Analytic Constraints on Electromagnetic Field Computations

R. H. T. BATES, MEMBER, IEEE

(*Invited Paper*)

Abstract—**It is attempted to provide a definitive statement of the theoretical bases and the computationally useful manifestations of integral equation formulations of field problems, and the expansion of fields in sequences of functions which are proper solutions of the wave equation. The reason for doing this is that it has become clear during the last ten years that sophisticated points of mathematical analysis have practical computational significance. For ease of exposition, only two-dimensional fields are treated in detail. The paper is in five parts. The first part (Sections I and II) is introductory. The second part (Sections III–V) deals with formal diffraction theory. Methods particularly suited to digital computation are presented in the third part (Sections VI–XI). The results of computational experience are assessed in the fourth part (Sections XII–XIV). The fifth part (Sections XV–XVII) discusses the types of investigation needed to increase the technological usefulness of existing techniques.**

I. INTRODUCTION

THE development of the digital computer has been the cause during the past twenty years of increased interest in the manipulation of Maxwell's equations into forms convenient for numerical evaluation. It is no longer a question of whether something can be computed, but how it can be computed most efficiently. Subtle points of mathematical analysis, which would have been disdained by engineers in the old slide-rule and desk-calculator days, have assumed practical computational importance.

This paper reviews in detail certain computational aspects of the interaction of electromagnetic fields with passive structures. It is apparent from the recent "global" reviews by Silvester and Csendes [1] and Ng [2] that the *differential approach* to computational problems in microwave engineering predominates over the *integral approach* and the *series approach*. The differential approach is based on direct solution of the wave equation and is typified by finite-difference and finite-element methods [3] and also the transmission-line matrix methods [4].

The concern here is with integral equation formulations (the integral approach) of field problems, and with the expansion of fields in sequences of functions which are proper solutions of the wave equation (the series approach). During the last ten years, many different, sometimes conflicting, techniques have been proposed, usually with impressive theoretical and/or computational supporting evidence. It has been difficult to assess the relative usefulness of the various suggestions because they often involve recondite mathematical notions which are not easy for engineers either to explain or to understand.

Three things are attempted here:

1) to present a unified development encompassing the existing methods;

2) to establish those results which are, at least, comparatively free from doubt;

3) to suggest the types of investigation which seem most urgent and appropriate for improving field computation techniques in radio and microwave engineering.

The literature contains sufficient authoritative treatments to ensure that items 1) and 2) are relatively free of the author's particular prejudices. The same cannot be said of item 3). However, it would be unduly timorous to neglect to suggest future research possibilities in a review of this sort.

In an effort to clarify the exposition and to concentrate on the physical meaning of the many intricate mathematical operations which have to be introduced, the analysis is restricted to fields which vary in only two dimensions. So it is the diffraction of cylindrical waves by cylindrical structures which is examined. There is no restriction, in principle, on the cross-sectional shapes or the material composition of the structures.

A full, three-dimensional treatment would have to be much more complicated and appreciably harder to understand. It is doubtful if it would add a great deal because the polarization of the field is not a quantity of overriding interest in the questions considered here. There is no coupling between those two-dimensional fields, termed electrically polarized and magnetically polarized, respectively, which have no magnetic-field component and no electric-field component parallel to the axis of the cylindrical wave system. Another useful aspect of the two-dimensional formulation is that it applies rigorously to acoustical fields.

The interaction of wide-band microwave signals with passive structures is attracting increasing interest, but there is no large literature as yet on the computation of the diffraction of modulated carrier waves. Accordingly, the treatment is here restricted to monochromatic waves. Reference is made to the remarkable "singularity expan-

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The author is with the Department of Electrical Engineering, University of Canterbury, Christchurch 1, New Zealand.

sion method" [5] which has been developed for the study of time-dependent scattering and which relies on analytical techniques related to some of those which are the main concern here.

This paper is divided into five parts. Part 1 (Sections I and II), which contains this introduction, sets the scene. Exact methods are developed in part 2 (Sections III–V). The theory of methods which are particularly suited for digital computation is discussed in part 3 (Sections VI–XI). The impact of theoretical constraints on practical computations is examined in part 4 (Sections XII–XIV). In part 5 (Sections XV–XVII), future possibilities are outlined.

II. PRELIMINARIES

Fig. 1 shows an arbitrary point P, having cylindrical polar coordinates ρ and φ, in a two-dimensional space Ω, which is the plane $z = 0$ of a three-dimensional space Y. The straight line $00'$ is an arbitrary datum from which the angular position of P is defined. A closed curve C, enclosing the origin 0 of coordinates, divides Ω into an interior part Ω_- and an exterior part Ω_+. The cylindrical polar coordinates of an arbitrary point Q on C are r and θ. The symbol C, besides standing for the curve itself, also denotes distance along the curve at Q, measured counterclockwise from the point (or from the innermost point, if there is more than one) where $00'$ intersects the curve. The ν direction, which is the outward normal to C at Q, makes an angle α with the extension of the line $0Q$. The distance between Q and P is denoted by R.

Set-theory notation tends to be unpopular with engineers, and rightly so when its only purpose seems to be to obscure the argument. But it has the advantage that it permits one to be precise concisely. The space Ω can be looked upon as a collection of points, a typical one of which is P. All the points in Ω_-, together with all the points on C, and all the points in Ω_+ make up all the points in Ω. This is written as

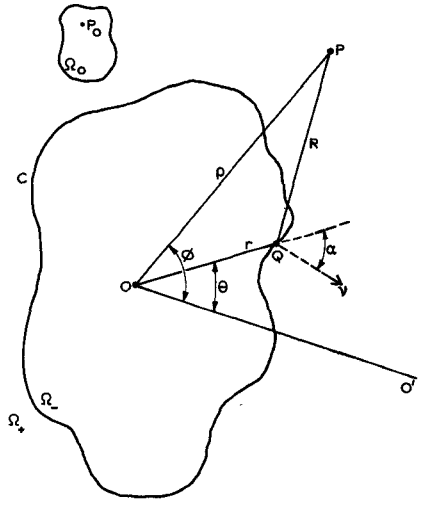

Fig. 1. Points, regions, and coordinate systems in Ω, the plane $z = 0$ of the three-dimensional space Y. The z direction is perpendicular to the paper.

$$\Omega = \Omega_- \cup C \cup \Omega_+. \tag{1}$$

If Ξ is a region, or a collection of intersecting or nonintersecting regions, within Ω, i.e., $\Xi \subset \Omega$, then the notation $P \in \Xi$ indicates that the point P can be anywhere within Ξ. Conversely, the notation $P \notin \Xi$ indicates that wherever P is, it cannot be in Ξ.

The part of space Ω_+ is partitioned into Ω_{+-} and Ω_{++}, interior and exterior, respectively, to the circle Γ_+, of radius r_+, centered on 0 circumscribing C. This is indicated in Fig. 2, which also shows that Ω_- is partitioned into Ω_{--} and Ω_{-+}, interior and exterior, respectively, to the circle Γ_-, of radius r_-, centerd on 0 inscribing C. This partitioning is summarized by

$$\begin{aligned} \Omega_- &= \Omega_{--} \cup \Gamma_- \cup \Omega_{-+} \\ \Omega_+ &= \Omega_{+-} \cup \Gamma_+ \cup \Omega_{++}. \end{aligned} \tag{2}$$

In Y there exists an electromagnetic field having an electric intensity $\boldsymbol{E}$, a magnetic intensity $\boldsymbol{H}$, and an angular frequency ω. The field exhibits no variations in the z direction and it is conveniently separated into two independent fields, for which H_z and E_z, respectively, are zero. Using complex exponential notation and suppressing the time factor $\exp(j\omega t)$, either of these fields is fully characterized by a single scalar [6] denoted here by $\Psi = \Psi(\rho,\varphi)$, which is conveniently defined as

$$\Psi = \begin{cases} E_z & \text{for electrically polarized field} \\ H_z & \text{for magnetically polarized field.} \end{cases} \tag{3}$$

A primary, or incident, field $\Psi_0 = \Psi_0(\rho,\varphi)$ emanates from impressed sources contained in a region $\Omega_0 \subset \Omega_+$, an arbitrary point of which is P_0 (see Fig. 1). A scattered, or reradiated, field $\Psi_1 = \Psi_1(\rho,\varphi)$ arises if parts of Ω are different from free space, so that the total field can be expressed as

$$\Psi = \Psi_0 + \Psi_1, \qquad P \in \Omega. \tag{4}$$

For the reader not overfamiliar with set-theory notation, this equation states: Ψ can be taken as the sum of Ψ_0 and Ψ_1, when the point P at which the field is observed is anywhere in the two-dimensional space Ω.

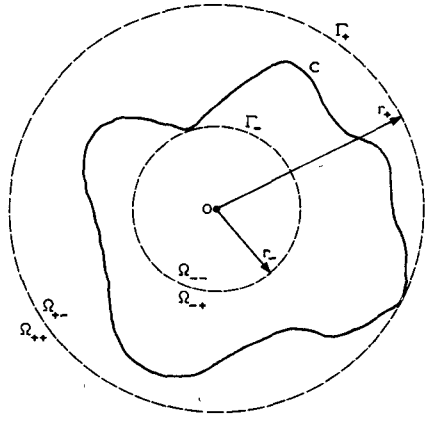

Fig. 2. Partitioning of Ω_- and Ω_+.

A convenient approach to diffraction and scattering of waves is the polarization-source formulation [7], in which all fields are taken to propagate as if they were in free space. Diffraction is accounted for, rigorously, by postulating equivalent (polarization) sources at all points where there are departures from free space. If $w_0 = w_0(\rho,\varphi)$ is the density of the impressed sources within Ω_0, then the incident field satisfies [7]

$$\nabla^2\Psi_0 + k^2\Psi_0 = -w_0, \qquad P \in \Omega \tag{5}$$

where k is the wavenumber or free-space propagation constant.

Space is taken to be free throughout all of Ω_+ apart from Ω_0. A time-invariant inhomogeneous propagating medium is permitted within $\Omega_- \cup C$. Consequently, the polarization-source density $w_1 = w_1(\rho,\varphi)$, which is computed directly from the variation throughout $\Omega_- \cup C$ of the permittivity, permeability, and conductivity of the medium [7], is zero outside C

$$w_1 = 0, \qquad P \in \Omega_+. \tag{6}$$

The reradiated field satisfies

$$\nabla^2\Psi_1 + k^2\Psi_1 = -w_1, \qquad P \in \Omega. \tag{7}$$

The total source density is defined by

$$w = w_0 + w_1. \tag{8}$$

It follows from (4)–(6) that

$$\nabla^2\Psi + k^2\Psi = -w, \qquad P \in \Omega. \tag{9}$$

If P' is a source point, i.e., a point in a part of space where $w \not\equiv 0$, and R' is the distance from P' to P, then Ψ is given by [7]

$$\Psi = \iint_{\Omega_0 \cup \Omega_- \cup C} w(\rho',\varphi') g_{P'P}\, d\Omega \tag{10}$$

where ρ' and φ' are the cylindrical polar coordinates of P', $d\Omega$ is an element of area, and g denotes the Green's function for two-dimensional waves [6]

$$g_{P'P} = g_{PP'} = -(j/4)H_0^{(2)}(kR') \tag{11}$$

which can be expanded in a form that is often useful by invoking the addition theorem for Bessel functions [8]. The expansion is illustrated conveniently in terms of the notation introduced in Fig. 1

$$\begin{aligned} j4g_{QP} &= H_0^{(2)}(kR) \\ &= \sum_{m=-\infty}^{\infty} J_m(kr)H_m^{(2)}(k\rho)\exp(jm[\varphi - \theta]), \quad \rho \geq r \\ &= \sum_{m=-\infty}^{\infty} H_m^{(2)}(kr)J_m(k\rho)\exp(jm[\varphi - \theta]), \quad \rho \leq r \end{aligned} \tag{12}$$

where $J_m(\cdot)$ and $H_m^{(2)}(\cdot)$ are, respectively, the Bessel function of the first kind and the Hankel function of the second kind, of order m.

