Integral Equations and Their Applications

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Outline

Section Content

4 main types of integral equations

- Fredholm Integral Equations
  - 1. kind

$$\int_a^b K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

  - 2. kind

$$\varphi(x) + \int_a^b K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

- Volterra Integral Equations
  - 1. kind

$$\int_a^x K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

  - 2. kind

$$\varphi(x) + \int_a^x K(x,y)\varphi(y)dy = g(x), \quad x \in I$$

Historical Remarks I

- Maxime Bôcher 1908
  The theory of integral equations may be regarded as dating back at least as far as the discovery by Fourier of the theorem concerning integrals which bears his name; for, though this was not the point of view of Fourier, this theorem may be regarded as a statement of the solution of a certain integral equation of the first kind.
  Fourier's inversion formula

$$g(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(x\xi) f(\xi) d\xi$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \cos(x\xi) g(\xi) d\xi$$

Historical Remarks II

- Abel's Integral 1826
  - a mechanical problem : a Tautochrone
  - the general accepted begin of the theory of integral equations
  - actually an inverse problem
  - The problem is to find the unknown path in the plane along which a particle will fall, under the influence of gravity alone, so that at each instant the time of fall is a known function of the distance fallen.

$$g(t) = \int_0^t \frac{f(y)}{\sqrt{2a(t-y)}} dy$$

Historical Remarks III

- Joachimstahl's attraction problem 1861
  - also an inverse problem
  - find the law of attraction if one knows the attraction force

$$\frac{g(h)}{2h} = \int_h^\infty \frac{f(r)}{\sqrt{r^2 - h^2}} dr$$

- at the turn of 20. century :
  - Volterra, Fredholm, Hilbert, Schmidt, ...
  - Introduction of Hilbert spaces
  - functional analysis

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Solutions of Integral Equations

- Over 99.99 . . . % of integral equations do not have a closed form solution.
- The solvability of integral equations is ensured by functional analytic approach.
- Numerical approximate solutions.

Linear Operators

- linear operator
  X, Y linear spaces. $A : X \to Y$ is linear iff for all $\alpha, \beta \in \mathbb{C}$

$$A(\alpha f + \beta g) = \alpha A(f) + \beta A(g), \quad \forall f, g \in X$$

- bounded operator
  X, Y are normed spaces. A is bounded if there exists a constant $C > 0$ such that

$$\|Af\| \leq C\|f\|, \quad \forall f \in X$$

- compact operator
  A is compact if it maps a bounded set to a relatively compact set.

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Integral Operators

- integral operator

$$(A\varphi)(x) := \int_G K(x,y)\varphi(y)dy, \quad x \in G \subset \mathbb{R}^m$$

where K is called the kernel of the integral operator.

- K is called weakly singular iff there exists a constant $M > 0$ and $\alpha \in (0, m]$ such that

$$|K(x,y)| \leq M|x-y|^{\alpha-m}, \quad \forall x, y \in G \subset \mathbb{R}^m, x \neq y$$

- A is compact if K is continuous or weakly singular.

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Section Content

Riesz Theory I

Consider the following integral equation of the second kind with a compact $A : X \to X$

$$\varphi - A\varphi = f$$

Let $L := I - A$.

- First Riesz Theorem
  The nullspace of L is a finite-dimensional subspace.
- Second Riesz Theorem
  The range of the operator L is a closed linear subspace.

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Riesz Theory II

- Third Riesz Theorem
  There exists a uniquely determined nonnegative integer r, called the Riesz number of A such that

$$\{0\} = N(L^0) \overset{\subset}{\neq} N(L^1) \overset{\subset}{\neq} \cdots \overset{\subset}{\neq} N(L^r) = N(L^{r+1}) = \ldots,$$

and

$$X = L^0(X) \overset{\supset}{\neq} L^1(X) \overset{\supset}{\neq} \cdots \overset{\supset}{\neq} L^r(X) = L^{r+1}(X) = \ldots,$$

Furthermore, we have the direct sum

$$X = N(L^r) \oplus L^r(X).$$

Fundamental Result of Riesz Theory

Theorem 1

Let $A : X \to X$ be a compact operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective, then its inverse operator $(I - A)^{-1}$ is bounded.

