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Summary

General inequalities of type (5) for eigenvalues were established as early as 1912 by H. Weyl [11] through consideration of integral equations; he used them for studying the asymptotic behaviour of λ_n when $n \to \infty$. These inequalities seem to have been forgotten, as some of them were independently re-discovered by several authors (see [4], p. 314 and [10], pp. 483–484). – Such inequalities are here derived and applied in various ways to special problems (vibrating strings, rods, membranes and plates). Explicit lower bounds for λ_1 are obtained in a somewhat similar manner in terms of the Green's function. For Schrödinger's equation (15′), the very simple inequalities (17′) are found. Analogous inequalities are established for a class of equilibrium problems. Many of the inequalities thus obtained express convexity properties (cf. the *Post-scriptum* above and the paper by Pólya and Schiffer quoted there).

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Green's Function for Laplace's Equation in a Circular Ring with Radiation Type Boundary Conditions

By James F. Heyda, Cincinnati, Ohio, USA[1])

1. Introduction

Let (r, θ) be the polar coordinates of a point of the circular ring $0 < r_1 < r < r_2$ and consider the problem of solving Poisson's equation,

$$\nabla^2 T \equiv \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\,\frac{\partial T}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 T}{\partial \theta^2} = -F(r, \theta)\,, \tag{1}$$

[1]) Aircraft Nuclear Propulsion Department, General Electric Company.

subject to the boundary conditions

$$\frac{\partial T}{\partial r} - h_1 T \Big|_{r=r_1} = \phi_1(r_1, \theta) \quad (h_1 \geqq 0) \,, \tag{2}$$

$$\frac{\partial T}{\partial r} + h_2 T \Big|_{r=r_2} = \phi_2(r_2, \theta) \quad (h_2 \geqq 0) \,, \tag{3}$$

where the source function F and the functions ϕ_1, ϕ_2 are given.

Equation (1) describes the flow of heat in a long hollow cylinder, the flow taking place in planes perpendicular to its axis, with $h_i = H_i/k$, where H_i is the coefficient of surface heat transfer at $r = r_i$ and k is the thermal conductivity of the cylinder material. We shall assume the h_i constant.

When $\phi_1 = \phi_2 \equiv 0$, the boundary conditions (2), (3) describe cooling by forced convection and are sometimes referred to as radiation boundary conditions.

The solution of the problem defined by equations (1), (2) and (3) is

$$T(r, \theta) = \iint\limits_R G(r, \theta; r_0, \theta_0)\, F(r_0, \theta_0)\, dA_0 + \int\limits_C \phi(r_0, \theta_0)\, G(r, \theta; r_0, \theta_0)\, ds_0 \,, \tag{4}$$

where $G(r, \theta; r_0, \theta_0)$ is the appropriate Green's function; dA_0 is an element of area of the plane annulus R: $r_1 < r < r_2$; C is the complete boundary of R; the function $\phi = \phi_i$ when $r_0 = r_i$ and ds_0 is an element of arclength on C.

2. Derivation of $G(r, \theta; r_0, \theta_0)$

It is convenient to denote the point (r, θ) by the vector $z = r\, e^{i\theta}$ and to write $G(r, \theta; r_0, \theta_0) \equiv G(z, z_0)$. The Green's function $G(z, z_0)$, where z, z_0 are distinct points of R, is defined to be the solution of

$$\nabla^2 G(z, z_0) = -\delta(z - z_0) \tag{5}$$

with the boundary conditions

$$\frac{\partial G}{\partial r} + (-1)^i h_i G \Big|_{r=r_i} = 0 \quad (i = 1, 2) \,. \tag{6}$$

It may be written

$$G = G^* + S \,, \tag{7}$$

where the singular component S is given by

$$S(z, z_0) = -\frac{1}{2\pi} \operatorname{Re} \log (r\, e^{i\theta} - r_0\, e^{i\theta_0}) \tag{8}$$

and the non-singular component G^* is a solution of $\nabla^2 G = 0$ subject to condition (6).

