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One approach to the derivation of exact integration formulae in the boundary element method

V.P. Fedotov, L.F. Spevak *

Institute of Engineering Science, Russian Academy of Sciences (Urals Branch), 34 Komsomolskaya Street, 620219 Ekaterinburg, Russian Federation

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ABSTRACT

An approach to the derivation of analytical formulae for exact integration in the boundary element solution of two-dimensional elasticity problems is proposed. The integration over an arbitrary boundary element reduces to the integration over a specific element, and this simplifies the derivation of the formulae. Integrals involving fundamental functions are considered, as well as integrals containing derivatives of the fundamental functions, the latter integrals being necessary for stress evaluation. Exact formulae have been obtained to calculate regular and singular integrals. Constant and discontinuous linear elements are considered. The accuracy of the solution obtained with the use of the formulae derived is verified against two test problems.

1. Introduction

Integral evaluation in the boundary element method is an important part of calculations; it influences the solution accuracy and computation speed. Originally [1,2], in boundary element solutions integrals were calculated numerically, generally by Gaussian quadrature formulae. Analytical integration was applied only to singular integrals. Lately, however, it has become possible to derive analytical formulae for the exact calculation of all the necessary integrals. The substitution of exact integration for numerical one offers higher computation accuracy and reduces calculation time significantly. Analytical integration formulae for two-dimensional potential problems were given in Refs. [3,4]. In Ref. [5] exact integration formulae for two-dimensional elastostatics were derived, but they were too lengthy. A new form of the fundamental functions f_{ij}^* proposed in Ref. [6] allows one to simplify integration and to derive more compact formulae. These functions were used in Refs. [7,8] to derive exact integration formulae for two-dimensional elastostatics. In Ref. [9] exact integration formulae were given for integrals involving fundamental function derivatives necessary to calculate stresses in the elastic region. In Refs. [10,11] the authors of the present paper derived compact exact integration formulae for constant elements using the original form of the fundamental functions given in Refs. [1,2]. The present paper proposes a new approach enabling one to derive exact integration formulae in a concise form suitable for programming. Coordinate transformation reduces integration over an arbitrary element to integration over a specific element, the same every time, which is a simpler problem. As a result, exact formulae for calculating regular and singular integrals involving the fundamental functions or their derivatives have been derived. Especially important is that formulae for integrals involving the fundamental function derivatives are as concise as those involving the fundamental functions. Constant and discontinuous linear elements are considered. The accuracy of computations made by the formulae is verified against two test problems.

2. The boundary element method as applied to two-dimensional elasticity problems

Consider a plane elastic region of an arbitrary geometry, with some loads or displacements specified on its boundary. It can be written for the displacement vector at any interior point that

$$u_i(\xi) = \int_S u_{ij}^*(\xi,x) f_j(x)\,dS(x) - \int_S f_{ij}^*(\xi,x) u_j(x)\,dS(x). \quad (1)$$

Here $x \in S$ is the boundary point of the region, ξ is the interior point of the region, $u_j(x)$ are the displacements and $f_j(x)$ are the boundary stresses. The fundamental functions $u_{ij}^*(\xi,x)$ and $f_{ij}^*(\xi,x)$ for the two-dimensional problem have the form [1,2]

$$u_{ij}^*(\xi,x) = c_1[c_2 \ln(r)\delta_{ij} - \Delta_i r \Delta_j r],$$
$$f_{ij}^*(\xi,x) = -\frac{c_3}{r}\left\{[c_4\delta_{ij} + 2\Delta_i r \Delta_j r]\frac{\partial r}{\partial n} - c_4(\Delta_i r n_j - \Delta_j r n_i)\right\}, \quad (2)$$

where, for plane strain, $c_1 = -1/8\pi\mu(1-\nu)$, $c_2 = 3-4\nu$, $c_3 = -1/4\pi(1-\nu)$, $c_4 = 1-2\nu$, ν is the Poisson ratio, μ is the shear modulus,

* Corresponding author. Tel.: +7 343 3753592; fax: +7 343 3745330.
E-mail address: lfs@imach.uran.ru (L.F. Spevak).

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$r = r(x,\xi)$ is the distance between points x and ξ, $\Delta_i r = \partial r/\partial x_i$, δ_{ij} is the Kronecker delta, n is unit outward normal.

For an arbitrary boundary point x_0, the boundary integral equation has the form

$$c_{ij}(x_0)u_j(x_0) = \int_S u^*_{ij}(x_0,x)f_j(x)\,dS(x) - \int_S f^*_{ij}(x_0,x)u_j(x)\,dS(x), \tag{3}$$

where $c_{ij}(x_0) = \delta_{ij}/2$ for the smooth boundary. This equation is used for the determination of unknown boundary displacements $u_j(x)$ and boundary stresses $f_j(x)$. Strains at any interior point are determined by the formulae

$$\varepsilon_{ij}(\xi) = \int_S w^*_{ijk}(\xi,x)f_k(x)\,dS(x) - \int_S g^*_{ijk}(\xi,x)u_k(x)\,dS(x), \tag{4}$$

where

$$w^*_{ijk}(\xi,x) = \frac{1}{2}\left(\frac{\partial u^*_{ik}}{\partial \xi_j} + \frac{\partial u^*_{jk}}{\partial \xi_i}\right),$$
$$g^*_{ijk}(\xi,x) = \frac{1}{2}\left(\frac{\partial f^*_{ik}}{\partial \xi_j} + \frac{\partial f^*_{jk}}{\partial \xi_i}\right). \tag{5}$$

Stresses are determined by Hook's law.