Solvability of a Second Kind Equation I

Theorem 2

If the homogeneous equation

$$\varphi - A\varphi = 0$$

has ony the trivial solution $\varphi = 0$, *then for each* $f \in X$ *the inhomogeneous equation*

$$\varphi - A\varphi = f$$

has a unique solution $\varphi \in X$ *and this solution depends continuously on* f.

Solvability of a Second Kind Equation II

Theorem 3

If the homogeneous equation

$$\varphi - A\varphi = 0$$

has nontrivial solution $\varphi \neq 0$*, then it has only a finite number m of linearly independent solutions* $\varphi_1, \varphi_2, \ldots, \varphi_m \in X$ *and the inhomogeneous equation is either unsolvable or its general solution is of the form*

$$\varphi = \tilde{\varphi} + \sum_{i=1}^{m} \alpha_i \varphi_i$$

where $\tilde{\varphi}$ *is a particular solution of the inhomogeneous equation.*

Remarks

- Reduction of the solvability of the equation to the solvability of the simpler homogeneous equation $\varphi - A\varphi = 0$.
- No answer to the question of whether the inhomogeneous equation $\varphi - A\varphi = f$ for a given inhomogenity is solvable in the case where the homogeneous equation has a nontrivial solution.

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Bilinear Forms

Definition 1

Let X, Y be linear spaces. A mapping $< \cdot, \cdot >: X \times Y \to \mathbb{C}$ is called a bilinear form if

$$< \alpha_1\varphi_1 + \alpha_2\varphi_2, \psi >= \alpha_1 < \varphi_1, \psi > +\alpha_2 < \varphi_2, \psi > \quad (1)$$

$$< \varphi, \beta_1\psi_1 + \beta_2\psi_2 >= \beta_1 < \varphi, \psi_1 > +\beta_2 < \varphi, \psi_2 > \quad (2)$$

for all $\varphi, \varphi_1, \varphi_2 \in X, \psi, \psi_1, \psi_2 \in Y$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$.
A bilinear form is called nondegenerated if

$$< \varphi, \psi >= 0, \quad \forall \varphi \in X \Longrightarrow \psi = 0$$

$$< \varphi, \psi >= 0, \quad \forall \psi \in Y \Longrightarrow \varphi = 0$$

Dual Systems

Definition 2

Two normed spaces X and Y equipped with a nondegenerated bilinear form $< \cdot, \cdot >: X \times Y \to \mathbb{C}$ are called a dual system and denoted by $< X, Y >$.

Example 1

$< C(G), C(G) >$ is a dual system with the bilinear form

$$< \varphi, \psi > := \int_G \varphi(x)\psi(x)dx, \quad \varphi, \psi \in C(G)$$

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Adjoint Operators

Definition 3

Let $< X_1, Y_1 >$ and $< X_2, Y_2 >$ be two dual systems. Then two operators $A : X_1 \to X_2, B : Y_2 \to Y_1$ are called adjoint (w.r.t. dual systems) if

$$< A\varphi, \psi > = < \varphi, B\psi >$$

for all $\varphi \in X_1, \psi \in Y_2$.

Theorem 4

Let $< X_1, Y_1 >$ and $< X_2, Y_2 >$ be two dual systems. If an operator $A : X_1 \to X_2$ has an adjoint $B : Y_2 \to Y_1$, then B is uniquely determined, and A and B are linear.

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Adjoint Integral Operators

Theorem 5

If A is compact, then the adjoint of A exists and is also compact.

Theorem 6

Let K be a continuous or a weakly singular kernel. Then in the dual system $< C(G), C(G) >$ the integral operators defined by

$$(A\varphi)(X) := \int_G K(x,y)\varphi(y)dy, \quad ,x \in G, \tag{3}$$

$$(B\psi)(X) := \int_G K(y,x)\psi(y)dy, \quad ,x \in G, \tag{4}$$

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are adjoint.