One shows readily that

$$
\left.\begin{aligned}
S(z, z_0) &= \frac{1}{2\pi}\left[-\log r + \sum_1^\infty \frac{1}{n}\left(\frac{r_0}{r}\right)^n \cos n(\theta-\theta_0)\right], \qquad (r_0 < r)\\
&= \frac{1}{2\pi}\left[-\log r_0 + \sum_1^\infty \frac{1}{n}\left(\frac{r}{r_0}\right)^n \cos n(\theta-\theta_0)\right], \qquad (r_0 > r),
\end{aligned}\right\} \tag{9}
$$

while the most general G^* in terms of circular harmonics is

$$
\left.\begin{aligned}
G^*(z, z_0) = A_0 + B_0 \log r + \sum_1^\infty (A_n r^n + B_n r^{-n}) \cos n\,\theta&\\
+ \sum_1^\infty (C_n r^n + D_n r^{-n}) \sin n\,\theta&,
\end{aligned}\right\} \tag{10}
$$

wherein the 'constants' $A_0, B_0, A_n, B_n, C_n, D_n$ $(n = 1, 2, \ldots)$ are now to be determined so that

$$
\frac{\partial G^*}{\partial r} + (-1)^i h_i G^*\Big|_{r=r_i} = -\frac{\partial S}{\partial r} + (-1)^{i-1} h_i S\Big|_{r=r_i} \qquad (i = 1, 2)\,. \tag{11}
$$

Finding and simplifying these 'constants', although somewhat of a prodigious task, is well worth the effort in view of the reasonably simple results. One finds, upon introducing the substitutions

$$
\frac{r}{r_1} = e^\eta, \quad \frac{r_0}{r_1} = e^{\eta_0}, \quad \frac{r_2}{r_1} = e^{\eta_2} \quad (0 \leqq \eta, \eta_0 \leqq \eta_2)\,, \tag{12}
$$

that

$$
A_0 = \frac{1}{2\pi}\,\frac{\left(h_1 \eta_0 + \frac{1}{r_1}\right)\left(h_2 \ln r_2 + \frac{1}{r_2}\right)}{h_1 h_2 \eta_2 + \frac{h_2}{r_1} + \frac{h_1}{r_2}}, \tag{13}
$$

$$
B_0 = \frac{h_1}{2\pi}\,\frac{h_2(\eta_2 - \eta_0) + \frac{1}{r_2}}{h_1 h_2 \eta_2 + \frac{h_2}{r_1} + \frac{h_1}{r_2}}, \tag{14}
$$

$$
A_n = \frac{-\cos n\,\theta_0}{2\pi r_2^n}\left[\frac{h_1\left(\frac{h_2}{n} - \frac{1}{r_2}\right)\sinh n\,\eta_0 + \frac{1}{r_1}\left(h_2 - \frac{n}{r_2}\right)\cosh n\,\eta_0}{\left(\frac{n^2}{r_1 r_2} + h_1 h_2\right)\sinh n\,\eta_2 + n\left(\frac{h_1}{r_2} + \frac{h_2}{r_1}\right)\cosh n\,\eta_2}\right], \tag{15}
$$

$$
B_n = \frac{-r_1^n \cos n\,\theta_0}{2\pi}\left[\frac{h_2\left(\frac{h_1}{n} - \frac{1}{r_1}\right)\sinh n(\eta_2 - \eta_0) + \frac{1}{r_2}\left(h_1 - \frac{n}{r_1}\right)\cosh n(\eta_2 - \eta_0)}{\left(\frac{n^2}{r_1 r_2} + h_1 h_2\right)\sinh n\,\eta_2 + n\left(\frac{h_1}{r_2} + \frac{h_2}{r_1}\right)\cosh n\,\eta_2}\right], \tag{16}
$$

$$
\frac{C_n}{A_n} = \frac{D_n}{B_n} = \tan n\,\theta_0\,. \tag{17}
$$

The final form of the GREEN'S function $G(r, \theta; r_0, \theta_0)$ may now be obtained by substituting the above evaluations of $A_0, B_0, A_n, B_n, C_n, D_n$ into equation (10) and combining the result with equation (9) in accordance with relation (7). After much algebraic refinement, the final result turns out to be:

For $r_1 \leqq r_0 < r \leqq r_2$, i.e., for $0 \leqq \eta_0 < \eta \leqq \eta_2$,

$$G(\eta, \theta; \eta_0, \theta_0) = \frac{1}{2\pi} \left\{ \frac{\left(h_1 \eta_0 + \frac{1}{r_1}\right)\left(h_2 [\eta_2 - \eta] + \frac{1}{r_2}\right)}{h_1 h_2 \eta_2 + \frac{h_2}{r_1} + \frac{h_1}{r_2}} \right.$$
$$\left. + 2 \sum_{n=1}^{\infty} \frac{\left(\frac{n}{r_1} \cosh n \eta_0 + h_1 \sinh n \eta_0\right)\left(\frac{1}{r_2} \cosh n (\eta_2 - \eta) + \frac{h_2}{n} \sinh n (\eta_2 - \eta)\right)}{\left(\frac{n^2}{r_1 r_2} + h_1 h_2\right) \sinh n \eta_2 + n \left(\frac{h_2}{r_1} + \frac{h_1}{r_2}\right) \cosh n \eta_2} \cos n (\theta - \theta_0) \right\}. \quad (18)$$

Since $G(z, z_0)$ is symmetric in z, z_0, the form of G when $r_1 \leqq r < r_0 \leqq r_2$ is obtained from equation (18) by interchanging η and η_0.

3. Green's Function Vanishing on the Ring Boundaries

By letting $h_1 \to \infty$, $h_2 \to \infty$ in formula (18) we obtain

$$G(\eta, \theta; \eta_0, \theta_0) = \frac{1}{2\pi} \left[\frac{\eta_0 (\eta_2 - \eta)}{\eta_2} + 2 \sum_{n=1}^{\infty} \frac{\sinh n \eta_0 \sinh n (\eta_2 - \eta)}{n \sinh n \eta_2} \cos n (\theta - \theta_0) \right], \quad (19)$$

a reasonably simple form of the GREEN'S function which vanishes on the outer circle $r = r_2$, i.e., $\eta = \eta_2$. Equation (19) applies when $0 \leqq \eta_0 < \eta \leqq \eta_2$; the applicable form when $0 \leqq \eta < \eta_0 \leqq \eta_2$ is obtained from (19) by interchanging η and η_0, thereby yielding the GREEN'S function which vanishes on the inner circle $\eta = 0$.

This case is treated by HILBERT and COURANT [1][2]) who derive the equivalent of (19) using complex variable methods. Although their analysis is short and painless, their final result, expressed in terms of theta functions, does not lend itself easily to analytical computations. It is of some interest to show that the two results, superficially very dissimilar, are indeed equivalent.

HILBERT-COURANT take $r_1 = q^{1/2}$, $r_2 = q^{-1/2}$, $(0 < q < 1)$, so that $r_1 r_2 = 1$. Free of the theta function notation, their result is

$$G(z, z_0) = -\frac{1}{2\pi} \left[\frac{1}{4} \log q + \log |z^{-\log z_0/\log q}| + \log \left| \sqrt{\frac{z}{z_0}} - \sqrt{\frac{z_0}{z}} \right| + \log |Q| \right], \quad (20)$$

[2]) Numbers in brackets refer to References, page 327.

