$$

where $\hat{z}$ is the unit vector along the z axis. Application of the boundary conditions at the ith point gives

$$\hat{n}_i \times [\hat{z} \times (\text{grad}_i' E_z^S - \text{grad}_i'' E_z^I)] = -\hat{n}_i \times [\hat{z} \times \text{grad}_i E_z^{\text{inc}}] \quad (24)$$

where $\hat{n}_i$ is the unit normal to the boundary at the ith point and grad_i', grad_i'', and grad_i are the gradient operators for the ith point in the (ρ_i',ϕ_i'), (ρ_i'',ϕ_i''), and (ρ_i,ϕ_i) coordinate systems, respectively (Fig. 5). When written out explicitly (22) and (24) constitute a matrix system for determining the unknown coefficients a_n and b_m of the scattered and internal fields, respectively.

Analytic continuation of the fields in the case of the dielectric cylinder is considerably more difficult and numerically time consuming than for the conducting cylinder. At least two new coordinate systems are needed for each point at which the field is matched and both the electric and magnetic fields must be calculated. For cases in which the dielectric cylinder is not too thin, a simpler representation of the fields near the scatterer is described in the next section.

Several advantages of analytic continuation over the usual methods are apparent:

1) No integrations are performed.
2) The size of the matrix to be inverted to give sufficient accuracy can be determined *a priori*.
3) The method works even for cases when the scatterer is resonant and surface current formulations fail.
4) No singularities in the fields arise in contrast to the singularities in surface current which arise, for example, at edges.
5) Upon solution of the matrix equation the scattered fields may be easily compared to the incident fields at the surface of the scatterer to determine how well the boundary conditions are met.

It should also be noted when comparing this method to the mode-matching method to be discussed in the next section, that analytic continuation is applicable to any scattering shape, whereas the mode-matching method works best for shapes which do not depart significantly from a circular cross section.

The major disadvantage of the method is the computational difficulty in calculating the matrix $[\beta_{in}]$. This is in part due to the time-consuming summations that must be performed for each matrix element plus the calculation of higher order Bessel and Hankel functions even when recursion relations are used to generate those functions. Furthermore, each point on a body requires the computation of several coordinates associated with it which must be carefully chosen to insure that the representation converges on the body.

The only analytic source of error which arises in this formulation is the truncation of the expansions in each coordinate system. As mentioned previously, the truncation size can be predetermined for each scatterer. Furthermore, upon solution of the problem we need only examine the rate and degree of decay of the scattering coefficients to be reasonably assured of the accuracy of the scattering pattern. In the surface current formulation, however, when moment methods with subsectional bases are used, the coefficients of expansion of the surface current are all of about the same order, and usually a larger matrix is solved to check convergence (Harrington [7]).

Mode Matching of Fields

For two-dimensional scatterers whose cross section does not depart significantly from a circular shape, the scattered field representation (1) may be used along the contour ∂S even if the formal series diverges for some points on the contour. However, the way (1) is used becomes critical here. It has been shown by several authors [8]–[10] that a truncation point N and a set of scattering coefficients $a_n(N)$ can always be found such that the mean-squared error in the scattered field representation on the contour,

$$\int_{\partial S} \left| E_z^S(\bar{\rho}) - \sum_{n=-N}^{N} a_n(N) j^{-n} H_n^{(2)}(k\rho) \exp(jn\phi) \right|^2 ds \quad (25)$$

can be made as small as desired. Thus we say that the field represented by the series in (25) converges in the mean (as N increases) to the true field in the region outside the scatterer S. The coefficients $a_n(N)$ have been written to show explicitly their dependence on N, because it is precisely this dependence which enables us to use this field representation everywhere without analytically continuing the fields. It is true that if we designate the exact scattered mode coefficient by a_n, the n $\lim_{N\to\infty} a_n(N) = a_n$. One finds in actual practice, however, that the higher order coefficients will not be relatively close to a_n in any truncated problem, and will generally be smaller in magnitude than the correct a_n. This relative error actually helps to keep the series field representation from diverging numerically. As N is increased, the coefficients in the first part of the series approach the correct coefficients a_n, so that the first part of the series begins to diverge in the near field. But this tendency is always corrected by the last few terms in the series whose coefficients are small, but whose scattered wave functions have a large contribution in the near field. The total finite sum actually does represent the field although the formal infinite sum representation may diverge.