By discretizing the region boundary into elements (this paper deals with straight elements), we arrive at a numerical procedure based on a boundary element and an arbitrary point affected by the stresses and displacements acting on this element. Thus, to obtain an accurate solution, we need to derive exact formulae for calculating the following integrals over an arbitrary line segment AB for an arbitrary point ξ on a plane:

$$I_k(u^*_{ij}) = \int_{AB} u^*_{ij}(\xi,x)N_k(x)\,dS(x),$$
$$I_k(f^*_{ij}) = \int_{AB} f^*_{ij}(\xi,x)N_k(x)\,dS(x),$$
$$I_k(u^*_{ij,m}) = \int_{AB} \frac{\partial u^*_{ij}(\xi,x)}{\partial \xi_m}N_k(x)\,dS(x),$$
$$I_k(f^*_{ij,m}) = \int_{AB} \frac{\partial f^*_{ij}(\xi,x)}{\partial \xi_m}N_k(x)\,dS(x). \tag{6}$$

Here, $k=0$ corresponds to the constant element, $N_0 = 1$; $k = 1$, 2 correspond to the linear element, where $u_i = \Sigma_{k=1}^2 u_i^k N_k(x)$, $f_i = \Sigma_{k=1}^2 f_i^k N_k(x)$, $N_1(x)$ and $N_2(x)$ are the shape functions, u_i^k and f_i^k are the nodal values of displacements and boundary stresses.

3. Analytical integration

In order to simplify the evaluation of the necessary integrals, we put the specific element and the corresponding source point in correspondence with an arbitrary boundary element and an arbitrary source point so that the integrals for the two instances are interrelated.

Consider a line segment AB on a plane, where $A(A_1,A_2)$ and $B(B_1,B_2)$ are arbitrary points, and an arbitrary source point $\xi(\xi_1,\xi_2)$, see Fig. 1. We suppose that the outward normal n is oriented as shown in Fig. 1. The displacements $u = (u_1,u_2)$ and the stresses $f = (f_1,f_2)$ acting on the segment AB are responsible for some displacement $u(\xi) = (u_1(\xi),u_2(\xi))$ at point ξ. Make coordinate transformation maintaining the distances and mapping point A into the origin $O(0,0)$ and point B into point $C(L,0)$, where L is the length of the segment AB. This transformation is parallel translation and rotation through the angle φ combined, see Fig. 1. The arbitrary point $x(x_1,x_2)$ on the plane is mapped into the point $\bar{x}(\bar{x}_1,\bar{x}_2)$ related to it through the relations

$$x = Q\bar{x} + A, \quad \bar{x} = Q^{-1}(x - A), \tag{7}$$

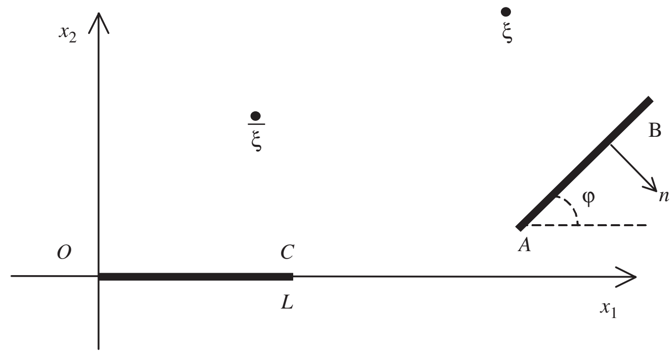

Fig. 1. Coordinate transformation.

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$, $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, Q is a rotation matrix:

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{8}$$

It is obvious that this transformation is a rigid displacement of the system of objects under study as a unit and that it retains the essence of elastic interaction. This means that, if the displacements $\bar{u} = Q^{-1}u$ and the stresses $\bar{f} = Q^{-1}f$ act on the segment OC, they are responsible for the displacements $u(\bar{\xi}) = Q^{-1}u(\xi)$ at point $\bar{\xi} = Q^{-1}(\xi - A)$. Using these relationships, we derive the relation between the integrals over the segment AB for the source point ξ and the corresponding integrals over the segment OC for the source point $\bar{\xi}$ as

$$\begin{pmatrix} I_k(u^*_{11}) & I_k(u^*_{12}) \\ I_k(u^*_{12}) & I_k(u^*_{22}) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} \bar{I}_k(u^*_{11}) & \bar{I}_k(u^*_{12}) \\ \bar{I}_k(u^*_{12}) & \bar{I}_k(u^*_{22}) \end{pmatrix} \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix},$$
$$\begin{pmatrix} I_k(f^*_{11}) & I_k(f^*_{12}) \\ I_k(f^*_{21}) & I_k(f^*_{22}) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \begin{pmatrix} \bar{I}_k(f^*_{11}) & \bar{I}_k(f^*_{12}) \\ \bar{I}_k(f^*_{21}) & \bar{I}_k(f^*_{22}) \end{pmatrix} \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix},$$
$$\begin{pmatrix} I_k(u^*_{11,m}) & I_k(u^*_{12,m}) \\ I_k(u^*_{12,m}) & I_k(u^*_{22,m}) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \left[q_{1m} \begin{pmatrix} \bar{I}_k(u^*_{11,1}) & \bar{I}_k(u^*_{12,1}) \\ \bar{I}_k(u^*_{12,1}) & \bar{I}_k(u^*_{22,1}) \end{pmatrix} + q_{2m} \begin{pmatrix} \bar{I}_k(u^*_{11,2}) & \bar{I}_k(u^*_{12,2}) \\ \bar{I}_k(u^*_{12,2}) & \bar{I}_k(u^*_{22,2}) \end{pmatrix} \right] \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix},$$
$$\begin{pmatrix} I_k(f^*_{11,m}) & I_k(f^*_{12,m}) \\ I_k(f^*_{21,m}) & I_k(f^*_{22,m}) \end{pmatrix} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \left[q_{1m} \begin{pmatrix} \bar{I}_k(f^*_{11,1}) & \bar{I}_k(f^*_{12,1}) \\ \bar{I}_k(f^*_{21,1}) & \bar{I}_k(f^*_{22,1}) \end{pmatrix} + q_{2m} \begin{pmatrix} \bar{I}_k(f^*_{11,2}) & \bar{I}_k(f^*_{12,2}) \\ \bar{I}_k(f^*_{21,2}) & \bar{I}_k(f^*_{22,2}) \end{pmatrix} \right] \begin{pmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{pmatrix}, \tag{9}$$