Fredholm Theory I

Let $< X, Y >$ be a dual system and $A : X \to X, B : Y \to Y$ be compact adjoint operators. We have the following theorems

- First Fredholm Theorem

$$\mathrm{dimN(I - A) = dimN(I - B)} < \infty$$

- Second Fredholm Theorem

$$(I - A)(X) = \{f \in X :< f, \psi >= 0, \psi \in N(I - B)\} \tag{5}$$

$$(I - B)(Y) = \{g \in Y :< \varphi, g >= 0, \varphi \in N(I - A)\} \tag{6}$$

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Fredholm Theory II

- Fredholm Alternative
  Either $I - A$ and $I - B$ are bijective
  or $I - A$ and $I - B$ have nontrivial nullspaces with finite dimension

$$\mathrm{dim N(I-A)} = \mathrm{dim N(I-B)} \in \mathbb{N}$$

  and the ranges are given by

$$(I - A)(X) = \{f \in X :< f, \psi >= 0, \psi \in N(I - B)\} \quad (7)$$

$$(I - B)(Y) = \{g \in Y :< \varphi, g >= 0, \varphi \in N(I - A)\} \quad (8)$$

Fundamental Results from Fredholm Theory I

Either the homogeneous integral equations

$$\varphi(x) - \int_G K(x,y)\varphi(y)dy = 0, \quad , x \in G, \tag{9}$$

$$\psi(x) - \int_G K(y,x)\psi(y)dy = 0, \quad , x \in G, \tag{10}$$

only have the trivial solutions $\varphi = 0$ and $\psi = 0$ and the inhomogeneous equations

$$\varphi(x) - \int_G K(x,y)\varphi(y)dy = f(x), \quad , x \in G, \tag{11}$$

$$\psi(x) - \int_G K(y,x)\psi(y)dy = g(x), \quad , x \in G, \tag{12}$$

have a unique solution $\varphi \in C(G)$ and $\psi \in C(G)$ for each right-hand side $f \in C(G)$ and $g \in C(G)$, respectively

Fundamental Results from Fredholm Theory II

or the homogeneous integral equations have the same finite number $m \in \mathbb{N}$ of linearly independent solutions and the inhomogeneous integral equations are solvable if and only if thr right-hand sides satisfy

$$\int_G f(x)\psi(x)dx = 0$$

for all solutions ψ of the homogeneous adjoint equation and

$$\int_G \varphi(x)g(x)dx = 0$$

for all solutions φ of the homogeneous equation, respectively.

Section Content

First Kind Fredholm Equations

Consider the following equation

$$A\varphi = f$$

In the case of a compact operator, we have the following theorem

Theorem 7

A compact operator defined in an infinite dimensional space can not have a bounded inverse.

This means that we are encountered with an ill-posed problem.

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Hadamard's Postulation of Well-posedness

Hadamard (1902)
A problem is called well-posed, if it has the following properties

1. Existence of a solution.
2. Uniqueness of the solution.
3. (Stability) Continuous dependence of the solution on the data.

otherwise it is called ill-posed.

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Examples of Ill-Posed Problems

Example 2 (Cauchy Problem for the Laplace Equation)

Find a harmonic function u in $D := \mathbb{R} \times [0, \infty]$ satisfying the following initial conditions

$$u(\cdot, 0) = 0, \quad \frac{\partial}{\partial y} u(\cdot, 0) = f,$$

where f is a given continuous function.

Let $f_n(x) = \frac{1}{n} \sin nx, x \in \mathbb{R}$.
For $n \in \mathbb{N}$, we obtain the solution

$$u_n(x, y) = \frac{1}{n^2} \sin nx \sinh ny, \quad (x, y) \in D.$$

Clearly, $(f_n) \to 0$, but (u_n) doesn't converge in any reasonable norm.

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Example 3 (Fredholm Integral Equation of the First Kind)

$$A\varphi(x) := \int_a^b K(x,y)\varphi(y)dy, \quad x \in [c,d]$$

Solving $A\varphi = f$ is ill-posed if, for example, the kernel K is continuous. If K is continuous, then the operator A will be compact. In this case, the operator A will not have a bounded inverse.