where

$$Q = \frac{\prod_{\nu=1}^{\infty}\left(1 - q^{2\nu}\frac{z}{z_0}\right)\left(1 - q^{2\nu}\frac{z_0}{z}\right)}{\prod_{\nu=1}^{\infty}\left(1 - q^{2\nu-1} z_0 z\right)\left(1 - \frac{q^{2\nu-1}}{z_0 z}\right)}. \tag{21}$$

It is sufficient to compare (19) and (20) for z_0 real, i.e. $z_0 = r_0$, $\theta_0 = 0$ or π. To be specific we take $\theta_0 = 0$. Equation (20) can then be rewritten, when $r_0 < r$, as

$$\left.\begin{aligned} 2\pi G = {} & \frac{1}{2}\left[\frac{\log r_0 \log r}{\log r_1} - \log\frac{r\,r_1}{r_0}\right] \\ & - \frac{1}{2}\log\left[1 + \left(\frac{r_0}{r}\right)^2 - 2\,\frac{r_0}{r}\cos\theta\right] - \log|Q|. \end{aligned}\right\} \tag{22}$$

It is then a simple matter to identify the term

$$\frac{1}{2}\left[\frac{\log r_0 \log r}{\log r_1} - \log\left(\frac{r\,r_1}{r_0}\right)\right]$$

with the term $\eta_0(\eta_2 - \eta)/\eta_2$ of equation (19). It remains then to show that

$$\left.\begin{aligned} & 2\sum_{n=1}^{\infty}\frac{\sinh n\,\eta_0 \sinh n\,(\eta_2 - \eta)}{n \sinh n\,\eta_2}\cos n\,\theta \\ & = -\frac{1}{2}\log\left[1 + \left(\frac{r_0}{r}\right)^2 - 2\left(\frac{r_0}{r}\right)\cos\theta\right] - \log|Q|. \end{aligned}\right\} \tag{23}$$

On the right side of equation (23) we have, for the first term, the known expansion

$$-\frac{1}{2}\log\left[1 + \left(\frac{r_0}{r}\right)^2 - 2\,\frac{r_0}{r}\cos\theta\right] = \sum_{n=1}^{\infty}\left(\frac{r_0}{r}\right)^n\frac{\cos n\,\theta}{n}. \tag{24}$$

From equation (21),

$$\left.\begin{aligned} \log|Q| = {} & \log\prod_{\nu=1}^{\infty}\left|1 - r_1^{4\nu}\frac{z}{r_0}\right| + \log\prod_{\nu=1}^{\infty}\left|1 - r_1^{4\nu}\frac{r_0}{z}\right| \\ & - \log\prod_{\nu=1}^{\infty}\left|1 - r_1^{4\nu-2} r_0 z\right| - \log\prod_{\nu=1}^{\infty}\left|1 - \frac{r_1^{4\nu-2}}{r_0 z}\right|. \end{aligned}\right\} \tag{25}$$

Consider the first term of equation (25). We may write

$$\left.\begin{aligned} \log\prod_{\nu=1}^{\infty}\left|1 - r_1^{4\nu}\frac{z}{r_0}\right| & = \sum_{\nu=1}^{\infty}\log\left|1 - r_1^{4\nu}\frac{z}{r_0}\right| \\ & = \sum_{\nu=1}^{\infty}\frac{1}{2}\log\left[1 + \left(r_1^{4\nu}\frac{r}{r_0}\right)^2 - 2\left(r_1^{4\nu}\frac{r}{r_0}\right)\cos\theta\right]. \end{aligned}\right\} \tag{26}$$