where

$$\bar{I}_k(u^*_{ij}) = \int_{OC} u^*_{ij}(\bar{\xi},x)\bar{N}_k(x)\,dS(x),$$
$$\bar{I}_k(f^*_{ij}) = \int_{OC} f^*_{ij}(\bar{\xi},x)\bar{N}_k(x)\,dS(x),$$
$$\bar{I}_k(u^*_{ij,m}) = \int_{OC} \frac{\partial u^*_{ij}(\bar{\xi},x)}{\partial \bar{\xi}_m}\bar{N}_k(x)\,dS(x),$$
$$\bar{I}_k(f^*_{ij,m}) = \int_{OC} \frac{\partial f^*_{ij}(\bar{\xi},x)}{\partial \bar{\xi}_m}\bar{N}_k(x)\,dS(x), \tag{10}$$

$\bar{N}_k(x)$ are the shape functions for the segment OC, which correspond to $N_k(x)$ for the segment AB.

Thus, we conclude that, to calculate the integrals over the arbitrary segment AB for the source point ξ, it would suffice to construct the matrix Q (8), to determine point $\bar{\xi}$ in terms of Eq. (7)

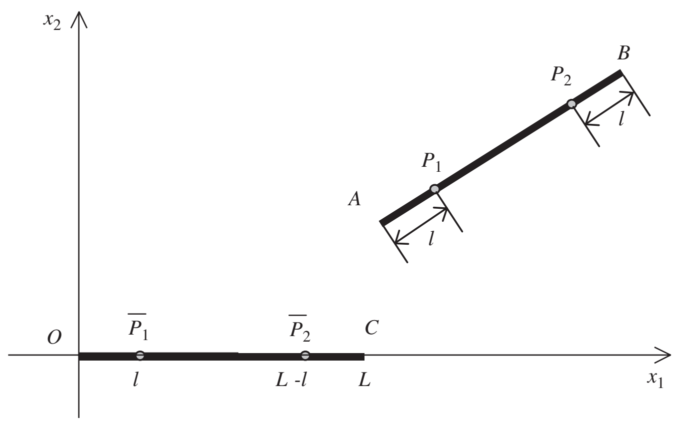

Fig. 2. Linear discontinuous element.

and to compute the integrals over the segment OC, and this is a simpler problem.

For the constant element, integrals I_0 are expressed by Eqs. (9) in terms of the integrals $\bar{I}_0$ over the segment OC, $\bar{N}_0 = 1$. Exact integration formulae for regular and singular integrals $\bar{I}_0$ are given in Appendix A1 and A2.

For the linear discontinuous element (Fig. 2), the shape functions for the segment OC have the following form: $\bar{N}_k = (x) = a_k + b_k x_1$, where

$$a_1 = \frac{l-L}{2l-L}, \quad b_1 = \frac{1}{2l-L}, \quad a_2 = \frac{l}{2l-L}, \quad b_2 = -\frac{1}{2l-L}. \qquad (11)$$

The function $\bar{N}_k(x)$ corresponds to the node P_k, see Fig. 2. Therefore, the integrals $\bar{I}_1$ and $\bar{I}_2$ can be presented as

$$\begin{aligned}
\bar{I}_k(u^*_{ij}) &= a_k \bar{I}_0(u^*_{ij}) + b_k \bar{I}^{(x)}(u^*_{ij}),\\
\bar{I}_k(u^*_{ij,m}) &= a_k \bar{I}_0(u^*_{ij,m}) + b_k \bar{I}^{(x)}(u^*_{ij,m}),\\
\bar{I}_k(f^*_{ij}) &= a_k \bar{I}_0(f^*_{ij}) + b_k \bar{I}^{(x)}(f^*_{ij}),\\
\bar{I}_k(f^*_{ij,m}) &= a_k \bar{I}_0(f^*_{ij,m}) + b_k \bar{I}^{(x)}(f^*_{ij,m}).
\end{aligned} \qquad (12)$$

Here, $k = 1,2$, and

$$\begin{aligned}
\bar{I}^{(x)}(u^*_{ij}) &= \int_{OC} x_1 u^*_{ij}(\bar{\xi}, x)\, \mathrm{d}S(x),\\
\bar{I}^{(x)}(f^*_{ij}) &= \int_{OC} x_1 f^*_{ij}(\bar{\xi}, x)\, \mathrm{d}S(x),\\
\bar{I}^{(x)}(u^*_{ij,m}) &= \int_{OC} x_1 \frac{\partial u^*_{ij}(\bar{\xi}, x)}{\partial \bar{\xi}_m}\, \mathrm{d}S(x),\\
\bar{I}^{(x)}(f^*_{ij,m}) &= \int_{OC} x_1 \frac{\partial f^*_{ij}(\bar{\xi}, x)}{\partial \bar{\xi}_m}\, \mathrm{d}S(x).
\end{aligned} \qquad (13)$$

Exact integration formulae for regular and singular integrals $\bar{I}^{(x)}$ are given in Appendix A1 and A2.

4. Test problems

4.1. A square plate with a circular hole

To verify the exact integration formulae derived, we apply them to the problem on a square plate with a circular hole under uniform tension. The plate is assumed to be under plane stress. The plate edge length is 20 m and the radius of the hole is 1 m. The elastic constants are as follows: $E = 2 \times 10^{11}$ Pa, $\nu = 0.33$.