III. SERIES EXPANSIONS

Fig. 3 shows a region $\tilde{\Omega}$ contained within Ω. A general expression for a particular field $\chi = \chi(\rho,\varphi)$ satisfying the free-space wave equation in $\tilde{\Omega}$ is [6]

$$\chi = \Phi_1 + \Phi_2 \tag{13}$$

$$\Phi_1 = \sum_{m=-\infty}^{\infty} A_m J_m(k\rho)\exp(jm\varphi) \tag{14}$$

$$\Phi_2 = \sum_{m=-\infty}^{\infty} B_m H_m^{(2)}(k\rho)\exp(jm\varphi) \tag{15}$$

where the A_m and B_m are constants. The expression (14) for Φ_1 can represent "standing waves," whereas the Hankel functions of the second kind in (15) imply that Φ_2 consists necessarily of "outward traveling waves" propagating away from 0. Another way of writing a general expression for χ is to separate it into "inward" and "outward" traveling waves. However, (14) and (15) are more convenient for the development pursued in this paper.

Since Hankel functions are singular when their arguments are zero [8] and since, on physical grounds, the field must be well behaved everywhere in free space, it follows that:

$$B_m \equiv 0, \qquad 0 \in \tilde{\Omega}. \tag{16}$$

It is of interest to define the class $\mathfrak{a}\{\tilde{\Omega}\}$ of field expansions which are entirely outgoing (irrespective of the values assumed by their expansion coefficients) throughout $\tilde{\Omega}$, with respect to the origin 0 of coordinates. If V is in this class, the notation

$$V \in \mathfrak{a}\{\tilde{\Omega}\} \tag{17}$$

is invoked. It follows that:

$$A_m \equiv 0, \qquad \chi \in \mathfrak{a}\{\tilde{\Omega}\} \tag{18}$$

since the A_m must assume special forms for Φ_1 to represent pure traveling waves; in fact, if the A_m are all real, for instance, then Φ_1 represents pure standing waves.

There are three fundamental principles which should be kept in mind when representing fields by infinite series

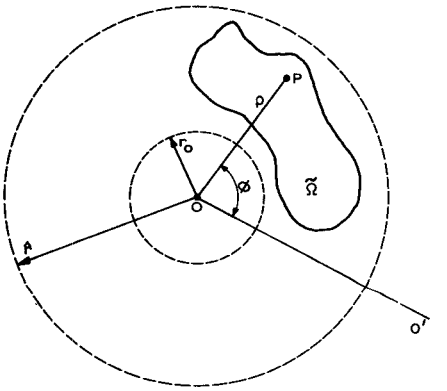

Fig. 3. The region $\tilde{\Omega}$.

of wave functions, such as the expressions on the right hand sides (RHS) of (14) and (15). The first principle can be stated without preamble.

First Fundamental Principle

An infinite series of wave functions is a useful representation of a field, within a region of space, only if the series has a unique value at each point within the region, i.e., if the series converges (in the sense that when the series is truncated to a sufficient number of terms, its value is negligibly increased by adding further terms) everywhere in the region.

Bessel functions of the first kind of integer order can be written as series in nonnegative powers of their arguments [8]. Consequently, the series for Φ_1 in (14) can be rearranged as a Taylor series, for ρ less than the *maximum radius of convergence* [9]. The magnitude of a Hankel function decreases monotonically as the magnitude of its argument increases [8], and all Hankel functions are singular when their arguments are zero, so that the expression for Φ_2 in (15) has a *minimum radius of convergence.* Defining the quantities $\hat{r}$ and r_0 by

$$\hat{r} = \text{maximum radius of convergence of RHS (14)} \quad (19)$$

$$r_0 = \text{minimum radius of convergence of RHS (15)} \quad (20)$$

it is seen that all of $\tilde{\Omega}$ must lie within the annulus $r_0 \leq \rho \leq \hat{r}$, as is illustrated in Fig. 3.

The concept of analytic continuation is most familiar in complex-variable theory [9], but it applies to real spaces of any number of dimensions [10]. It has been applied with effect to electromagnetic theory by Waterman [11], by Mittra and Wilton [12], Imbriale and Mittra [13], by Weston *et al.* [14], and Weston and Boerner [15]. The second principle, the importance of which has been emphasized by Millar [16], is based on the uniqueness of analytic continuation [10].

Second Fundamental Principle

Consider two regions $\tilde{\Omega}$ and Ξ, such that $\tilde{\Omega} \subset \Xi$, where Ξ is contained within the part of space where a particular field exists. If a series representation of the field is known to be valid within the region $\tilde{\Omega}$, but is subsequently found to converge within the wider region Ξ, then the uniqueness of analytic continuation ensures that the series is a valid representation of the field throughout Ξ.

It is a necessary corollary to this second principle that Ψ_0 can be expressed as

$$\Psi_0 = \sum_{m=-\infty}^{\infty} a_m J_m(k\rho) \exp(jm\varphi), \qquad P \in \Omega_{--} \quad (21)$$

where the a_m are constants. In Section II, the sources of Ψ_0 are defined as existing entirely in $\Omega_0 \subset \Omega_+$, so that Ψ_0 is analytic at the origin 0. Consequently, (21) agrees with (16) in that no series such as the one in (15) can be part of a representation of Ψ_0 throughout Ω_{--}. The maximum radius of convergence of the series in (21) cannot be less than r_- (see Fig. 2), by the second principle, because Ψ_0 can only cease to be analytic within Ω_0. So Ψ_0 must be expressible as in (21) within Ω_{--}.

Is it worth recalling that if Ψ_0 has the form of a plane wave, incoming from infinity at an angle ϑ to the datum 00′ (see Fig. 1), then [8]

$$a_m = j^m \exp(jm\vartheta). \quad (22)$$

Although a field must have a unique value at any point in a homogeneous source-free region, there is an infinite number of possible series of wave functions which can be used to represent the field within a finite region containing that point. Note that the functional dependence of almost all of these wave functions will be unseparated, because there are only eleven separable coordinate systems.

Suppose that there are two sets of wave functions, $\{V_m^{(1)}(\rho,\varphi)\}$ and $\{V_m^{(2)})\rho,\varphi)\}$, both of which satisfy the free-space wave equation. It will be found that a particular field $V = V(\rho,\varphi)$ can be expressed as

$$V = \sum_{m=-\infty}^{\infty} C_m^{(1)} V_m^{(1)}(\rho,\varphi), \qquad P \in \Xi_1$$

$$= \sum_{m=-\infty}^{\infty} C_m^{(2)} V_m^{(2)}(\rho,\varphi), \qquad P \in \Xi_2 \quad (23)$$

where the $C_m^{(1)}$ and $C_m^{(2)}$ are constants, and Ξ_1 and Ξ_2, respectively, are the widest regions throughout which the two series converge. Millar's studies [16] demonstrate the importance of the third principle.

Third Fundamental Principle

Within $\Xi_1 \cap \Xi_2$, i.e., the hatched region shown in Fig. 4, it is useful to rearrange the $V_m^{(1)}$ and $V_m^{(2)}$ in terms of each other, so that the set of constants $\{C_m^{(2)}\}$ can be determined from $\{C_m^{(1)}\}$, and vice versa. However, within the nonintersecting parts of Ξ_1 and Ξ_2, such a rearrangement cannot lead to a unique result for these sets of constants.

Suppose that there is a field $V = V(\rho,\varphi)$ which is entirely outgoing in a region Ξ_{1+}, i.e.,

$$V \in \mathfrak{A}\{\Xi_{1+}\} \quad (24)$$

where Ξ_{1+} is all space outside the circle Γ_{01} of radius r_{01} (see Fig. 5). Then V can be written as

$$V = \sum_{m=-\infty}^{\infty} C_{1,m} H_m^{(2)}(k\rho_1) \exp(jm\varphi_1), \qquad \rho_1 > r_{01} \quad (25)$$

where the $C_{1,m}$ are constants, and ρ_1 and φ_1 are polar

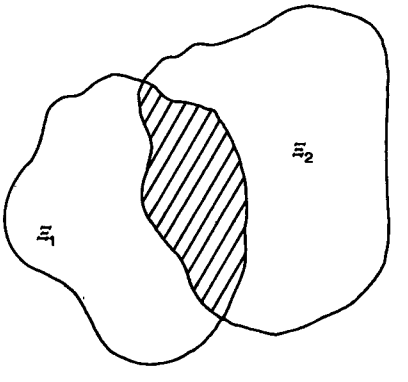

Fig. 4. Intersecting regions.

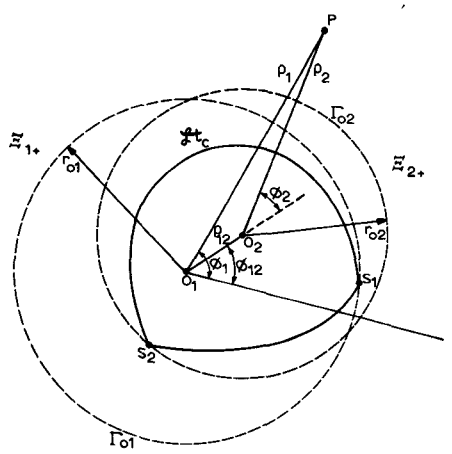

Fig. 5. Change of coordinate origin.

coordinates referred to origin O_1, which is the center of Γ_{01}. If r_{01} is the minimum radius of convergence of RHS (25), then there must be at least one point, S_1, say, on Γ_{01} at which V is singular (i.e., either V does not exist at S_1, or derivatives of V of an order higher than some finite order do not exist at S_1).

In Fig. 5, the point O_2 is the center of the circle Γ_{02} of radius r_{02}, and is origin for the polar coordinates ρ_2 and φ_2. Also, Ξ_{2+} is all space outside Γ_{02}. Now the addition theorem (12) for Bessel functions can be extended to functions of any order [8], so that when the position of P is such that $\rho_2 > \rho_{12}$, it is found that (25) can be rewritten as

$$V = \sum_{m=-\infty}^{\infty} C_{2,m} H_m^{(2)}(k\rho_2) \exp(jm\varphi_2) \tag{26}$$

$$C_{2,m} = \exp(jm\varphi_{12}) \sum_{n=-\infty}^{\infty} C_{1,m-n} J_n(k\rho_{12}) \cdot \exp(-jn[\varphi_{12} + \pi]) \tag{27}$$

which is, in effect, a rearrangement of terms of the kind referred to in the third fundamental principle. So (27) has meaning only in the part of space for which RHS (25) and (26) both converge. Denoting the minimum radius of convergence of RHS (26) by r_{02}, it is seen that (26) applies only for $\rho > r_{02}$, and the rearrangement of terms expressed by (26) and (27) is only valid in $\Xi_{1+} \cap \Xi_{2+}$. Also, there must be at least one point, S_2, say, on Γ_{02} at which V is singular.