Assuming the field may be so represented, the solution of (25) may be obtained in an approximate sense if the mean-squared error is minimized over a set of points on the body rather than over the entire surface. The solution of the perfectly conducting scatterer follows then as in (13) if now we redefine

$$\beta_{in} = j^{-n} H_n^{(2)}(k\rho_i) \exp(jn\phi_i). \quad (26)$$

The method may be extended to include the solution of the dielectric scattering problem by using (21) throughout the interior of the scatterer and applying boundary conditions (22) and (24) to obtain a matrix equation for the unknown scattering and internal cylindrical mode coefficients.

As the cross section of the scatterer is made thinner, one finds that the method described requires more and more coefficients to accurately represent the fields. Thus our *a priori* estimate of the number of coefficients required becomes less accurate. While only the first few coefficients may actually contribute to the far-field pattern, in order to obtain them accurately, a large matrix must be inverted. Furthermore, the larger matrix will have somewhat less desirable characteristics for computational purposes because of the order of the singularities in the higher order Hankel functions. For thin bodies therefore, the analytic continuation procedure may be more useful despite the large number of coordinate transformations required.

Scattering by Parallel Cylinders of Arbitrary Cross Section

Either the analytic continuation method or the method of direct mode matching of the fields may be extended to solve the problem of multiple scatterers. We illustrate the method by considering the scattering of two parallel perfectly conducting cylinders with a TM wave incident. One simply writes the scattered field in terms of two series of outgoing wavefunctions, each scatterer containing a coordinate origin for one set of wave functions. Then analytic continuation or direct mode matching is used to minimize the total tangential electric field on the boundary of both of the scatterers. (If one or both of the scatterers is dielectric, the field inside each dielectric scatterer is expanded in a series of regular wave functions and the difference between the tangential components of the total interior and exterior electric and magnetic fields is minimized on the contour of the scatterer.)

If, however, the inverse or pseudoinverse of the matrix $[\beta_{in}]$ of either (10) or (26) is already known for each scatterer one may formulate the problem such as to write the solution directly in terms of these matrices, which are independent of the incident field and depend only on the shape of the scatterer and the frequency. Hence we make the following definitions which refer to Fig. 6:

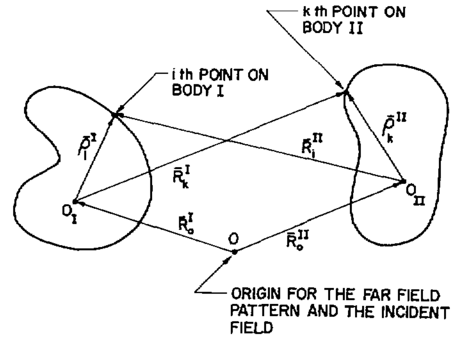

Fig. 6. Geometry for two cylindrical conducting scatterers.

$$E_{\rm I} = [E_i^{\rm I}]$$

$$E_i^{\rm I} = E_z^{\rm inc}(\bar{R}_0^{\rm I} + \bar{\rho}_i^{\rm I}) \quad (27a)$$

$$E_{\rm II} = [E_k^{\rm II}]$$

$$E_k^{\rm II} = E_z^{\rm inc}(\bar{R}_0^{\rm II} + \bar{\rho}_k^{\rm II}) \quad (27b)$$

$$H_{\rm I} = [H_{kn}^{\rm I}]$$

$$H_{kn}^{\rm I} = j^{-n} H_n^{(2)}(kR_k^{\rm I}) \exp(jn\phi_k^{\rm I}) \quad (27c)$$

$$H_{\rm II} = [H_{im}^{\rm II}]$$

$$H_{im}^{\rm II} = j^{-m} H_m^{(2)}(kR_i^{\rm II}) \exp(jm\phi_i^{\rm II}) \quad (27d)$$

$$G_{\rm I} = [G_{pn}^{\rm I}]$$

$$G_{pn}^{\rm I} = \exp[-jkR_0^{\rm I}\cos(\Phi_0^{\rm I} - \phi_p)] \exp(jn\phi_p) \quad (27e)$$

$$G_{\rm II} = [G_{pm}^{\rm II}]$$

$$G_{pm}^{\rm II} = \exp[-jkR_0^{\rm II}\cos(\Phi_0^{\rm II} - \phi_p)] \exp(jm\phi_p). \quad (27f)$$

The vectors $E_{\rm I}$ and $E_{\rm II}$ represent values of the incident field on the surfaces of two scatterers. The matrix $H_{\rm I}$ is a propagation matrix which relates the scattering coefficients from body I to field values at the appropriate points on body II and similarly for $H_{\rm II}$. $G_{\rm I}$ and $G_{\rm II}$ relate the scattering coefficients from body I and II, respectively, to the value of the scattered far-field pattern at angle ϕ_p with respect to the origin 0. We will designate by $\beta_{\rm I}^{-1}$ and $\beta_{\rm II}^{-1}$ the inverses or pseudoinverses of the β matrices which are presumed to be known for body I and body II, respectively.