Firstly, we consider a quarter of the plate, see Fig. 3a. For an infinite plate, the circumferential stress along the x_1-axis

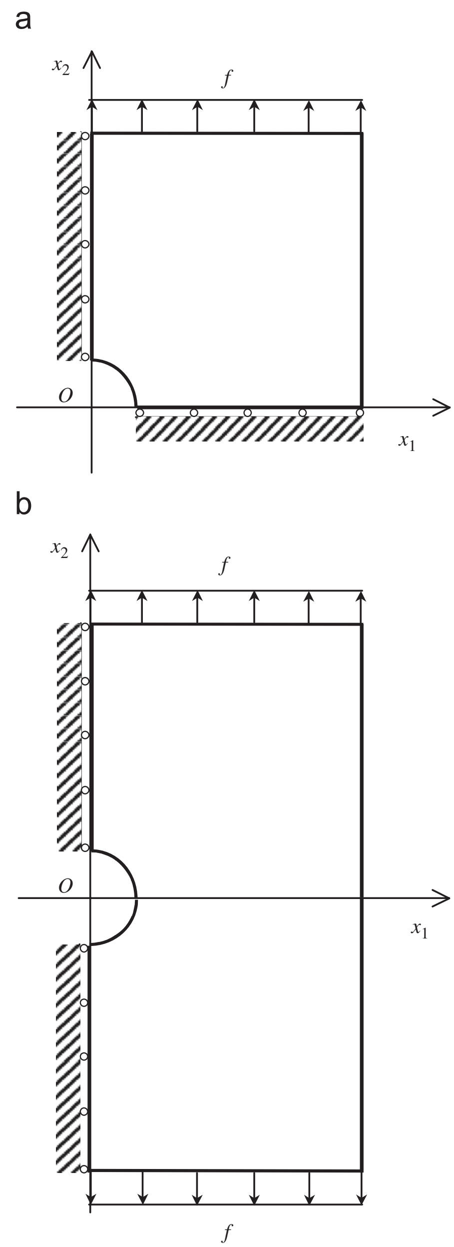

Fig. 3. A square plate with a circular hole: (a) quarter model; (b) half model.

is given in [12] as

$$\sigma_\theta = \frac{f}{2}\left(2 + \frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right), \qquad (14)$$

where f is uniform pressure, R is the radius of the hole, r is the distance from the origin of the coordinates. We compare the stress from Eq. (14) to the boundary stress ($-f_2$) obtained by solving the set of linear algebraic equations created from Eq. (3) with the use of exact integration. Calculations are made for 200 constant elements and for 100 linear discontinuous elements for which $l = \frac{L}{4}$, see Fig. 2. The results for $f = 1\,\mathrm{N/m^2}$ are shown in Fig. 4.

Secondly, we consider a half of the plate, Fig. 3b, and calculate the stress σ_{22} along x_1 using Eq. (4) for interior points and exact integration for the integrals involving the fundamental function derivatives. Calculations are made again for 200 constant elements and for 100 linear discontinuous elements. Fig. 5 compares the calculation results with Eq. (14).

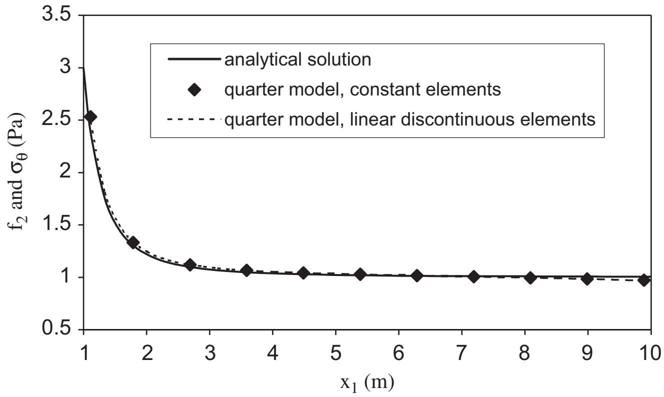

Fig. 4. The analytical circumferential stress along the x_1-axis compared with the boundary stress.

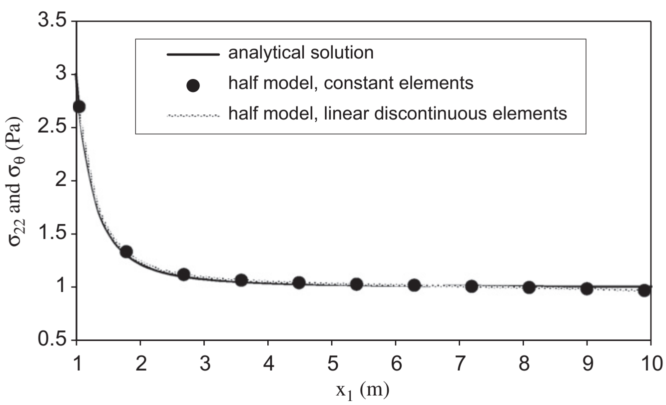

Fig. 5. The analytical circumferential stress along the x_1-axis compared with the stress σ_{22}.

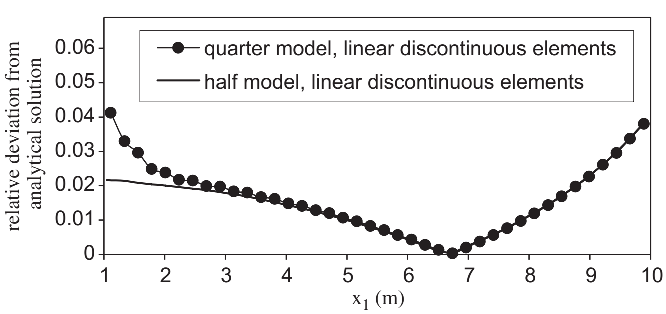

Fig. 6. Relative deviations of the boundary element solutions from the analytical one.