Fredholm integral Equations 2. Kind

$$\varphi(x) - \frac{1}{2}\int_0^1 (x+1)e^{-xy}\varphi(y)dy = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}, \quad 0 \le x \le 1$$

Trapzoidal rule

n	$x=0$	$x=0.5$	$x=1$
4	-0.007146	-0.010816	-0.015479
8	-0.001788	-0.002711	-0.003882
16	-0.000447	-0.000678	-0.000971
32	-0.000112	-0.000170	-0.000243

Simpson's rule

n	$x=0$	$x=0.5$	$x=1$
4	-0.00006652	-0.00010905	-0.00021416
8	-0.00000422	-0.00000692	-0.00001366
16	-0.00000026	-0.00000043	-0.00000086
32	-0.00000002	-0.00000003	-0.00000005

Fredholm integral Equations 1. Kind

$$\int_0^1 (x+1)e^{-xy}\varphi(y)dy = 1 - e^{-(x+1)}, \quad 0 \le x \le 1$$

Trapzoidal rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	0.4057	0.3705	0.1704
8	-4.5989	14.6094	-4.4770
16	-8.5957	2.2626	-153.4805
32	3.8965	-32.2907	22.5570
64	-88.6474	-6.4484	-182.6745

Simpson's rule

n	$x = 0$	$x = 0.5$	$x = 1$
4	0.0997	0.2176	0.0566
8	-0.5463	6.0868	-1.7274
16	-15.4796	50.5015	-53.8837
32	24.5929	-24.1767	67.9655
64	23.7868	-17.5992	419.4284

Well-posedness in Tikhonov Sense

Tikhonov (1943)
The solving of the problem

$$A : X \to Y, \quad A\varphi = f$$

is called conditionally correct if

1. Existence of a solution in a subset $M \subset X$.
2. Uniqueness of the solution in that subset M.
3. (Stability) Continuous dependence of the solution on the data from the set AM.

Ill-Posed Problems : Regularization

Definition 4 (Regularization)

Assume X, Y are normed spaces.
Let the operator $A : X \to Y$ be linear, bounded and injective.
A family of bounded linear operators $R_\alpha : Y \to X, \alpha > 0$ is called a regularization scheme for

$$A\varphi = f,$$

if it satisfies the following pointwise convergence

$$\lim_{\alpha \to 0} R_\alpha A\varphi = \varphi, \text{ for all } \varphi \in X$$

In this case, the parameter α is called the regularization parameter.

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Regularization : Error

Find a stable approximation to the equation

$$A\varphi = f$$

The regularized approximation

$$\varphi_\alpha^\delta := R_\alpha f^\delta$$

The total approximation error

$$\varphi_\alpha^\delta - \varphi = R_\alpha f^\delta - R_\alpha f + R_\alpha A\varphi - \varphi$$

We have

$$\|\varphi_\alpha^\delta - \varphi\| \leq \delta\|R_\alpha\| + \|R_\alpha A\varphi - \varphi\|$$

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$$\|\varphi_\alpha^\delta - \varphi\| \le \delta \|R_\alpha\| + \|R_\alpha A\varphi - \varphi\|$$

Regularization : Methods

How to choose the regularization parameter α ?

1. a priori choice based on some information of the solution. In general not available
2. a posteriori choice based on the data error level δ

Discrepancy Principle of Morozov :

$$\|AR_\alpha f^\delta - f^\delta\| = \gamma\delta, \qquad \gamma \geq 1$$

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Regularization : Example

X, Y Hilbert spaces.

Theorem 8

Assume $A : X \to Y$ compact and linear.
Then for every $\alpha > 0$, the operator

$$\alpha I + A^* A : X \to X$$

is bijective and has a bounded inverse.
Furthermore, if the operator A is injective, then

$$R_\alpha := (\alpha I + A^* A)^{-1} A^*, \qquad \alpha > 0$$

describes a regularization scheme with $\|R_\alpha\| \leq \frac{1}{2\sqrt{\alpha}}$.