It is important to realize that S_1 must lie either on Γ_{02} or inside it. By the third fundamental principle, RHS (25) is identically equal to RHS (26) throughout $\Xi_{1+} \cap \Xi_{2+}$. So, if S_1 were in Ξ_{2+}, then RHS (26) would have to be singular in Ξ_{2+}, but this cannot be true because the definition of r_{02} ensures that RHS (26) is analytic throughout Ξ_{2+}. Similarly, S_2 must lie either on or inside Γ_{01}. This ensures that the field V must be singular at S_1 and S_2, whatever expressions are used to represent the field. By continuously changing the origin of coordinates, the continuous, closed, convex curve $\mathcal{C}_c$ can, in principle, be traced out where $\mathcal{C}_c$, which is called the "convex hull" of the singularities, is the convex envelope of the singularities of V. Note that the origin of coordinates must follow one of a special set of paths if the true convex hull is to be traced out.

IV. INTEGRAL FORMULATIONS

In Section II, when considering the whole of Ω, the seat of the reradiated field Ψ_1 is taken to be the polarization-source density within $\Omega_- \cup C$. However, when considering only Ω_+, Ψ_1 can be expressed conveniently in terms of equivalent sources on C, as is demonstrated by, for instance, Baker and Copson [17]. Refer to (11) and the definitions preceeding it and then consider the quantity $\mathfrak{F}_1 = \mathfrak{F}_1(\rho,\varphi)$ which is defined by

$$\mathfrak{F}_1 = \iint_{\Omega_+} [g_{SP}\nabla^2\Psi_1 - \Psi_1\nabla^2 g_{SP}]\, d\Omega \tag{28}$$

where the point S lies within the element of area $d\Omega$, and P is the point in Ω at which $\mathfrak{F}_1$ is computed. Now g_{SP} is analytic (i.e., regular or well behaved) unless P coincides with S, so that the integrand in (28) can only fail to be analytic in the neighborhood (denoted by $\mathfrak{N}_P$) of P. The Green's function g_{SP} can be thought of as the field radiated by a point source at S, so that [6]

$$\nabla^2 g_{SP} + k^2 g_{SP} = -\delta_{SP}, \qquad P \in \Omega \tag{29}$$

where δ_{SP} is the two-dimensional delta function

$$\iint_{\Omega} \delta_{SP}\, d\Omega = \iint_{\mathfrak{N}_P} \delta_{SP}\, d\Omega = 1. \tag{30}$$

If follows from (6), (7), (29), and (30) that (28) can be rewritten as:

$$\mathfrak{F}_1(\rho,\varphi) = \iint_{\mathfrak{N}_P} \Psi_1\delta_{SP}\, d\Omega = \Psi_1(\rho,\varphi), \qquad P \in \Omega_+. \tag{31}$$

Since

$$g\nabla^2\Psi = \nabla\cdot(g\nabla\Psi) - \nabla g\cdot\nabla\Psi \tag{32}$$

one can rewrite (28) as

$$\mathfrak{F}_1 = \iint_{\Omega_+} \nabla\cdot(g_{SP}\nabla\Psi_1 - \Psi_1\nabla g_{SP})\, d\Omega. \tag{33}$$

On referring to the notation introduced in Fig. 1, it is seen that this equation can be transformed with the aid of the divergence theorem into

$$\mathfrak{F}_1 = \int_C \left[\Psi_1(r,\theta)\frac{\partial g_{QP}}{\partial \nu} - g_{QP}\frac{\partial \Psi_1(r,\theta)}{\partial \nu}\right] dC, \quad P \in \Omega_+ \tag{34}$$

where dC is the element of arc along C. When transforming from the surface integral over Ω to the line integral along the contour enclosing Ω, there is a line integral along a circle at infinity as well as the line integral along C. But the integral at infinity vanishes because both g_{SP} and Ψ_1 are outgoing at infinity. This is the famous "Sommerfeld radiation condition" *ausstrahlungsbedingung* [6], [17]. Combining (31) and (34) gives

$$\Psi_1(\rho,\varphi) = \int_C \left[\Psi_1(r,\theta)\,\frac{\partial g_{QP}}{\partial \nu} - g_{QP}\,\frac{\partial \Psi_1(r,\theta)}{\partial \nu}\right] dC, \quad P \in \Omega_+. \tag{35}$$

Using (30) and the notation of Fig. 1, (5) can be written as

$$\nabla^2\Psi_0 + k^2\Psi_0 = -\iint_{\Omega_0} w_{P_0}\delta_{P_0P}\,d\Omega \tag{36}$$

where w_{P_0} denotes the value of w_0 at the point P_0 in Ω_0. On replacing the subscript 1 in (28) with the subscript 0, and retracing the reasoning which leads from (28) to (31), it transpires that

$$\mathfrak{F}_0 = 0, \qquad P \in \Omega_+ \tag{37}$$

because the contributions from the sources of Ψ_0 in Ω_0 cancel the term

$$\iint_{\mathfrak{N}_P} \Psi_0\delta_{SP}\,d\Omega.$$

By analogy with (32)–(34) it follows that:

$$\int_C \left[\Psi_0(r,\theta)\,\frac{\partial g_{QP}}{\partial \nu} - g_{QP}\,\frac{\partial \Psi_0(r,\theta)}{\partial \nu}\right] dC = 0, \quad P \in \Omega_+. \tag{38}$$

Combining (35) and (38) gives an expression for the reradiated field in terms of the total field on C and its normal derivative there. The ratio of the field to its normal derivative can be looked upon as a normalized impedance, denoted here by

$$Z = \Psi(r,\theta) \Big/ \frac{\partial \Psi(r,\theta)}{\partial \nu} \tag{39}$$

which in general varies along C

$$Z = Z(r,\theta) = Z(C) = Z_Q. \tag{40}$$

These different notations expressing the functional dependence of Z are useful later.

Combining (4), (35), and (38)–(40) gives

$$\Psi_1 = \int_C [g_{QP} - Z_Q\partial g_{QP}/\partial \nu]F(C)\,dC, \qquad P \in \Omega_+ \tag{41}$$

where the linear density of reradiating sources $F(C)$ is defined by

$$F(C) = -\partial\Psi(r,\theta)/\partial \nu. \tag{42}$$

It should be noted that $F(C)$ is proportional to the density of equivalent surface currents on the wall of the infinite cylinder of cross section C. For electrically and magnetically polarized fields, respectively, $F(C)$ is proportional to the equivalent electric and magnetic surface-current densities. For perfectly conducting cylinders, $F(C)$ is proportional to the actual surface-current density. Remember that there can be no magnetic surface current on a perfect conductor, in which case $Z_QF(C)$ is proportional to the electric surface-current density when the field is magnetically polarized.

To remove any ambiguity concerning the meaning of the integrand in (41) when $Z = \infty$, the following difinition is necessary:

$$F(C) = 0 \quad \text{and} \quad Z_QF(C) = G(C) \quad \text{when} \quad Z_Q = \infty \tag{43}$$

where $G(C)$, an alternative definition of the density of reradiating sources on C, is a well-behaved function of C.

Z depends upon the form assumed by the field in Ω_-, which means that, in general, the interior and exterior (with respect to C) fields have to be treated in conjunction before either can be computed accurately. However, if the medium occupying $\Omega_- \cup C$ is highly reflecting or very lossy, Z can be specified usefully *a priori* [18]. In particular, if the medium is totally reflecting (e.g., C is the cross section of an infinite, perfectly conducting cylinder) then Z is zero or infinity, respectively, depending upon whether the field is electrically or magnetically polarized.

When Z and Ψ_0 are specified, the determination of Ψ_1 in Ω_+ is a properly formulated boundary value problem which can be posed as an integral equation, derived from (39)–(41)

$$\left[\frac{\partial \Psi_{0Q'}}{\partial \nu'} + \int_C \left[\frac{\partial g_{QQ'}}{\partial \nu'} - Z_Q\,\frac{\partial^2 g_{QQ'}}{\partial \nu \partial \nu'}\right] F(C)\,dC\right] Z_{Q'}$$

$$= \Psi_{0Q'} + \int_C [g_{QQ'} - Z_Q\partial g_{QQ'}/\partial \nu]\,F(C)\,dC \tag{44}$$

where $\Psi_{0Q'}$ is the value on C of Ψ_0 at the point Q', at which the outward normal direction to C is the ν' direction. The unknown in (44) is $F(C)$ which can be evaluated numerically by a number of different moment methods [19], [20]. Once $F(C)$ is determined, Ψ_1 can be computed by substituting $F(C)$ into (41).

It is often convenient to expand Ψ_1 in an angular Fourier series, which is what the physicists call its "partial wave expansion." To do this, it is first necessary to divide C into parts on which the radial coordinate of Q is less than, or greater than, the radial coordinate of the point P at which the field is being examined

$$C = L_- \cup L_+ \tag{45}$$

$$Q \in L_- \quad \text{when} \quad \rho < r \tag{46}$$

$$Q \in L_+ \quad \text{when} \quad \rho \geq r. \tag{47}$$

Note that both L_- and L_+ are functions of the position of P

$$L_\pm = L_\pm(\rho). \tag{48}$$

By inspection of Fig. 2 it is seen that

$$L_- = \phi \quad \text{and} \quad L_+ = C, \qquad P \in \Omega_{++} \tag{49}$$

$$L_+ = \phi \quad \text{and} \quad L_- = C, \qquad P \in \Omega_{--} \tag{50}$$

where ϕ denotes the null set (e.g., the region of space with no points in it).

It is convenient to return to the quantity χ, which is introduced in Section III, and to define it throughout all

Ω by

$$\chi = \int_C [g_{QP} - Z_Q \partial g_{QP}/\partial \nu] F(C)\, dC, \qquad P \in \Omega. \quad (51)$$

Comparison with (41) shows that

$$\chi = \Psi_1, \qquad P \in \Omega_+. \quad (52)$$

It is now necessary to widen the definitions of the quantities Φ_1 and Φ_2, introduced in (14) and (15), respectively. The series in (14) and (15) are quite general, but it is only in particular regions, such as $\tilde{\Omega}$, that it is useful to choose the expansion coefficients (the A_m and the B_m) as constants. Define

$$\Phi_1 = \sum_{m=-\infty}^{\infty} \lambda_m^-(\rho) J_m(k\rho) \exp(jm\varphi) \quad (53)$$

$$\Phi_2 = \sum_{m=-\infty}^{\infty} \lambda_m^+(\rho) H_m^{(2)}(k\rho) \exp(jm\varphi). \quad (54)$$

It follows from the notation of Fig. 1 and from (12), (40), (45)–(48), and (51) [and also from (106)] that:

$$\lambda_m^-(\rho) = \int_{L-(\rho)} F(C) \tilde{H}_m^{(2)}(k,r) \exp(-jm\theta)\, dC \quad (55)$$

$$\lambda_m^+(\rho) = \int_{L+(\rho)} F(C) \tilde{J}_m(k,r) \exp(-jm\theta)\, dC \quad (56)$$

where the tilde on H and J is a shorthand notation

$$j4\tilde{W}_m(k,r) = W_m(kr) - \tfrac{1}{2}k[W_{m-1}(kr) \exp(j\alpha) - W_{m+1}(kr) \exp(-j\alpha)]Z(C) \quad (57)$$

where W denotes either J or $H^{(2)}$. The easiest way to confirm the preceding results seems to be to expand g_{QP} in (51) by the addition theorem (12) before differentiating with respect to ν. Reference to (49) and (50) shows that

$$\lambda_m^-(\rho) = A_m \quad \lambda_m^+(\rho) = 0, \qquad P \in \Omega_{--} \quad (58)$$

$$\lambda_m^+(\rho) = B_m \quad \lambda_m^-(\rho) = 0, \qquad P \in \Omega_{++} \quad (59)$$

where A_m and B_m are introduced in (14) and (15). It then follows from (52) that:

$$\Psi_1 \in \mathfrak{A}\{\Omega_{++}\} \quad (60)$$

on making use of the notation introduced in (17). On referring to (18) and (52)–(59), it is seen that (60) implies that

$$\Psi_1 = \sum_{m=-\infty}^{\infty} B_m H_m^{(2)}(k\rho) \exp(jm\varphi), \qquad P \in \Omega_{++}. \quad (61)$$

The analytic continuation procedure by which (26) is got from (25) can only trace out the convex hull $\mathfrak{C}_c$ of the singularities of a field. A different procedure [12]–[15] is needed to isolate the concavities (if there are any) in the actual hull $\mathfrak{C}$ of the singularities.