If we designate by $a_{\rm I} = [a_n^{\rm I}]$ and $a_{\rm II} = [a_m^{\rm II}]$ the column vectors for the scattering mode coefficients for fields scattered from bodies I and II, respectively, we see that the boundary conditions may be written

$$\beta_{\rm I} a_{\rm I} + H_{\rm II} a_{\rm II} = -E_{\rm I}$$

$$H_{\rm I} a_{\rm I} + \beta_{\rm II} a_{\rm II} = -E_{\rm II}. \quad (28)$$

It should be noted that (28) includes all the interactions between the two scatterers and is therefore exact to the extent of the approximation that the scattering of the bodies may be completely characterized by the various matrices in (28). The matrices β_{I} and β_{II} must have inverses (or pseudoinverses) by the uniqueness theorem of the single-body scattering problem. Thus we find the solution of (28) to be

$$a_{\mathrm{I}} = \beta_{\mathrm{I}}^{-1}[U - H_{\mathrm{II}}\beta_{\mathrm{II}}^{-1}H_{\mathrm{I}}\beta_{\mathrm{I}}^{-1}]^{-1}[-E_{\mathrm{I}}+H_{\mathrm{II}}\beta_{\mathrm{II}}^{-1}E_{\mathrm{II}}]$$
$$a_{\mathrm{II}} = \beta_{\mathrm{II}}^{-1}[U - H_{\mathrm{I}}\beta_{\mathrm{I}}^{-1}H_{\mathrm{II}}\beta_{\mathrm{II}}^{-1}]^{-1}[-E_{\mathrm{II}} + H_{\mathrm{I}}\beta_{\mathrm{I}}^{-1}E_{\mathrm{I}}] \tag{29}$$

where U designates the identity matrix. The values of the pattern at various angles are given by the column vector

$$G_{\mathrm{I}}a_{\mathrm{I}} + G_{\mathrm{II}}a_{\mathrm{II}} \tag{30}$$

or the coefficients a_{I} and a_{II} may be used to calculate the near fields when these are of interest.

Numerical Results

The techniques discussed here have been applied to a number of example problems and where possible these results have been compared to results found in the literature. Some of the more important results are given here.

Fig. 7 shows the scattered pattern produced by an elliptical cylinder of semimajor axis $ka = 1.0$ and semiminor axis $kb = 0.5$. Both inside and outside expansions and the direct mode-matching methods were used and the results are compared to the method of Burke and Twersky [11]. The same problem was also solved using an integral equation approach and good agreement was obtained.

We also show in Fig. 7 the scattered pattern of a square cylinder of half-length $ka = 1.0$. Also shown is the result obtained by Mei and Van Bladel [12]. Here, the sides of the cylinder have infinite radius of curvature and only the inside expansion could be used. It should be noted here that no special precautions need be taken at the cylinder corners in this formulation of the problem since the unknown is the field rather than the current at the corner and the field is finite at the edge. Furthermore, it was demonstrated for this problem that the matrix resulting from the integral equation formulation for the current became ill-conditioned near the resonant frequencies for the square cylinder [6]. This does not occur when the problem is formulated with the field as the unknown. Several different shapes of rectangular cylinders were investigated and the results compared closely to those of Mei and Van Bladel [12] and with results obtained by the integral equation formulation.

Fig. 7 also shows the scattered pattern of an elliptical dielectric cylinder of semimajor axis $ka = 1.2$ and semiminor axis $kb = 1.0198$ with a dielectric constant $\epsilon_d/\epsilon_0 = 2.0$. The result shown was obtained using the direct mode-matching method. It appears that sufficient data does not exist in the literature to accurately check the results in this instance. However, the elliptical dielectric scatterer was also solved for a number of cases with very small wavenumbers and the results agreed with the well-known low-frequency Rayleigh approximations [13]. It should be remarked that even though the geometry of the dielectric elliptic cylinder conforms to a separable coordinate system, a closed form solution in this instance cannot be obtained. Even in elliptic cylindrical coordinates, one obtains an infinite set of linear equations for the modal coefficients, so that one might prefer to solve the problem numerically in circular cylindrical coordinates in order to use Bessel functions rather than Mathieu functions.