The graphs in Figs. 4 and 5 show a good agreement between the boundary element solutions and the analytical one, and they verify the accurate calculation by the formulae derived. To demonstrate the difference between the two types of solutions obtained for the same number of nodes, 200, we compare their relative deviation from the analytical solution presented in Eq. (14). The comparison shown in Fig. 6 indicates that the results obtained for interior points in the half model are closer to the analytical solution than the quarter model results.

4.2. A rectangular plate with a crack

Another example is a problem on a rectangular plate with a crack under uniform tension, Fig. 7. Due to symmetry, we consider again a quarter of the plate, see the grey part of the plate shown in Fig. 7. The plate is assumed to be under plane stress; the plate height is 20 m, the plate width $2b=2$ m and the crack length $2d=0.2$ m. The elastic constants are the same as in the previous problem. For an infinite plate, the stress σ_{22} near the crack tip for $x_1>d$ and $x_2=0$ is given in [13] as

$$\sigma_{22}=\frac{f\sqrt{d}(1-\lambda+0.326\lambda^2)}{\sqrt{2(x_1-d)(1-\lambda)}}, \tag{15}$$

where f is uniform pressure, $\lambda=d/b$.

The boundary element solutions obtained for 360 constant elements (100 elements on the crack) and for 180 linear discontinuous elements (50 elements on the crack), $l=L/4$, and the analytical solution for $f=1\,\text{N/m}^2$ are shown in Fig. 8. The boundary element solutions are seen to agree well with the analytical one near the crack tip.

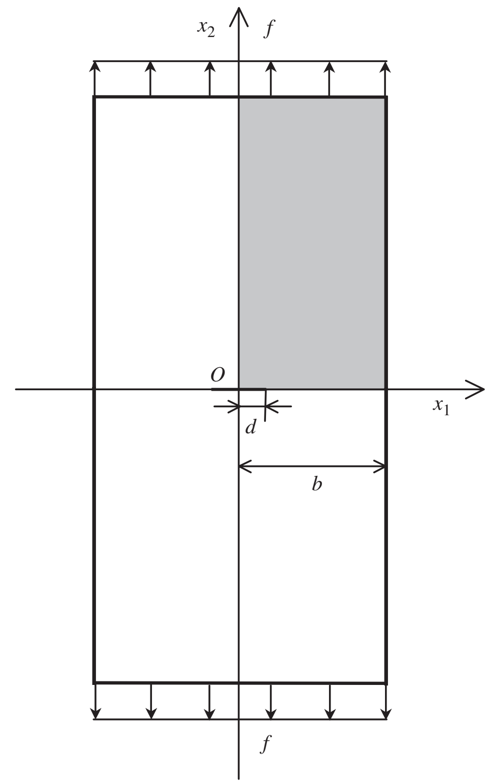

Fig. 7. A rectangular plate with a crack.

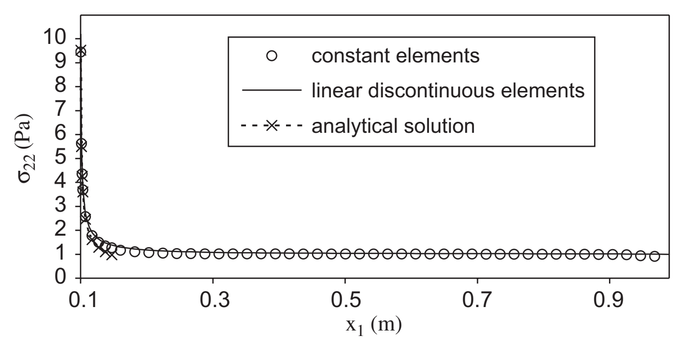

Fig. 8. A rectangular plate with a crack, stress σ_{22} along the x_1-axis.

5. Conclusion

The exact integration formulae for all integrals in 2D elastostatic problems are derived using the specific element and the coordinate transformation. All the formulae have concise form, including ones for integrals involving the fundamental function derivatives. Test problem solutions made with the use of the formulae derived have a high accuracy, and this shows the effectiveness of the formulae.

6. Appendix A1. Regular integrals

$\bar{\xi}_2 \neq 0$, i.e., the source point ξ does not lie on the straight line containing the element AB over which we integrate:

$$\bar{I}_0(u^*_{11}) = c_1\left((c_2+1)(\bar{\xi}_2 Q_3 - L) + \frac{c_2}{2}(LQ_1 + \bar{\xi}_1 Q_2)\right), \tag{A.1}$$

$$\bar{I}_0(u^*_{12}) = -\frac{1}{2}c_1\bar{\xi}_2 Q_2, \tag{A.2}$$

$$\bar{I}_0(u^*_{22}) = c_1\left((c_2-1)\bar{\xi}_2 Q_3 + c_2\left(\frac{1}{2}(LQ_1 + \bar{\xi}_1 Q_2) - L\right)\right), \tag{A.3}$$

$$\bar{I}_0(f^*_{11}) = c_3((c_4+1)Q_3 + \bar{\xi}_2 d_1), \tag{A.4}$$

$$\bar{I}_0(f^*_{12}) = c_3\left(-\frac{1}{2}c_4 Q_2 - \bar{\xi}_2 d_2\right), \tag{A.5}$$

$$\bar{I}_0(f^*_{21}) = c_3\left(\frac{1}{2}c_4 Q_2 - \bar{\xi}_2 d_2\right), \tag{A.6}$$