It follows from the second fundamental principle of Section III that RHS (61) is a valid representation of the analytic continuation of Ψ_1 for all of Ω outside the circle Γ_0, centered at 0, and of radius r_0 (see Fig. 6), where r_0 is the minimum radius of convergence of RHS (61). However, a different expression is needed to represent the analytic continuation of Ψ_1 throughout the interior of the circle $\tilde{\Gamma}$, centered at $\tilde{0}$, and of radius $\tilde{r}$, shown in Fig. 6. Note that $\tilde{\Gamma}$ intersects both Γ_0 and $\mathfrak{C}_c$. Using the addition theorem referred to in the derivation of (26), the analytic continuation of RHS (61) into the interior of $\tilde{\Gamma}$ is found to be

$$\sum_{m=-\infty}^{\infty} \tilde{A}_m J_m(k\tau) \exp(jm\psi), \qquad \tau < \tilde{r} \quad (62)$$

$$\tilde{A}_m = \sum_{n=-\infty}^{\infty} B_n H_{n+m}^{(2)}(k\tilde{\rho}) \exp(jn\tilde{\varphi}) \quad (63)$$

where the $\tilde{A}_m$ are determined by comparing (61) and (62) within the intersection of the interiors of Γ_0 and $\tilde{\Gamma}$ (refer to the third fundamental principle of Section III).

If $\tilde{r}$ is the maximum radius of convergence of RHS (62), then $\tilde{\Gamma}$ must be tangent to $\mathfrak{C}$ at one point at least. Since, as was indicated in Section III, the singularities of fields are unique, $\tilde{\Gamma}$ cannot intersect the interior of $\mathfrak{C}$. In fact, $\tilde{\Gamma}$ can only intersect the interior of $\mathfrak{C}_c$ if $\mathfrak{C}$ itself possesses concavities.

By successive applications of the addition theorem, Ψ_1 can be continued analytically throughout the exterior of $\mathfrak{C}$.

When C has a sharp corner, as at Q_0 in Fig. 7, then $\mathfrak{C}$ almost always touches C at the corner. Using the polar coordinates τ and ψ introduced in Fig. 7, the total field in Ω_+ in the neighborhood of Q_0 must be expressible in the form [6]

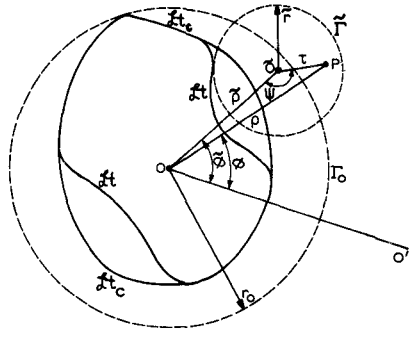

Fig. 6. Analytic continuation inside the convex hull of singularities.

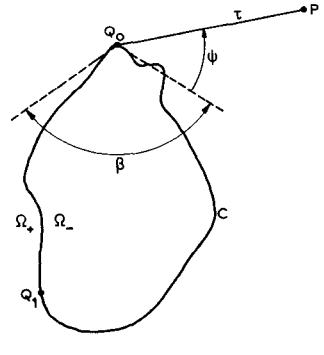

Fig. 7. Curve C having a corner.

$$\Psi = \sum_{m=1}^{\infty} c_m J_{\mu_m}(k\tau) \sin(\mu_m \psi + \Theta_m) \tag{64}$$

where the c_m are constants, and the μ_m satisfy the boundary conditions set by the surface impedances Z_1 at $\psi = 0$, and Z_2 at $\psi = (2\pi - \beta)$, in the neighborhood of Q_0. So the μ_m and the Θ_m are the values of μ and Θ, respectively, which satisfy

$$\mu Z_1 = \tan(\Theta) \quad \mu Z_2 = \tan([2\pi - \beta]\mu + \Theta) \tag{65}$$

where, because the physical field Ψ must be finite at $\tau = 0$, the real part of μ must be nonnegative. Measuring distance along C from Q_0, the surface-source density is given, from (42) and (64), by

$$F(C) = \frac{1}{C} \sum_{m=1}^{\infty} F_m^- J_{\mu_m}(-kC), \qquad C < 0$$

$$= \frac{1}{C} \sum_{m=1}^{\infty} F_m^+ J_{\mu_m}(kC), \qquad C > 0 \tag{66}$$

in, at least, a neighborhood of Q_0, where the constants $F_m^{\pm}$ are given by

$$F_m^- = -\mu_m c_m \cos(\Theta_m) \tag{67}$$

$$F_m^+ = \mu_m c_m \cos([2\pi - \beta]\mu_m + \Theta_m). \tag{68}$$

Provided that C is analytic and Z varies analytically along C then, by the second fundamental principle of Section III, the series in (66) are valid representations for $F(C)$ for values of C for which the series converge. If C_1 is the distance along C from Q_0 to Q_1, and if C_1 is the radius of convergence of the second series in (66), then

$$F(C) = \frac{1}{C} \sum_{m=1}^{\infty} F_m^+ J_{\mu_m}(kC), \qquad 0 < C < C_1. \tag{69}$$

It is important to realize that the nature of Bessel functions [8] is such that, unless all the μ_m are positive integers, $F(C)$ must cease to be analytic at Q_0.

V. THE EXTINCTION THEOREM

On repeating the derivation of the quantities $\mathfrak{F}_0$ and $\mathfrak{F}_1$, as presented in Section IV, but for Ω_-, as opposed to Ω_+, it is found that [17]

$$\chi = -\Psi_0, \qquad P \in \Omega_- \tag{70}$$

where χ is defined in (51). Now χ represents the reradiations, throughout all Ω, from the equivalent surface sources on C. So it follows from (70) that these reradiations extinguish the incident field in Ω_-. This is obvious, on physical grounds alone, when C is perfectly conducting; but it is nevertheless true in the general case, as seems to have been noticed first by Love [21]. However, this "extinction theorem," as it is called, dates back to 1915 in the optical literature, in which it is usually associated with the names of Ewald and Oseen. It has been reexamined in detail by optical scientists recently [22], [23]. Waterman [11] has developed it, under the name of the "extended boundary condition" (EBC), into a systematic technique for solving boundary value problems. There have been a number of extensions of Waterman's methods and these have been reviewed very recently [24], [25].

Since $\Omega_{--} \subset \Omega_-$ (see Fig. 2), it follows from (70) that

$$\chi = -\Psi_0, \qquad P \in \Omega_{--}. \tag{71}$$

At any radial distance ρ from the origin 0, the angular coordinate φ spans the interval $0 \leq \varphi < 2\pi$ continuously within Ω_{--}. This means that within Ω_{--} individual angular Fourier coefficients of χ and $(-\Psi_0)$ can be equated. It then follows from (13), (21), (53), and (54) that:

$$\lambda_m^-(\rho) J_m(k\rho) + \lambda_m^+(\rho) H_m^{(2)}(k\rho) = -a_m J_m(k\rho), \qquad 0 \leq \rho < r_+ \tag{72}$$

for all integers m. It is useful to introduce a notation for the set of integers I_1 to I_2, inclusive

$$\mathfrak{g}(I_1, I_2) = \{I_1, I_1 + 1, I_1 + 2, \cdots, I_2 - 1, I_2\}. \tag{73}$$

Reference to (49), (50), (55), and (56) then shows that (72) reduces to (note that $\lambda_m^+(\rho) = 0$ for $\rho < r_-$)

$$\lambda_m^-(r_-) \doteq \int_C F(C) \tilde{H}_m^{(2)}(k,r) \exp(-jm\theta)\, dC = -a_m, \qquad m \in \mathfrak{g}(-\infty, \infty) \tag{74}$$

which is an infinite set of nonsingular integral equations for $F(C)$, the null-field equations [26]. They should be compared with the single, singular integral equation (44). It is singular because $g_{QQ'}$ ceases to be analytic as Q tends to Q'. The relative merits of (44) and (74) are discussed in Sections XII–XIV.

Choosing a suitable set $\{f_n(C);\ n \in \mathfrak{g}(-\infty, \infty)\}$ of basis functions, $F(C)$ can be expressed as

$$F(C) = \sum_{n=-\infty}^{\infty} F_n f_n(C) \tag{75}$$

where the F_n are constants. By the first fundamental principle of Section III, this expansion is only useful if it converges everywhere on C, except at those points, such as corners, where $F(C)$ must be singular (refer to the final paragraph of Section IV). Substituting (75) into (74) gives

$$\sum_{n=-\infty}^{\infty} F_n K_{n,m} = -a_m, \qquad m \in \mathfrak{g}(-\infty, \infty) \tag{76}$$

$$K_{n,m} = \int_C f_n(C) \tilde{H}_m^{(2)}(k,r) \exp(-jm\theta)\, dC, \qquad m,n \in \mathfrak{g}(-\infty, \infty). \tag{77}$$

The numerical solution of (76) is discussed in Section IX.

When $F(C)$ satisfies (74), then χ, which is defined by (51), satisfies

$$\chi + \Psi_0 = 0, \qquad P \in \Omega_{--} \tag{78}$$

which is merely another way of writing (71). But both χ

and Ψ_0 are analytic throughout Ω_-, because the sources of χ are on C, and the sources of Ψ_0 are in $\Omega_0 \subset \Omega_+$. The uniqueness of analytic continuation [10] then ensures that

$$\chi + \Psi_0 = 0, \qquad P \in \Omega_- \tag{79}$$

in agreement with (70).

It is worth noting that χ actually does cease to be analytic on C, since [refer to (51)] g_{QP} ceases to be analytic as P tends to Q. The point is that χ represents different quantities on either side of C. This is important, because if it were not true, the uniqueness of analytic continuation would force χ and $(-\Psi_0)$ to be equal in Ω_+. To belabor this: the integral in (51) does not have to cease to exist (i.e., become indeterminate) as P approaches C; it is merely that derivatives, of (an) order higher than some finite order, of this integral with respect to ρ and/or φ must cease to exist as P approaches C.

VI. THE RAYLEIGH HYPOTHESIS

In the nineteenth century Rayleigh [27] realized that theoretical descriptions of diffraction phenomena would be much simplified if any reradiated field could be represented by a single expansion everywhere outside the object scattering the incident field. This conjecture is now known as the Rayleigh hypothesis. It can be stated concisely using the notation introduced in Section III.

Rayleigh Hypothesis

$$\Psi_1 \in \mathfrak{A}\{\Omega_+\}.$$

It has already been demonstrated [refer to (60)] that $\Psi_1 \in \mathfrak{A}\{\Omega_{++}\}$. It remains to examine Ψ_1 in Ω_{+-} (see Fig. 2).

It is helpful to introduce a "critical circle," denoted by Γ_0 in Fig. 8. This is the same as the Γ_0 shown in Fig. 6. Its radius r_0, the "critical radius," is the minimum radius of convergence of the series on RHS (61). By the second fundamental principle of Section III, this series is a valid representation of Ψ_1 for $\rho > r_0$, and it follows from (60) that

$$r_0 \leq r_+. \tag{80}$$

It is clear that the Rayleigh hypothesis is valid if and only if $r_0 \leq r_-$. This means that $\mathfrak{K}$ (see Fig. 6) must not intersect Ω_{-+} (see Fig. 2).