Since

$$r_1^{4\nu}\frac{r}{r_0} < r_1^{4\nu}\frac{1}{r_1^2} = r_1^{4\nu-2} \leqq r_1^2 < 1 \text{ for } \nu = 1, 2, \dots,$$

we may rewrite (26) as

$$\left.\begin{aligned} \log\prod_{\nu=1}^{\infty}\left|1 - r_1^{4\nu}\frac{z}{r_0}\right| &= -\sum_{\nu=1}^{\infty}\sum_{n=1}^{\infty} r_1^{4\nu n}\left(\frac{r}{r_0}\right)^n \frac{\cos n\,\theta}{n} \\ &= -\sum_{n=1}^{\infty}\frac{r_1^{4n}}{1-r_1^{4n}}\left(\frac{r}{r_0}\right)^n \frac{\cos n\,\theta}{n}, \end{aligned}\right\} \tag{27}$$

where we have used an expansion as in (24), interchanged orders of summation (which is permissible here) and summed the resulting geometric series. Observing that $\eta_2 = -2\log r_1$, we may write finally

$$\log\prod_{\nu=1}^{\infty}\left|1 - r^{4\nu}\frac{z}{r_0}\right| = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{r}{r_0}\right)^n \frac{r_1^{2n}\cos n\,\theta}{n\sinh n\,\eta_2}\,. \tag{27}$$

Proceeding similarly with the other terms of (25) one easily obtains the result

$$-\log|Q| = \sum_{n=1}^{\infty}\frac{r_1^{2n}\cosh\left(n\log\frac{r}{r_0}\right)}{n\sinh n\,\eta_2}\cos n\,\theta - \sum_{n=1}^{\infty}\frac{\cosh(n\log r\,r_0)}{n\sinh n\,\eta_2}\cos n\,\theta\,, \tag{29}$$

whence it is a simple matter to show that when (29) and (24) are added and simplified, the result is the required left member of (23).

Other specializations of the general form (18) are also of interest. Thus one might keep h_1 finite and let $h_2 \to \infty$, as would be required in determining the steady temperature in a long hollow cylinder with cooling at the inner surface and temperature 0 at the outer surface.

The form (19) is only slightly more involved when the ring boundaries are non-concentric. This result, involving bi-polar coordinates, was previously presented by the writer in reference [2].

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Zusammenfassung

Es wird die Greensche Funktion der Laplaceschen Gleichung für einen Kreisring hergeleitet, wobei Randbedingungen vom Strahlungstyp angenommen sind. In demjenigen Spezialfall, in dem die Randbedingungen das Verschwinden der Greenschen Funktion verlangen, wird gezeigt, dass die gefundene Darstellung der Greenschen Funktion übereinstimmt mit der Formel, wie sie im Buch von HILBERT-COURANT unter Verwendung von Thetafunktionen hergeleitet ist.

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A Boundary-Layer in a Non-Newtonian Fluid

By JOHN R. JONES, Swansea, Great Britain[1])

1. Introduction

The boundary-layer theory was developed by PRANDTL [1][2]) for fluids of constant viscosity, initially for the case of laminar flow, and later was extended to include compressible viscous fluids and turbulent flow in the boundary-layer. In recent years experiments have shown that several phenomena observable in fluids are not predicted by the classical theory of viscous flow. This has led to the formulation of more complicated rheological equations of state, either deduced from a molecular picture or microscopic model of the material, or based on idealizations of simple experiments. The invariant forms of rheological equations of state for a homogeneous continuum, suitable for application to all conditions of motion and of stress, have been discussed generally by OLDROYD [2]; restrictions are imposed on their form by the physical conditions that the rheological properties they describe are independent of the frame of reference and independent of the motion of the material element in space. With the growing interest in non-Newtonian behaviour it seems that one should now, if possible, extend the boundary-layer theory so as to include a wider range of fluids. In the present paper the effect of a variation of viscosity with rate of shear and of a normal-stress coefficient (representing 'cross-viscosity') are examined in a simple type of two-dimensional boundary-layer. For reasons of mathematical convenience the case which is worked out in detail, by way of illustration, is one in which the viscosity coefficient is a linear function of the rate of shear, the normal-stress coefficient is arbitrary and the mainstream velocity is proportional to the one-third power of distance measured from a stagnation point. It is of interest to note

[1]) Department of Applied Mathematics, University College of Swansea.

[2]) Numbers in brackets refer to References, page 343.