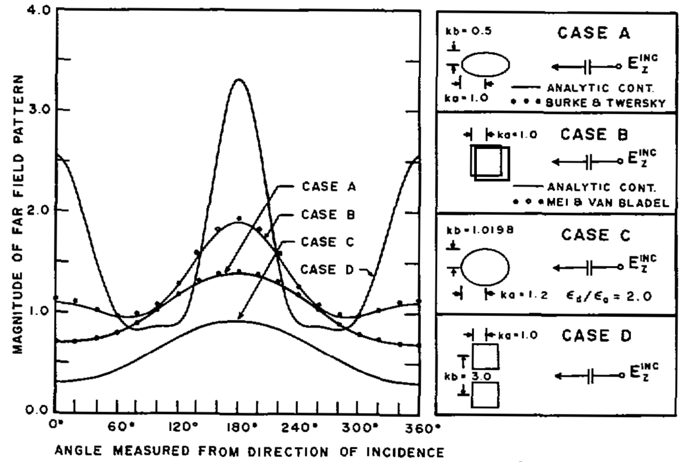

Fig. 7. Scattering patterns for conducting elliptic cylinder, conducting square cylinder, dielectric elliptic cylinder, and two conducting square cylinders.

TABLE I

Pattern Values for Two Circular Cylinders Calculated by Multiple Scattering Approach and by Method of Row [14].

Angle of Observation (Degrees)	Pattern Values $g(\phi)$ Multiple scattering	Row's Method
0	3.136	3.1359
20	2.591	2.5912
40	1.511	1.5105
60	0.8793	0.8792
80	0.8011	0.8011
100	0.8064	0.8064
120	1.029	1.0293
140	1.576	1.5760
160	2.170	2.1700
180	2.429	2.4290

Fig. 7 also gives the scattering pattern of two square cylinders of the type considered and spaced a distance $kb = 3.0$ apart. This pattern was obtained using the matrix which resulted from solving the scattering problem of an isolated square cylinder. Although data for this case do not exist, the method was checked using the same program to calculate scattering by two circular cylinders, the results for which are known (Row [14]) and are given by Imbriale and Mittra [15]. Table I gives a comparison of the pattern values for the two circular cylinders of radius $ka = 1.0$ and spaced $kb = 3.0$ apart for various angles. The configuration is similar to that of the square cylinders in Fig. 7.

Conclusions

It has been demonstrated that the scattered field of a two-dimensional cylindrical scatterer can be determined directly without first solving for the induced current. The method proves to be useful for scatterers in the range $0 < kd < 20$, d being the largest dimension of the scatterer and k the wavenumber. It should be noted that for scatterers of this size the physical or geometrical optics approximations are not applicable and one must resort to inverting an integral or matrix operator.

We have also demonstrated a method for calculating scattering from two parallel perfectly conducting cylinders. The method may be extended to the case of three-dimensional and dielectric scatterers.

It should be mentioned that in cases when the induced surface current or the polarization current is the quantity of interest, the problem should be formulated with the current as the unknown. This is because the higher order multipole terms contribute substantially to the near fields and it is the higher order Fourier pattern coefficients which have the greatest relative error, even though the coefficients themselves are very small compared to the low-order coefficients. Thus the far-field pattern may be quite accurate, whereas the near field may be somewhat in error resulting in larger errors in the surface or polarization current. The lower order coefficients are relatively more accurate because the higher order multipole terms do not couple significantly with the lower order terms when the mean-square error in the total field is minimized on the body contour.

The formulation of the scattering problem in terms of modal wavefunctions also has certain computational advantages over the integral formulation. In particular, one does not have to consider singularities in either the induced current or the kernel function in the integral. Furthermore, numerical instabilities due to the resonances of the interior problem do not arise. Partially offsetting these advantages, however, is the disadvantage of calculating the higher order multipole wave functions.

In the case of the dielectric scatterers, it appears that the modal formulation has an additional advantage. In the integral equation formulation, the integral equation is satisfied at a sufficient number of points per wavelength internal to the scatterer to insure a reasonable approximation to the polarization current. Thus the number of unknowns is proportional to the cross sectional area. In the modal formulation, however, the number of unknowns, the far-field coefficients, is proportional to the largest dimension of the scatterer. For moderate size scatterers in terms of wavelengths, the difference in the number of unknowns required in the two formulations can be significant. It is apparently for this reason that few results appear in the literature for the dielectric scatterer.

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