What is required is an *a priori* method for gauging the validity of the Rayleigh hypothesis. The analytic continuation procedures introduced in Sections III and IV are of no use for this because they permit $\mathfrak{K}$ to be traced out only *a posteriori*. However, as has been discussed in physical terms by Nevière *et al.* [28], and as is examined in a little detail which follows, the method of conformal transformation can be adapted to estimate the positions of (at least some of) the singularities of the field.

In Fig. 8 the point P is accorded Cartesian coordinates x and y as well as polar coordinates ρ and φ. It is convenient to think of a complex plane [see Fig. 9(a)] superimposed upon the physical x,y plane. This complex plane is called the z plane, where

$$z = x + jy. \tag{81}$$

Since the complex number z is employed only in this section, there need be no confusion with the Cartesian coordinate z introduced in Section I. If $\tilde{\Omega}_+$ is all of Ω_+ apart from Ω_0, then it follows from (4)–(9) that:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + k^2\Psi = 0, \qquad P \in \tilde{\Omega}_+ \tag{82}$$

which can be rewritten as

$$4\frac{\partial^2\Psi}{\partial z \partial z^*} + k^2\Psi = 0, \qquad P \in \tilde{\Omega}_+ \tag{83}$$

on using (81). The asterisk denotes the complex conjugate.

A new complex variable ζ is introduced

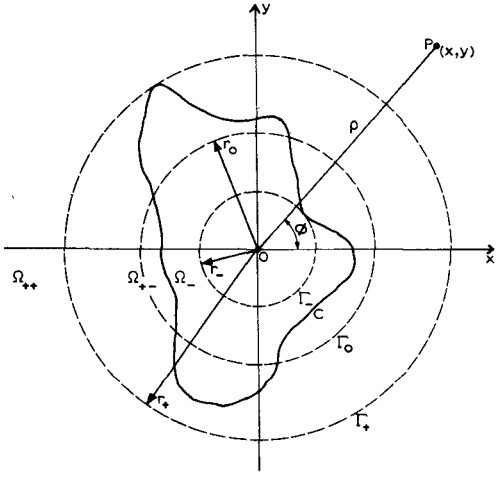

Fig. 8. Regions and coordinate systems needed for examining the Rayleigh hypothesis.

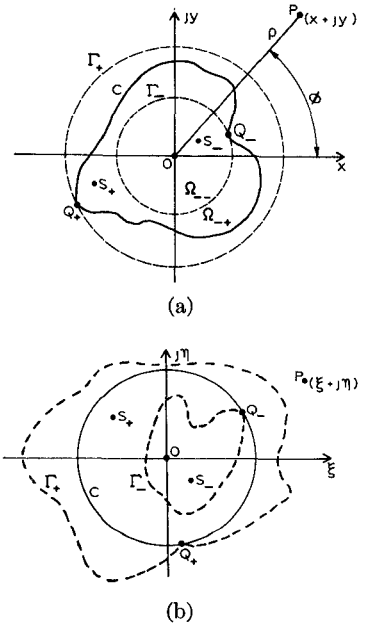

Fig. 9. Conformal transformation of C. (a) C mapped onto complex z plane. (b) C transformed into circle in complex ζ plane.

$$\zeta = \xi + j\eta. \tag{84}$$

This variable ζ is thought of as a function of z, and z is also thought of as a function of ζ

$$\zeta = \zeta(z) \quad z = z(\zeta). \tag{85}$$

The form of $\zeta(z)$ is chosen so that C is transformed into a circle in the ζ plane [see Fig. 9(b)], and Ω_+ is transformed single valuedly onto the exterior of this circle. Now

$$\frac{\partial^2\Psi}{\partial z \partial z^*} = \left|\frac{\partial\zeta}{\partial z}\right|^2 \frac{\partial^2\Psi}{\partial\zeta\partial\zeta^*} \tag{86}$$

so that (82) is transformed into

$$\frac{\partial^2\Psi}{\partial\xi^2} + \frac{\partial^2\Psi}{\partial\eta^2} + k^2\left|\frac{\partial z}{\partial\zeta}\right|^2\Psi = 0, \qquad P \in \tilde{\Omega}_+ \tag{87}$$

which implies that the original problem of an arbitrarily shaped diffracting body embedded in a homogeneous space has been exchanged for a body of simple shape (a circle) embedded in an inhomogeneous space. The factor $|\partial z/\partial\zeta|$ plays the part of a variable refractive index.

It seems that provided Ω_- is simply connected, $z(\zeta)$ is analytic throughout Ω_+. Algorithms useful for computing $z(\zeta)$ are known [28]–[30]. It is obvious on purely physical grounds that a field must exhibit singular behavior at points in a medium where the refractive index ceases to be analytic. Consequently, at least some of the singularities of the analytic continuation of Ψ_1 into Ω_- (see Section IV) must coincide with the singularities of $|\partial z/\partial\zeta|$.

Suppose that points $S_\pm$ (see Fig. 9) are singular points of $|\partial z/\partial\zeta|$. Note that the positions of these can be computed in either the z plane or the ζ plane, depending upon which of $z(\zeta)$ or $\zeta(z)$ is the easier to examine. If any points such as S_+ exist, then the Rayleigh hypothesis must fail because [refer to Fig. 9(a)] these are singularities in Ω_{-+}. The Rayleigh hypothesis may be valid if the only singularities of $|\partial z/\partial\zeta|$ are at points such as S_-. It should be kept in mind that it is not yet clear how important are singularities other than those of $|\partial z/\partial\zeta|$.

Known algorithms [28]–[30] for numerical computation of conformal transformations seem suitable for computing the points in the z and ζ planes where $\partial\zeta/\partial z$ and $\partial z/d\zeta$, respectively, are zero. All these points are singularities because, on physical grounds, the field must be singular wherever the refractive index is zero or infinite. Consequently, there should be no difficulty in computing the positions of all points such as S_+.

VII. STRAIGHTFORWARD POINT MATCHING

When the Rayleigh hypothesis holds, RHS (61) is a valid representation of Ψ_1 everywhere in Ω_+. A useful approximate expression for Ψ_1 is then

$$\Psi_1(\rho,\varphi) \approx \sum_{m=-M}^{M} B_m H_m^{(2)}(k\rho)\exp(jm\varphi), \qquad P \in \Omega_+ \tag{88}$$

where the positive integer M is large enough that Ψ_1 is approximated to within some required accuracy.

A set $\{Q_n\}$ of $(2M+1)$ points is chosen on C. The polar coordinates of Q_n are r_n and θ_n.

The normalized impedance, introduced in (39), can be looked upon as a boundary condition. So if both Ψ_0 and Z are known (or specified) at each Q_n, the B_m can be found by solving the $(2M+1)$ linear algebraic equations

$$Z(r_n,\theta)\frac{\partial\Psi(r_n,\theta_n)}{\partial\nu_n} = \Psi(r_n,\theta_n), \qquad n \in \mathcal{I}(1,2M+1) \tag{89}$$

where (40) and (73) have been used, and the outward normal direction to C at Q_n is the ν_n direction. It follows from (4) and (88) that:

$$\sum_{m=-M}^{M} B_m\left[Z(r_n,\theta_n)\frac{\partial(H_m^{(2)}(kr_n)\exp(jm\theta_n))}{\partial\nu_n} - H_m^{(2)}(kr_n)\exp(jm\theta_n)\right] = \Psi_0(r_n,\theta_n) - Z(r_n,\theta_n)\frac{\partial\Psi_0(r_n,\theta_n)}{\partial\nu_n}, \quad n \in \mathcal{I}(1;2M+1) \tag{90}$$

from which the B_m can be found numerically by standard elimination procedures. This straightforward point-matching (SPM) method, the electromagnetic applications of which are due initially to Mullin *et al.* [31] and Yee and Audeh [32], has been reviewed lately [33], [34]. It is discussed further in Sections XII–XIV, together with other more sophisticated point-matching techniques.

VIII. BOUNDARY SMOOTHING

Since Ψ_1 is analytic throughout Ω_+, there must exist a set $\{V_m = V_m(\rho,\varphi);\ m \in \mathcal{I}(-\infty,\infty)\}$ of wave functions such that

$$\Psi_1 = \sum_{m=-\infty}^{\infty} C_m V_m, \qquad P \in \Omega_+ \tag{91}$$

where the C_m are constants, and each of the V_m satisfies the impedance boundary condition (39)

$$Z(r,\theta) = V_m(r,\theta)/(\partial V_m(r;\theta)/\partial\nu), \quad m \in \mathcal{I}(-\infty,\infty) \tag{92}$$

where use has been made of (40). Full mathematical justification of (91) is probably difficult, but the existence of an infinite number of V_m, each satisfying (92), is a physical necessity to ensure that variations in the expansion coefficients, the a_m [refer to (21)], of the incident field result in corresponding, unique variations in the expansion coefficients, the C_m [refer to (91)], of the scattered field.

By the third fundamental principle of Section III, the B_m in (61) can be found in terms of the C_m in (91), provided that RHS (61) and RHS (91) are compared in Ω_{++}. Remember that (61) is exact, because of (60). If P travels around a circle centered at the origin 0 of coordinates, it does not cross C if $\rho > r_+$, i.e., if P is in Ω_{++}. So it follows from (61) and (91) that

$$B_m = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{C_n}{H_m^{(2)}(k\rho)} \int_0^{2\pi} V_n(\rho,\varphi) \exp(-im\varphi)\, d\varphi, \qquad \rho > r_+. \quad (93)$$

When C and Z are arbitrary there is no simple way of constructing the V_m, which means that (93) is not useful in computational practice. However, neither is (61) useful in computational practice because it contains an infinite number of unknowns. Recourse must be had to (88) which implies that the expression

$$\Psi_1 = \sum_{m=-M}^{M} C_m V_m, \qquad P \in \Omega_+ \quad (94)$$

should be used instead of (91).

Each of the $H_m^{(2)}(k\rho)$ is analytic throughout Ω_+, so that a finite sum of these functions can never cease to be analytic within Ω_+. Somewhat similar reasoning has led Ikuno and Yasuura [35] to suggest that (88) should be valid throughout Ω_+, whether or not RHS (61) converges throughout $\Omega_+ -$. In order to overcome the deficiences of SPM in cases where the Rayleigh hypothesis fails, the boundary condition (39) is satisfied in a least squares sense on C. This accounts for the title "boundary smoothing" given to this section. Combining (4), (39), and (88) gives

$$\sum_{m=-M}^{M} B_m \left[\frac{\partial}{\partial \nu} (H_m^{(2)}(kr) \exp(jm\theta)) Z(r,\theta) - H_m^{(2)}(kr) \exp(jm\theta) \right] = \Psi_0(r,\theta) - Z(r,\theta)\, \partial\Psi_0(r,\theta)/\partial\nu. \quad (95)$$

There are several ways of estimating the B_m from (95). A convenient method is begun by expressing r as a function of θ. It may not always be possible to do this single valuedly in $0 \leq \theta < 2\pi$. However, provided that the curve C is singular at no more than a finite number of points, it is always possible to divide C into a finite number i, of parts throughout each of which r is single valued in θ. On the nth part, r can be written unambiguously as

$$r = r(\theta) \quad \theta_{n1} \leq \theta < \theta_{n2}, \qquad n \in \mathcal{I}(1,i). \quad (96)$$

The set $\{\theta_{n1} \leq \theta < \theta_{n2}$, where $n \in \mathcal{I}(1,i)\}$ of i intervals spans the complete interval $0 \leq \theta < 2\pi$ without any gaps, but intervals must overlap (if it is found that no intervals overlap then r is, in fact, single valued in θ; and i can be reduced to unity). After (96) is substituted into (95), the form of the latter is suitable for direct evaluation of its trigonometrical Fourier coefficients. From those of order $-M$ through M, a set of $(2M + 1)$ inhomogeneous linear algebraic equations is obtained for the B_m. These can be solved numerically by standard elimination procedures. The B_m computed in this way are optimum in a least squares sense, because of the orthogonality of trigonometrical functions in the interval $0 \leq \theta < 2\pi$.