$$\bar{I}_0(f^*_{22}) = c_3((c_4+1)Q_3 - \bar{\xi}_2 d_1), \tag{A.7}$$

$$\bar{I}^{(x)}(u^*_{11}) = c_1\left(c_2\left(\frac{L^2}{4}(Q_1-1) + \frac{\bar{\xi}_1^2 - \bar{\xi}_2^2}{4}Q_2 + \bar{\xi}_1\bar{\xi}_2 Q_3 - \frac{\bar{\xi}_1 L}{2}\right) - \frac{\bar{\xi}_2^2}{2}Q_2 + \bar{\xi}_1\bar{\xi}_2 Q_3 - \frac{L^2}{2}\right), \tag{A.8}$$

$$\bar{I}^{(x)}(u^*_{12}) = c_1\left(-\frac{\bar{\xi}_1\bar{\xi}_2}{2}Q_2 - \bar{\xi}_2^2 Q_3 + \bar{\xi}_2 L\right), \tag{A.9}$$

$$\bar{I}^{(x)}(u^*_{22}) = c_1\left(c_2\left(\frac{L^2}{4}(Q_1-1) + \frac{\bar{\xi}_1^2 - \bar{\xi}_2^2}{4}Q_2 + \bar{\xi}_1\bar{\xi}_2 Q_3 - \frac{\bar{\xi}_1 L}{2}\right) + \frac{\bar{\xi}_2^2}{2}Q_2 - \bar{\xi}_1\bar{\xi}_2 Q_3\right), \tag{A.10}$$

$$\bar{I}^{(x)}(f^*_{11}) = c_3\left(c_4\left(-\frac{\bar{\xi}_2}{2}Q_2 + \bar{\xi}_1 Q_3\right) - \bar{\xi}_2 Q_2 + \bar{\xi}_1 Q_3 + d_6\right), \tag{A.11}$$

$$\bar{I}^{(x)}(f^*_{12}) = c_3\left(-c_4\left(\frac{\bar{\xi}_1}{2}Q_2 + \bar{\xi}_2 Q_3 - L\right) - \bar{\xi}_2 Q_3 + d_7\right), \tag{A.12}$$

$$\bar{I}^{(x)}(f^*_{21}) = c_3\left(c_4\left(\frac{\bar{\xi}_1}{2}Q_2 + \bar{\xi}_2 Q_3 - L\right) - \bar{\xi}_2 Q_3 + d_7\right), \tag{A.13}$$

$$\bar{I}^{(x)}(f^*_{22}) = c_3\left(c_4\left(-\frac{\bar{\xi}_2}{2}Q_2 + \bar{\xi}_1 Q_3\right) + \bar{\xi}_1 Q_3 - d_6\right), \tag{A.14}$$

$$\bar{I}_0(u^*_{11,1}) = c_1\left(\frac{1}{2}c_2 Q_2 + \bar{\xi}_2 d_2\right), \tag{A.15}$$

$$\bar{I}_0(u^*_{11,2}) = c_1((c_2+1)Q_3 + \bar{\xi}_2 d_1), \tag{A.16}$$

$$\bar{I}_0(u^*_{12,1}) = c_1\bar{\xi}_2 d_1, \tag{A.17}$$

$$\bar{I}_0(u^*_{12,2}) = c_1\left(-\frac{1}{2}Q_2 - \bar{\xi}_2 d_2\right), \tag{A.18}$$

$$\bar{I}_0(u^*_{22,1}) = c_1\left(\frac{1}{2}c_2 Q_2 - \bar{\xi}_2 d_2\right), \tag{A.19}$$

$$\bar{I}_0(u^*_{22,2}) = c_1((c_2-1)Q_3 - \bar{\xi}_2 d_1), \tag{A.20}$$

$$\bar{I}_0(f^*_{11,1}) = c_3(c_4 d_2 + d_3), \tag{A.21}$$

$$\bar{I}_0(f^*_{11,2}) = c_3((c_4+2)d_1 + d_4), \tag{A.22}$$

$$\bar{I}_0(f^*_{12,1}) = c_3(c_4 d_1 + d_4), \tag{A.23}$$

$$\bar{I}_0(f^*_{12,2}) = c_3(-(c_4+2)d_2 + d_5), \tag{A.24}$$

$$\bar{I}_0(f^*_{21,1}) = c_3(-c_4 d_1 + d_4), \tag{A.25}$$

$$\bar{I}_0(f^*_{21,2}) = c_3((c_4-2)d_2 + d_5), \tag{A.26}$$

$$\bar{I}_0(f^*_{22,1}) = c_3((c_4+2)d_2 - d_3), \tag{A.27}$$

$$\bar{I}_0(f^*_{22,2}) = c_3(c_4 d_1 - d_4), \tag{A.28}$$

$$\bar{I}^{(x)}(u^*_{11,1}) = c_1\left(c_2\left(\frac{\bar{\xi}_1}{2}Q_2 + \bar{\xi}_2 Q_3 - L\right) + \bar{\xi}_2 Q_3 - d_7\right), \tag{A.29}$$

$$\bar{I}^{(x)}(u^*_{11,2}) = c_1\left(c_2\left(-\frac{\bar{\xi}_2}{2}Q_2 + \bar{\xi}_1 Q_3\right) - \bar{\xi}_2 Q_2 + \bar{\xi}_1 Q_3 + d_6\right), \tag{A.30}$$

$$\bar{I}^{(x)}(u^*_{12,1}) = c_1\left(-\frac{1}{2}\bar{\xi}_2 Q_2 + d_6\right), \tag{A.31}$$

$$\bar{I}^{(x)}(u^*_{12,2}) = c_1\left(-\frac{1}{2}\bar{\xi}_1 Q_2 - 2\bar{\xi}_2 Q_3 + L + d_7\right), \tag{A.32}$$