This method is equivalent to Bolomey and Wirgin's [36] total support moment treatment of the Rayleigh hypothesis. Its theoretical justification and its practical computational significance are much the same as Davies' boundary residual method [37] and the simplified approach to the EBC (see Section IX). Ikuno and Yasuura [35] evaluate the integrals by which the Fourier coefficients of (95) are obtained by a point-matching approximation.

IX. EXTENDED BOUNDARY CONDITION

To obtain numerical values for the quantity $F(C)$ discussed in Section V, the summation in (75) must be truncated

$$F(C) \approx \sum_{n=-N}^{N} F_n f_n(C) \quad (97)$$

so that (74) becomes

$$\sum_{n=-N}^{N} F_n K_{n,m} = -a_m, \qquad m \in \mathcal{I}(-N,N) \quad (98)$$

where the $K_{n,m}$ are defined in (77). The integer m in (98) is restricted to $\mathcal{I}(-N,N)$ so that a linear system of $(2N + 1)$ equations in $(2N + 1)$ unknowns is obtained. Provided the integrations in (77) are carried out with sufficient numerical accuracy, the F_n can be determined from (98) by standard elimination procedures to the accuracy inherent in (97). This accuracy can be improved by increasing N.

Waterman [11] shows how to compute the scattered field without first having to evaluate $F(C)$ explicitly. This computation can be simplified when the Rayleigh hypothesis is valid. Using (4), (21) with the summation truncated to $(2M + 1)$ terms, (42), and (88), it follows that all the expansion coefficients in (97) can be replaced by unity, i.e.,

$$F_n = 1, \qquad n \in \mathcal{I}(-N,N) \quad (99)$$

provided that $N = M$ and the basis functions have the form

$$-j4f_n(C) = a_n \partial(J_n(kr) \exp(jn\theta))/\partial\nu - B_n \partial(H_n^{(2)}(kr) \exp(jn\theta))/\partial\nu. \quad (100)$$

It is convenient to define

$$K_{n,m}^{(0)} = \delta_{n,m} + \int_C \frac{\partial(J_n(kr) \exp(jn\theta))}{\partial\nu} \tilde{H}_m^{(2)}(k,r) \cdot \exp(-jm\theta)\, dC \quad (101)$$

$$K_{n,m}^{(1)} = \int_C \frac{\partial(H_n^{(2)}(kr) \exp(jn\theta))}{\partial\nu} \tilde{H}_m^{(2)}(k,r) \cdot \exp(-jm\theta)\, dC \quad (102)$$

where $\delta_{p,q}$ is the Kronecker delta

$$\left.\begin{aligned} \delta_{p,q} &= 1, \quad p = q \\ &= 0, \quad p \neq q \end{aligned}\right\}, \qquad p,q \in \mathcal{I}(-\infty,\infty). \quad (103)$$

Substituting (99)–(102) into (98) and (78) gives

$$\sum_{n=-M}^{M} B_n K_{n,m}^{(1)} = \hat{a}_m, \qquad m \in \mathcal{J}(-M,M) \tag{104}$$

$$\hat{a}_m = \sum_{n=-M}^{M} a_m K_{n,m}^{(0)}. \tag{105}$$

If the incident field and the scattering body are given, $\hat{a}_m$ can be computed directly from (21) and (101), so that the B_n (which characterize the scattered field completely) can be obtained from (104) by standard elimination procedures.

X. STATE-VARIABLE FORMULATIONS

Vincent and Petit [38], Petit and Maystre [39], and Hizal and Tosun [40], [41] have noticed that wave scattering can be cast in a generalized Riccati equation form, so that the extensive theoretical and computational apparatus associated with state variables [42] can be invoked.

Inspection of Fig. 1 indicates that

$$\frac{\partial}{\partial \nu} = \cos(\alpha)\frac{\partial}{\partial r} - \sin(\alpha)\frac{\partial}{r\,\partial\theta}. \tag{106}$$

Combining (4), (21), (13), and (52)–(54) leads to

$$\Psi = \sum_{m=-\infty}^{\infty} [(a_m + \lambda_m^{-}(\rho))J_m(k\rho) + \lambda_m^{+}(\rho)H_m^{(2)}(k\rho)] \cdot \exp(jm\varphi), \qquad P \in \Omega_+. \tag{107}$$

Substituting (106) and (107) into (42), and using the recurrence relations for Bessel functions [8], gives

$$\begin{aligned} F(C) = {} & \eta(r,\theta) \\ & - \cos(\alpha) \sum_{m=-\infty}^{\infty} [\lambda_m^{-\prime}(r)J_m(kr) \\ & + \lambda_m^{+\prime}(r)H_m^{(2)}(kr)] \exp(jm\theta) \\ & + \tfrac{1}{2}k \sum_{m=-\infty}^{\infty} ([\lambda_m^{-}(r)J_{m+1}(kr) \\ & + \lambda_m^{+}(r)H_{m+1}^{(2)}(kr)] \exp(j\alpha) \\ & - [\lambda_m^{-}(r)J_{m-1}(kr) + \lambda_m^{+}(r)H_{m-1}^{(2)}(kr)] \\ & \cdot \exp(-j\alpha)) \exp(jm\theta) \end{aligned} \tag{108}$$

where the prime denotes differentiation with respect to r, and

$$\eta(r,\theta) = \tfrac{1}{2}k \sum_{m=-\infty}^{\infty} a_m[J_{m+1}(kr)\exp(j\alpha)J_{m-1}(kr)\exp(-j\alpha)] \cdot \exp(jm\theta). \tag{109}$$

Inspection of (45)–(48), (55), and (56) indicates that

$$\lambda_m^{-\prime}(r) = -F(C)\tilde{H}_m^{(2)}(k,r)\exp(-jm\theta)\csc(\alpha) \tag{110}$$

$$\lambda_m^{+\prime}(r) = F(C)\tilde{J}_m(k,r)\exp(-jm\theta)\csc(\alpha) \tag{111}$$

where use has been made of

$$dC/dr = \csc(\alpha) \tag{112}$$

which follows from inspection of Fig. 1. On postulating that

$$\lambda_m^{-}(r) \approx \lambda_m^{+}(r) \approx 0, \qquad |m| > M = M(r) \tag{113}$$

where M is a finite positive integer, it is seen that substituting (108) into (109) and (110) leads to a generalized Riccati equation for the $\lambda_m^{\pm}$.

A circle of radius r and centered at 0, in general, intersects C at an even number of points, each of which is denoted by $Q_q(r)$ where $q \in \mathcal{J}(1,2l)$ and where, as is seen from (96),

$$l = l(r) \leq i. \tag{114}$$

The $Q_q(r)$ are ordered counterclockwise around the circle. It is convenient to denote the values, at $Q_q(r)$, of $\lambda_m^{\pm}$, θ, and α by $\lambda_{m,q}^{\pm}$, θ_q, and α_q, respectively. Note that

$$\theta_q = \theta_q(r) \qquad \alpha_q = \alpha_q(r). \tag{115}$$

It follows from (108)–(114) that the Riccati equation can be written as:

$$(I + X)\Lambda' + Y\Lambda = W \tag{116}$$

where I is the identity matrix, X and Y are square matrices, and Λ and W are column matrices, all of order $4(2M+1)l$. The elements of Λ and W are defined by

$$\left.\begin{aligned} \Lambda_{2pl+q} &= \lambda_{-M+p,q}^{+}(r) \\ \Lambda_{2(2M+1)l+2pl+q} &= \lambda_{-M+p,q}^{+}(r) \end{aligned}\right\}, \quad p \in \mathcal{J}(0,2M)$$

$$\left.\begin{aligned} W_{2pl+q} &= -\eta(r,\theta_q)\tilde{H}_m^{(2)}(k,r) \cdot \exp(-jm\theta_q)\csc(\alpha_q) \\ W_{2(2M+1)l+2pl+q} &= \eta(r,\theta_q)\tilde{J}_m(k,r) \cdot \exp(-jm\theta_q)\csc(\alpha_q) \end{aligned}\right\}, \quad q \in \mathcal{J}(1,2l). \tag{117}$$

The elements of X and Y are now found by inspection of (108)–(111).

Inspection of Figs. 1 and 2 shows that (116) needs to be solved in the range $r_- \leq r \leq r_+$. Two general methods of solution are now considered.

A. Two-Point Boundary Value Problem

On referring to (58), (59), Figs. 1 and 2, and (117) it is seen that

$$\Lambda_{2(2M+1)l+2pl+q}(r_+) = \Lambda_{2pl+q}(r_-) = 0, \qquad p \in \mathcal{J}(0,2M), \qquad q \in \mathcal{J}(1,2l). \tag{118}$$

The solution of the Ricatti equation gives the initially unknown quantities A_m and B_m quoted in (58) and (59). Because the boundary conditions (118) are specified only partly at each end of the range $r_- \leq r \leq r_+$, a two-point boundary value problem results, as in the work of Vincent

and Petit [38], Petit and Maystre [39], and Hizal and Tosun [40].

B. Initial Value Boundary Value Problem

On using (74) and the part of the boundary condition (118) referring to $r = r_-$, the column matrix Λ is completely specified at $r = r_-$, so that, as in the work of Hizal [41], an initial value boundary value problem results

$$\left.\begin{aligned} \Lambda_{2pl+q}(r_-) &= 0 \\ \Lambda_{2(2M+1)l+2pl+q}(r_-) &= -a_p \end{aligned}\right\}, \quad \begin{aligned} &p \in \mathcal{I}(0,2M) \\ &q \in \mathcal{I}(1,2l). \end{aligned} \tag{119}$$

Note that, as reference to (21) shows, the a_p specify the incident field.

XI. FORMULAS FOR CLOSED EMPTY WAVEGUIDES

The methods so far discussed have practical application to the computation of the characteristics of closed, empty, metallic waveguides, for which the fields are confined to Ω_-. This section lists those pertinent formulas which differ from the ones already presented.

Because the fields do not vary in the z direction, the formulas apply to waveguides only at cutoff. But this is not a significant restriction from a practical point of view, because the behavior of a propagating mode in a closed waveguide can be deduced straightforwardly from its cutoff characteristics [43].

A line source running the whole length of a closed waveguide is rarely of interest. What matters are the modes which can be supported by the waveguide. Because common metals in their commercially available states are so highly conducting, modal characteristics can usually be computed to acceptable accuracy by assuming that the waveguide wall is perfectly conducting

$$Z = \begin{cases} 0 & \text{for electrically polarized field} \\ \infty & \text{for magnetically polarized field} \end{cases} \tag{120}$$

which means that the currents running in the waveguide wall can be looked upon as the sole seat of the field

$$\Psi = \Psi_1 \quad \Psi_0 \equiv 0. \tag{121}$$

The hull $\hat{\mathcal{K}}$ of the singularities of Ψ is unique and lies in $\Omega_+ \cup C$. It can be traced out by the same types of analytic continuation procedure which are appropriate for tracing out $\mathcal{K}_c$ and $\mathcal{K}$ (see Sections III and IV, respectively).