$$\bar{I}^{(x)}(u^*_{22,1}) = c_1\left(c_2\left(\frac{\bar{\xi}_1}{2}Q_2 + \bar{\xi}_2 Q_3 - L\right) - \bar{\xi}_2 Q_3 + d_7\right), \tag{A.33}$$

$$\bar{I}^{(x)}(u^*_{22,2}) = c_1\left(c_2\left(-\frac{\bar{\xi}_2}{2}Q_2 + \bar{\xi}_1 Q_3\right) + \bar{\xi}_2 Q_2 - \bar{\xi}_1 Q_3 - d_6\right), \tag{A.34}$$

$$\bar{I}^{(x)}(f^*_{11,1}) = c_3\left(c_4\left(Q_3 - \frac{d_7}{\bar{\xi}_2}\right) + Q_3 - d_8\right), \tag{A.35}$$

$$\bar{I}^{(x)}(f^*_{11,2}) = c_3\left(c_4\left(-\frac{Q_2}{2} + \frac{d_6}{\bar{\xi}_2}\right) - Q_2 + \frac{2d_6}{\bar{\xi}_2} + d_9\right), \tag{A.36}$$

$$\bar{I}^{(x)}(f^*_{12,1}) = c_3\left(-c_4\left(\frac{Q_2}{2} - \frac{d_6}{\bar{\xi}_2}\right) + d_9\right), \tag{A.37}$$

$$\bar{I}^{(x)}(f^*_{12,2}) = c_3\left(-c_4\left(Q_3 - \frac{d_7}{\bar{\xi}_2}\right) - Q_3 + d_8\right), \tag{A.38}$$

$$\bar{I}^{(x)}(f^*_{21,1}) = c_3\left(c_4\left(\frac{Q_2}{2} - \frac{d_6}{\bar{\xi}_2}\right) + d_9\right), \tag{A.39}$$

$$\bar{I}^{(x)}(f^*_{21,2}) = c_3\left(c_4\left(Q_3 - \frac{d_7}{\bar{\xi}_2}\right) - Q_3 + d_8\right), \tag{A.40}$$

$$\bar{I}^{(x)}(f^*_{22,1}) = c_3\left(c_4\left(Q_3 - \frac{d_7}{\bar{\xi}_2}\right) + Q_3 - \frac{2d_7}{\bar{\xi}_2} + d_8\right), \tag{A.41}$$

$$\bar{I}^{(x)}(f^*_{22,2}) = c_3\left(c_4\left(-\frac{Q_2}{2} + \frac{d_6}{\bar{\xi}_2}\right) - d_9\right). \quad (A.42)$$

In the above equations,

$$Q_1 = \ln D_1, \quad Q_2 = \ln D_2 - \ln D_1, \quad (A.43)$$

$$Q_3 = \mathrm{arctg}\left(\frac{\bar{\xi}_1}{\bar{\xi}_2}\right) - \mathrm{arctg}\left(\frac{\bar{\xi}_1 - L}{\bar{\xi}_2}\right), \quad (A.44)$$

$$d_1 = \frac{\bar{\xi}_1 - L}{D_1} - \frac{\bar{\xi}_1}{D_2}, \quad d_2 = \bar{\xi}_2\left(\frac{1}{D_2} - \frac{1}{D_1}\right), \quad (A.45)$$

$$d_3 = 2\bar{\xi}_2\left(\frac{\bar{\xi}_1^2}{D_2^2} - \frac{(\bar{\xi}_1 - L)^2}{D_1^2}\right), \quad d_4 = 2\bar{\xi}_2^2\left(\frac{\bar{\xi}_1}{D_2^2} - \frac{\bar{\xi}_1 - L}{D_1^2}\right), \quad (A.46)$$

$$d_5 = 2\bar{\xi}_2^3\left(\frac{1}{D_2^2} - \frac{1}{D_1^2}\right), \quad d_6 = \frac{\bar{\xi}_2 L(\bar{\xi}_1 - L)}{D_1}, \quad (A.47)$$

$$d_7 = \frac{\bar{\xi}_2^2 L}{D_1}, \quad d_8 = d_7 d_2 + d_6 d_1 + \frac{d_7}{\bar{\xi}_2}, \quad (A.48)$$

$$d_9 = d_6 d_2 - d_7 d_1, \quad (A.49)$$

$$D_1 = (\bar{\xi}_1 - L)^2 + \bar{\xi}_2^2, \quad D_2 = \bar{\xi}_1^2 + \bar{\xi}_2^2. \quad (A.50)$$

7. Appendix A2. Singular integrals

$\bar{\xi}_2 = 0$, i.e., the point ξ lies on the straight line containing the element AB:

$$\bar{I}_0(u^*_{11}) = c_1(c_2(L(Q^*_1 - 1) + \bar{\xi}_1 Q^*_2) - L), \quad (A.51)$$

$$\bar{I}_0(u^*_{22}) = c_1 c_2(L(Q^*_1 - 1) + \bar{\xi}_1 Q^*_2), \quad (A.52)$$

$$\bar{I}_0(u^*_{12}) = \bar{I}_0(f^*_{11}) = \bar{I}_0(f^*_{22}) = 0, \quad (A.53)$$

$$\bar{I}_0(f^*_{12}) = -\bar{I}_0(f^*_{21}) = -c_3 c_4 Q^*_2, \quad (A.54)$$

$$\bar{I}^{(x)}(u^*_{11}) = \frac{c_1}{2}\left(c_2\left(L^2 Q^*_1 + \bar{\xi}_1^2 Q^*_2 - \frac{L^2}{2} - L\bar{\xi}_1\right) - L^2\right), \quad (A.55)$$

$$\bar{I}^{(x)}(u^*_{22}) = \frac{c_1 c_2}{2}\left(L^2 Q^*_1 + \bar{\xi}_1^2 Q^*_2 - \frac{L^2}{2} - L\bar{\xi}_1\right), \quad (A.56)$$