The singular integral equation corresponding to (44) is

$$\int_C \left[\frac{\partial g_{QQ'}}{\partial \nu'} - Z_Q \frac{\partial g_{QQ'}}{\partial \nu \partial \nu'}\right] F(C)\, dC Z_{Q'} = \int_C [g_{QQ'} - Z_Q \partial g_{QQ'}/\partial \nu] F(C)\, dC \tag{122}$$

which is the genotype of the equations studied by Spielman and Harrington [44].

The extinction theorem requires that

$$\Psi = 0, \qquad P \in \Omega_+ \tag{123}$$

so that the null-field equations, corresponding to (74), are [45]

$$\int_C F(C) \tilde{J}_m(k,r) \exp(-jm\theta)\, dC = 0, \qquad m \in \mathcal{I}(-\infty,\infty). \tag{124}$$

An internal Rayleigh hypothesis [33] can be postulated

$$\Psi = \sum_{m=-\infty}^{\infty} A_m J_m(k\rho) \exp(jm\varphi), \qquad P \in \Omega_- \tag{125}$$

which is equivalent to assuming that $r_0 \geq r_+$, where r_0 is the radius of convergence of RHS (125). Whether or not the analytical continuation of Ψ into Ω_+ has singularities in Ω_{+-} can be estimated by the conformal transformation procedure described in Section VI. However, in this case, Ω_- is mapped single valuedly onto the interior of the circle in the ζ plane; and singular points $S_\pm$ are in Ω_+, S_+ being in Ω_{+-}, and S_- in Ω_{++}.

The SPM equations, which are based on (125), become [33]

$$\sum_{m=-M}^{M} A_m \left[Z(r_n,\theta_n) \frac{\partial (J_m(kr_n) \exp(jm\theta_n))}{\partial \nu_n} - J_m(kr_n) \exp(jm\theta_n) \right] = 0, \quad n \in \mathcal{I}(1,2M+1) \tag{126}$$

where the coordinates r_n and θ_n are defined in the second paragraph of Section VII. The boundary-smoothing method is still expressed by (95), but with zero on the RHS and with B_m replaced by A_m. For the simplified EBC method (based on the internal Rayleigh hypothesis) the basis functions appearing in (97) are given by

$$f_n(C) = \tfrac{1}{4} j A_n\, \partial (J_n(kr) \exp(jn\theta))/\partial \nu \tag{127}$$

and H is replaced by J in the integrand in (102). The B_n are then found from

$$\sum_{n=-M}^{M} B_n K_{n,m}^{(1)} = 0. \tag{128}$$

The Riccati equation, derived from the state-variable formulation, is still (116) but with

$$W \equiv 0. \tag{129}$$

The boundary conditions for the two-point problem are still expressed by (118). For the initial value problem, however, the boundary condition is

$$\Lambda(r_-) = 0. \tag{130}$$

When any of the methods are reduced to computable form, a set of homogeneous equations results, so that only trivial solutions exist except when k takes on one of its eigenvalues (i.e., a cutoff wavenumber for one of the modes appropriate to the waveguide of cross section C).

XII. DISCUSSION OF SOME PARADOXES

It has only been during the last fifteen years that there have been systematic digital computational approaches to the solution of diffraction problems of interest in radio engineering. For the last ten of these, mild controversies have arisen over the validity of certain methods which were introduced for their conputational convenience. This section discusses several of the apparent paradoxes from which the controversies arose.

A. Symmetries and Straightforward Point Matching

Yee and Audeh's [32] SPM method is equivalent to the numerical technique of Mullin *et al.* [31]. It is known that in certain cases the fields computed by SPM are very inaccurate, even though the cutoff wavenumbers are given with an accuracy acceptable for some engineering purposes [33].

SPM involves finding those values of k which force to zero the determinant of the coefficients of the A_m in (126). For any one of these cutoff wavenumbers, (126) is transformed into an inhomogeneous set of $2M$ equations, in $2M$ unknown A_m, by fixing the value of one of the A_m, i.e., it is usual to set $A_0 = 1$. After the A_m are found by standard elimination procedures, the field is computed from (125), with $A_m = 0$ for $|m| > M$.

It is clear from the first fundamental principle of Section III and from the discussion presented in Section VI that the SPM fields are suspect if the (internal and/or external) Rayleigh hypothesis is invalid. But why are the cutoff wavenumbers more trustworthy?

The earliest comment [46] on SPM was unfavorable because of the reliance of SPM on the Rayleigh hypotheses. It was then suggested [47] that the excellent results obtained with SPM were due to certain symmetries exhibited by the waveguides used in illustrative examples [32], [48]. It was afterwards found that satisfactory results could be got from an asymmetrical waveguide [49]. It was finally noticed [50] that the crucial symmetry was not the waveguide cross section, but rather a correspondence between the equations expressing approximate and rigorous approaches to the problem. The determinant, which is nulled to obtain the cutoff wavenumbers, is essentially the same whether it is derived from the SPM equations (126) or from the complete point-matching (CPM) equations, themselves obtained by a point-matching approximation to the rigorous null-field equations (124). The only difference between the SPM and the CPM determinants is that their rows and columns are interchanged.

A final point worth making is that the accuracy of the SPM cutoff wavenumbers falls significantly when the internal Rayleigh hypothesis fails, even though they are the same as the CPM wavenumbers; and the CPM is derived from rigorous equations. It has been established [33], [45] that the reason for this is the crudity of the CPM approximation to the null-field equations. The integrand in (124) tends to vary more rapidly with C in cases for which the internal Rayleigh hypothesis fails, which implies that when using the null-field method, the numerical integrations should be performed with increasing care as C becomes increasingly complicated, as has been confirmed in a study of the ridge waveguide [45].

B. Singularities of the Field

Millar's analysis [16] establishes that the analytic continuation of Ψ_1 into Ω_- has unique singularities there. However, James and Gallett [34] have put forward an argument based on Millar's discussion of scattering from a perfectly conducting elliptical cylinder [51]. Millar points out that Ψ_1 can be expanded in Mathieu functions, all of which cease to be analytic at the left and right foci, f_L and f_R, respectively, of the ellipse (see Fig. 10). James and Gallett reply that this is certainly true, but if Ψ_1 is expressed by RHS (88), then its analytic continuation only becomes singular at the origin 0. This counter is irrefutable as long as Ψ_1 is expressed by a *finite* expansion. However, a rigorous expansion involves an infinite number of terms, as in RHS (61). It then follows from the third fundamental principle of Section III that the minimum radius of convergence of RHS (61) is r_0 (see Fig. 10). Note that if Γ_0 intersects Ω_+, the Rayleigh hypothesis is invalid.

C. Simplified Extended Boundary Condition

The null-field equations (74) follow rigorously from the extinction theorem, so that if N is a large enough, $F(C)$ can be found from (97) and (98) to arbitrary accuracy. The simplified approach (to computing the scattered field) based on the Rayleigh hypothesis (refer to Section IX) is not, of course, universally valid [36]. However, it does not seem to have been generally realized that this simplified EBC method is inherently arbitrarily accurate, if it is first used to compute the surface-source density. Express $F(C)$ by RHS (97) and choose the basis functions to have the form of (100). Assume for the moment that the solution to the problem is known so that the correct value of B_n, for each $n \in \mathcal{I}(-N,N)$, can be inserted in (100). The F_n can be calculated from (97) by standard elimination procedures. If the Rayleigh hypothesis is valid, (99) will be found to be satisfied. However, if the Rayleigh hypothesis is invalid, so that the form of Ψ_1 in the neighborhood of all points Q on C cannot be computed directly

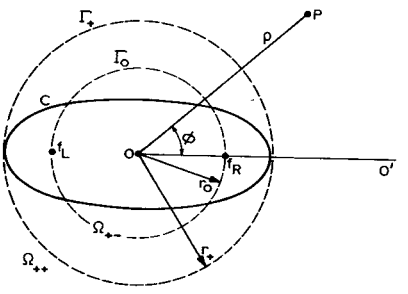

Fig. 10. Elliptical scatterer.

from (88), the F_n will be found to have values different from those given by (99), but if these F_n are substituted into (97), and the B_n are computed from (56) and (59), with m replaced by n, then they must be the same as the B_n in (100), because no approximation has been made apart from the truncation (97) of the exact summation (75). This emphasizes a point that Millar has made [16]: any complete set of basis functions is satisfactory for expressing $F(C)$. However, if RHS (61) does not converge throughout Ω_{+-}, $F(C)$ cannot be equated with $(j/4) \cdot \partial(-\Psi)/\partial\nu$ at each point Q on C, when Ψ and Ψ_1 are given by (4) and (88), respectively.

XIII. CONCLUSIONS DRAWN FROM COMPUTATIONAL PRACTICE

As is confirmed by Ng's Table I [2], SPM [31]–[34] is by far the most economical method from the points of view of programming effort and computer time. But it can only be used with confidence when C is such that the (external or internal) Rayleigh hypothesis is valid. It is now clear that when C is such that the Rayleigh hypotheses are invalid, the field computed by SPM applies to a boundary curve quite different from C. As might be expected, the Rayleigh hypothesis appears to be valid for the new curve [33], [45].

If C has corners for which all the μ_m (refer to the final paragraph of Section IV) are not integers, and if all these corners do not touch Γ_+ or Γ_-, then the internal, or external, Rayleigh hypothesis fails necessarily. Note that for the internal hypothesis, the angle β in Fig. 7 is replaced by $(2\pi - \beta)$ because it is the field in Ω_- which is of interest.

There does not seem to have been any systematic use as yet of conformal transformation for estimating *a priori* the validity of the Rayleigh hypothesis (refer to Section VI). Certain computations of the analytic continuation of fields confirm that the Rayleigh hypothesis can be expected to fail if C has appreciable concavities [33], and the occurrence of singularities is suggested by numerical instabilities. It is the present lack of a general method of estimating r_0 *a priori* from the shape of C that is perhaps the chief difficulty regarding fitting SPM into a general computational scheme.

There are sophisticated point-matching methods which, while being almost as economical as SPM, are uniformly valid, e.g., the methods of overlapping domains [52] and of intermediate expansions [53], and the extended point-matching method [33]. However, every C represents a special case, and the accuracy and efficiency of the computations depend upon the ingenuity of the user.

The various boundary-smoothing methods, which are listed at the end of Section VIII, are much more widely applicable than SPM and are as easy to use. They are much less economical because of the numerical integrations which are necessary, e.g., abstracting the Fourier coefficients of (95). They cannot be used with the same confidence as the methods discussed in the previous paragraph because they rely on the Rayleigh hypotheses. Ikuno and Yasuura [35] have modified the Rayleigh hypothesis such that the scattered far field can be computed with useful accuracy in many situations where the conventional Rayleigh hypothesis fails. In their particular boundary-smoothing procedure they (effectively) invoke the finite summations (88) and (94), so that they can rearrange terms within Ω_{+-} even when the conventional Rayleigh hypothesis fails. This is not allowable in the limit when $M = \infty$, by the third fundamental principle of Section III. Consequently, the larger M becomes, the less confidence the procedure inspires. In fact, while they note the wider applicability of boundary smoothing in comparison with SPM, Bolomey and Wirgin [36] observe significant deficiencies in some results computed by boundary smoothing when the Rayleigh hypothesis is invalid. This is reminiscent of Lewin's [54] comment on the null-field method: that the more complicated C is, the more accurately must $F(C)$ be computed to ensure that the extinction theorem is obeyed by χ in Ω_{-+}, for scattering problems, or in Ω_{+-}, for closed waveguides. The final sentence of Section XII-A makes the same point.

An immediate corollary to the arguments raised in the previous paragraph is that it is usually easier to compute far fields, rather than surface sources, to a specified accuracy. This is because surface sources often exhibit intricate "wiggles" which are needed to represent the near field accurately in order to satisfy the boundary conditions, but which have negligible effect in the far field. Consequently, far fields computed by boundary smoothing, and even by SPM methods are often satisfactory when the corresponding computations of near fields and surface sources are inadequate.