$$\bar{I}^{(x)}(u^*_{12}) = \bar{I}^{(x)}(f^*_{11}) = \bar{I}^{(x)}(f^*_{22}) = 0, \quad (A.57)$$

$$\bar{I}^{(x)}(f^*_{12}) = -\bar{I}^{(x)}(f^*_{21}) = -c_3 c_4(\bar{\xi}_1 Q^*_2 - L), \quad (A.58)$$

$$\bar{I}_0(u^*_{11,1}) = \bar{I}_0(u^*_{22,1}) = c_1 c_2 Q^*_2, \quad (A.59)$$

$$\bar{I}_0(u^*_{12,1}) = \bar{I}_0(u^*_{11,2}) = \bar{I}_0(u^*_{22,2}) = 0, \quad (A.60)$$

$$\bar{I}_0(u^*_{12,2}) = -c_1 Q^*_2, \quad (A.61)$$

$$\bar{I}_0(f^*_{11,1}) = \bar{I}_0(f^*_{22,1}) = \bar{I}_0(f^*_{12,2}) = \bar{I}_0(f^*_{21,2}) = 0, \quad (A.62)$$

$$\bar{I}_0(f^*_{12,1}) = -\bar{I}_0(f^*_{21,1}) = \bar{I}_0(f^*_{22,2}) = \frac{c_3 c_4 L}{\bar{\xi}_1(\bar{\xi}_1 - L)}, \quad (A.63)$$

$$\bar{I}_0(f^*_{11,2}) = \frac{c_3(c_4 + 2)L}{\bar{\xi}_1(\bar{\xi}_1 - L)}, \quad (A.64)$$

$$\bar{I}^{(x)}(u^*_{11,1}) = \bar{I}^{(x)}(u^*_{22,1}) = c_1 c_2(\bar{\xi}_1 Q^*_2 - L), \quad (A.65)$$

$$\bar{I}^{(x)}(u^*_{12,1}) = \bar{I}^{(x)}(u^*_{11,2}) = \bar{I}^{(x)}(u^*_{22,2}) = 0, \quad (A.66)$$

$$\bar{I}^{(x)}(u^*_{12,2}) = -c_1(\bar{\xi}_1 Q^*_2 - L), \quad (A.67)$$

$$\bar{I}^{(x)}(f^*_{11,1}) = \bar{I}^{(x)}(f^*_{22,1}) = \bar{I}^{(x)}(f^*_{12,2}) = \bar{I}^{(x)}(f^*_{21,2}) = 0, \quad (A.68)$$

$$\bar{I}^{(x)}(f^*_{12,1}) = -\bar{I}^{(x)}(f^*_{21,1}) = \bar{I}^{(x)}(f^*_{22,2}) = c_3 c_4\left(\frac{L}{\bar{\xi}_1 - L} - Q^*_2\right), \quad (A.69)$$

$$\bar{I}^{(x)}(f^*_{11,2}) = c_3(c_4 + 2)\left(\frac{L}{\bar{\xi}_1 - L} - Q^*_2\right). \quad (A.70)$$

In Eqs. (A.51)–(A.70),

$$Q^*_1 = \ln|L - \bar{\xi}_1|, \quad Q^*_2 = \ln|\bar{\xi}_1| - \ln|L - \bar{\xi}_1|. \quad (A.71)$$

References

[1] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques. Berlin, Neidelberg, New York, Tokyo: Springer; 1984.
[2] Banerjee PK, Batterfield R. Boundary element methods in engineering science. UK: McGraw-Hill Book Company Ltd; 1981.
[3] Yoon SS, Heister SD. Analytic solution for fluxes at interior points for the 2D Laplace equation. Eng Anal Bound Elem 2000;24(2):155–60.
[4] Fratantonio M, Rencis JJ. Exact boundary element integrations for two-dimensional Laplace equation. Eng Anal Bound Elem 2000;24(4):325–42.
[5] Padhi GS, Shenoi RA, Moy SSJ, McCarthy MA. Analytic integration of kernel shape function product integrals in the boundary element method. Comput Struct 2001;79(14):1325–33.
[6] Ghosh N, Rajiyah H, Ghosh S, Mukherjee S. A new boundary element formulation for linear elasticity. ASME J Appl Mech 1986;53:69–76.
[7] Zhang XS, Zhang XX. Exact integration in the boundary element method for two-dimensional elastostatic problems. Eng Anal Bound Elem 2003;27(10):987–97.
[8] Zhang XS, Zhang XX. Exact integrations of two-dimensional high-order discontinuous boundary elements of elastostatics problems. Eng Anal Bound Elem 2004;28(7):725–32.
[9] Zhang XS, Zhang XX. Exact integration for stress evaluation in the boundary element analysis of two-dimensional elastostatics. Eng Anal Bound Elem 2004;28(9):997–1004.
[10] Fedotov VP, Spevak LF, Trukhin VB, Privalova VV, Dumsheva TD, Zenkova ES. Studying the convergence of the numerical-analytical technique for solving elasticity, diffusion and heat conduction problems. Vestn SamGTU Fiz-mat Nauki 2004;30:24–32 [in Russian].
[11] Dumsheva TD, Zenkova ES, Fedotov VP, Spevak LF, Privalova VV. Numerical-analytical algorithm for solving elasticity, heat conduction and diffusion problems. Algorithms Program Meanings Parallel Calculations (Algoritmy Programniye Sredstva Parallelnykh Vychisleniy) 2003;7:70–86 [in Russian].
[12] Timoshenko SP, Goodier JN. Theory of elasticity. 3rd ed. New York: McGraw-Hill; 1987.
[13] Hellan K. Introduction to fracture mechanics. New York: McGraw-Hill; 1984.

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