Methods based on state space formulations, singular integral equations, and the null-field equations always give accurate results, provided that proper care is taken in computing the coefficients in the equations from which numerical solutions are obtained. When a solution is obtained by a standard elimination procedure from a system of linear algebraic equations of order N, say, the required computer time is proportional to N^3. This puts a premium on techniques for reducing the value of N needed to produce a solution to a particular accuracy. The most significant existing technique is the employment of basis functions, such as the $f_n(C)$ introduced in (97), which satisfy physical requirements of the problem. If C is an analytic curve and $Z(C)$ is an analytic function, then the $f_n(C)$ should be analytic. Because of the singular behavior of $g_{QQ'}$ it is often inconvenient to use smooth basis functions, and the $f_n(C)$ are frequently chosen to be pulselike [20], [44]. An advantage of methods based on the extinction theorem is that smooth basis functions can be incorporated straightforwardly. Considerable decreases in N, for a given accuracy of the final solution, are observed when smooth basis functions are used [33], [45]. When C has corners, the electromagnetic edge conditions [6] require $F(C)$ to exhibit singular behavior

at the corners, and it is found that N decreases significantly when singular basis functions, such as the $J_{\mu_m}(kC)$ introduced in (64), are used instead of either smooth or pulselike basis functions [33], [45], [55]–[57]. It is suggested that the shapes of those scattering bodies which caused Waterman [11] difficulties (which surprised him) may be such that Waterman's basis functions—a set of functions derived by Gram–Schmidt orthogonalization of the appropriate null-field equations—were computationally awkward. Similar difficulties have been noted in another context [24].

Spurious internal resonances can contaminate numerical solutions to field problems posed for Ω_+. As Jones [71] remarks, explicit use of the extinction theorem prevents this. It should be noted that the null-field equations (74) (see also [26]) represent a "reliable systematic procedure" of the sort which Jones hopes could replace the ad hoc procedures discussed by him (Jones' [67] new approach is interesting in this context).

XIV. INHOMOGENEOUS MEDIA

The practical computational experience reflected in Sections XII and XIII is based almost entirely on studies of diffraction by perfectly conducting bodies embedded in homogeneous media. Certain problems involving inhomogeneous media have been tackled successfully by the differential approach (refer to the second paragraph of Section I). The polarization-source formulation, which is an integral approach first developed as a systematic computational method for microwave engineering by Richmond [58], has led to a number of useful solutions [7], and, in particular, it provides convenient formulas for dielectric-loaded waveguides.

Recent results of James and Gallett [53], [59] may have considerable computational significance because they hint that the consequences of the failure of the Rayleigh hypotheses are less severe for penetrable media than for perfectly conducting structures.

Enormous computational effort is often required for accurate numerical evaluation of diffraction problems involving inhomogeneous media, so that methods of accelerating the computations are welcome. James and Gallett's method [59] of correcting point-matched solutions for the fields of dielectric waveguides is interesting in this context. It should be mentioned that this method would become highly significant if the eigenvalues as well as the fields could be corrected (without increasing the computation time to that required for a numerical solution based on, for example, the rigorous polarization-source formulation).

XV. NEEDED INVESTIGATIONS

To obtain numerical solutions of integral equations, such as (44) or the infinite set (74), it is necessary to expand the unknown function [i.e., $F(C)$ in the integral equations just alluded to] in a finite number of basis functions. Using, for example, the expansion (97) the number of basis functions is seen to be $(2N + 1)$. The most serious deficiency of existing numerical techniques is the lack of means for estimating *a prori* how large N must be for the solution to possess a specified accuracy. In practice, the only way of checking whether numerical convergence is *apparently* occurring is to repeat the computations several times, increasing N successively [60], [71]. This is a wasteful process since the number of operations required to invert a matrix of order $(2N + 1)$ is proportional to $(2N + 1)^3$. As a result, numerical convergence is sometimes disregarded and N is chosen on the basis of previous computational experience [61]: an acceptable engineering compromise, provided that new problems to which the method is applied do not differ too much from the ones that were originally solved and checked against measurements.

Jones [62] shows how to modify the kernel of an integral equation so that the solution of a truncated system of algebraic equations, derived from the integral equation, is less in error than that obtained with the original kernel. A bound on the error can be computed. To be useful, the error bound needs to be sharpened, but this approach or something similar may well lead to useful *a priori* estimates of how large N should be to achieve a specified accuracy.

When N is large, perturbational solutions of integral equations can be expected to be more economical computationally than solutions by matrix inversion. Hashimoto and Fujisawa [19] show that a matrix inversion solution, obtained with a *sufficiently large* N, can be used to start a convergent iterative scheme. However, it seems to be very difficult to obtain an accurate *a priori* estimate of how large is *sufficiently large*. But it might be worthwhile attempting to combine this approach with that of Jones [62].

It seems both possible and worthwhile to develop the conformal transformation method described in Section VI so that it can be applied to a curve C of arbitrary shape, to estimate whether the internal and/or external Rayleigh hypothesis is valid for that particular C. Unfortunately, complex-variable theory is necessarily two dimensional. It is not clear how to develop methods which might be useful for three-dimensional diffracting bodies.

The most promising aspect of the state-variable formulations described in Section X is that the dimension of the state vectors, needed to give a specified accuracy, can be estimated while the computations are being performed. This does not apply to the two-point boundary value problem (Section X-A) since, before it can be solved, a matrix has to be inverted, and there seems to be no *a priori* means of gauging how large its order should be to give a specified accuracy. However, the initial value problem (Section X-B) is solved serially, and at each value of r the value of $M(r)$ [refer to (113)] can be estimated in terms of the final, required accuracy. It must be remarked, however, that Hizal (in private correspondence with the author) now feels that analysis more intricate than that outlined in Section X is required to establish a viable initial value boundary value procedure.

On occasion, as with extended point matching [33] for example, more than one set of expansion functions

must be truncated simultaneously. This leads to what are called relative convergence problems [60], all of which have so far been treated as special cases. Since relative convergence can be critical as regards numerical accuracy, general methods of examining it are needed.

XVI. ADVANCED METHODS

The literature boasts a number of methods [56], [63]–[67] which are more sophisticated, or advanced, than the methods described in Sections III–XI. These advanced methods are powerful and illuminating, but they do not offer any striking computational advantages, nor do they provide even partial solutions to the problems noted in Section XV.

In three dimensions, the only thing that is significantly different—apart from the impossibility, already noted in Section XV, of applying standard complex-variable theory in relation to conformal transformation—is that the linear dimensions of the largest diffracting body that can be handled conveniently by a given computer are appreciably smaller than in two dimensions. Mittra [60] has organized a comprehensive collection of three-dimensional treatments which incorporate comparisons of various computational approaches.

The diffraction of modulated carrier waves can be treated by first solving the problem monochromatically for a number of frequencies, spaced sufficiently closely to satisfy the sampling theorem, thereby obtaining the "transfer function" of the diffracting body. The time history of the diffracted field is then found by Fourier transforming the transfer function. This is a tedious undertaking which the proponents of the singularity expansion method [5], [68] claim can be accelerated by replacing the Fourier integral by a contour integral in the complex frequency plane. In those cases for which the only singularities of the transfer function are, or appear to be, simple poles, the contour integral reduces to a conveniently computable form. For certain diffracting bodies, in particular, antennas and scatterers constructed from thin wires, the singularity expansion method seems to be competitive with direct Fourier transformation of the transfer function. However, for diffracting bodies of arbitrary shape, the location of the singularities of the transfer function is an enormous computational task. It may be significant that lately Tesche [69] has reverted to the direct Fourier transform.

Something which so far does not seem to have been attempted is to formulate the singularity expansion method assuming that the Rayleigh hypothesis is valid. This might lead to computational advantages for diffracting bodies of suitable shape.

XVII. ENVOI

The methods and approaches discussed in this paper are mere intellectual gymnastics unless they can form the basis of straightforward computational procedures which can be applied with *complete* confidence to any diffracting body that is small enough (in terms of the wavelength) to be handled by an available digital computer. No engineer is going to use a complicated, expensive technique if he has *any* doubts about its validity.

As remarked in Section I, the differential approach seems most popular at present for treating localized fields (e.g., fields associated with guiding structures and cavities). In fact, several generally applicable program packages, based on the differential approach, already exist and are in use [70]. So it might be argued that there is no real need to bother with the integral and series approaches since they have "missed the boat." However, it must be emphasized that, for a given diffracting body and a specified accuracy of solution, the orders of the matrices and determinants obtained with integral and series approaches are dramatically smaller than those obtained with differential approaches [2]. The significance of this will increase as the power of computers increases, permitting the investigation of bodies of increasing size. It should also be remembered that the differential approach is inconvenient for some scattering and antenna problems [71].

The series approach would become generally adopted if computationally convenient methods could be developed for estimating *a priori* the validity of the Rayleigh hypothesis; and if necessary, modifications to SPM, in cases for which the shape of the diffracting body makes the Rayleigh hypothesis invalid, could be incorporated systematically into a computational scheme.

The most urgent requirement for integral approaches is an accurate technique for estimating *a priori* the order of the matrices and determinants required to provide particular accuracies of solution.

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Numerical Solution of Steady-State Electromagnetic Scattering Problems Using the Time-Dependent Maxwell's Equations

ALLEN TAFLOVE AND MORRIS E. BRODWIN, SENIOR MEMBER, IEEE

***Abstract*—A numerical method is described for the solution of the electromagnetic fields within an arbitrary dielectric scatterer of the order of one wavelength in diameter. The method treats the irradiation of the scatterer as an initial value problem. At $t = 0$, a plane-wave source of frequency f is assumed to be turned on. The diffraction of waves from this source is modeled by repeatedly solving a finite-difference analog of the time-dependent Maxwell's equations. Time stepping is continued until sinusoidal steady-state field values are observed at all points within the scatterer. The envelope of the standing wave is taken as the steady-state scattered field. As an example of this method, the computed results for a dielectric cylinder scatterer are presented. An error of less than ± 10 percent in locating and evaluating the standing-wave peaks within the cylinder is achieved for a program execution time of 1 min. The extension of this method to the solution of the fields within three-dimensional dielectric scatterers is outlined.**

I. INTRODUCTION

THE accurate determination of the electromagnetic fields within an arbitrary, inhomogeneous, dielectric scatterer is both an important theoretical problem and a practical objective of workers investigating the effects of microwaves upon living tissue. Exact analytical solutions are obtained only for simple scatterers like the sphere and the circular cylinder, which may be solved using separation of variables. For complicated scatterers like most body organs, we must resort to some numerical method if an accurate model is to be examined.

The computer techniques relevant to this problem that have appeared in the literature may be called, as a class, frequency-domain methods. These methods are based upon the assumption of an exp $(j2\pi ft)$ time dependence in the fundamental Maxwell's equations. In general, methods of this type derive a set of linear equations for either field variables or field expansion coefficients, and then solve the linear system with a suitable matrix-inversion scheme.

Wu and Tsai [1] solve two-dimensional scattering by an arbitrary dielectric cylinder. They develop a coupled integral equation pair for the electric field and its normal derivative at the surface of the scatterer. They then derive a corresponding set of linear equations for the surface fields using the moment method of Harrington [2]. Solution of this set of equations allows computation of the interior fields using Huygens' integrals. This method allows the very accurate solution of a homogeneous dielectric cylinder, about one free-space wavelength in circumference, by inverting an 80-by-80 matrix.

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The authors are with the Department of Electrical Engineering, Technological Institute, Northwestern University, Evanston, Ill. 60201.