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Tikhonov Regularization

Theorem 9

Let $A : X \to Y$ be a linear and bounded operator. Assme $\alpha > 0$. Then for each $f \in Y$ there exists a unique $\varphi_\alpha \in X$ such that

$$\|A\varphi_\alpha - f\| + \alpha\|\varphi_\alpha\| = inf_{\varphi \in X} \left\{ \|A\varphi - f\|^2 + \alpha\|\varphi\|^2 \right\}$$

The minimizer φ_α is given by the unique solution of the equation

$$\alpha\varphi_\alpha + A^* A\varphi_\alpha = A^* f$$

and depends continuously on f.

Approximate Solution

Definition 5 (Minimum Norm Solution)

Let $A : X \to Y$ be a bounded linear operator and let $\delta > 0$. For a given $f \in Y$ an element $\varphi_0 \in X$ is called a minimum norm solution of $A\varphi = f$ with discrepancy δ if $\|A\varphi_0 - f\| \leq \delta$ and

$$\|\varphi_0\| = \inf_{\|A\varphi - f\| \leq \delta} \|\varphi\|$$

Remark

φ_0 is a minimal norm solution to $A\varphi = f$ with discrepancy δ if and only if φ_0 is a best approximation to the zero element of X with respect to $U_f := \{\varphi \in X : \|A\varphi - f\| \leq \delta\}$.

Approximate Solution

Definition 5 (Minimum Norm Solution)

Let $A : X \to Y$ be a bounded linear operator and let $\delta > 0$. For a given $f \in Y$ an element $\varphi_0 \in X$ is called a minimum norm solution of $A\varphi = f$ with discrepancy δ if $\|A\varphi_0 - f\| \leq \delta$ and

$$\|\varphi_0\| = \inf_{\|A\varphi - f\| \leq \delta} \|\varphi\|$$

Remark

φ_0 is a minimal norm solution to $A\varphi = f$ with discrepancy δ if and only if φ_0 is a best approximation to the zero element of X with respect to $U_f := \{\varphi \in X : \|A\varphi - f\| \leq \delta\}$.

Theorem 10

Let $A : X \to Y$ be a linear and bounded operator with dense range. For $\delta > 0$, there exists for every $f \in Y$ a unique minimal norm solution of $A\varphi = f$ with discrepancy δ.
Furthermore, the parameter α can be so chosen, that φ_0 is the solution of

$$\alpha\varphi_\alpha + A^*A\varphi_\alpha = A^*f$$

with $\|A\varphi_0 - f\| = \delta$.

Theorem 11

Assume $A : X \to Y$ is a linear, bounded and injective operator with dense range. $\delta > 0, f \in A(X)$. For $f^\delta \in Y$ with $\|f^\delta - f\| \leq \delta$ and $\delta < \|f^\delta\|$ we have

$$\varphi^\delta \to A^{-1}f, \qquad \delta \to 0,$$

where φ^δ is the minimal norm solution with discrepancy δ

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Section Content

Inverse Problems : Definition

Keller, 1976

Two problems are inverse to each other if the formulation of each of them requires all or partial knowledge of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former problem is called the direct problem, while the latter is called the inverse problem.

(Inverse Problems , Joseph B. Keller,
The American Mathematical Monthly, Vol. 83, No. 2. (Feb., 1976), pp. 107-118.)

Inverse Problems : Examples

Example 4

What are the questions to which the answers are

1. "Washington Irving" ?
2. "Nine W"?
3. "Chicken Sukiyaki"?

1. What is the capital of the United States, Max ?
2. Do you spell your name with a "V", Herr Wagner ?
3. What is the name of the sole surving Kamikaze pilot ?

Inverse Problems : Examples

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Example 5 (Differentiation)

Direct problem (DP): Given $\varphi \in C([0,1])$, solve

$$(T\varphi)(x) := \int_0^x \varphi(t)dt, \qquad x \in [0,1]$$

Inverse problem (IP) : Given $g \in C([0,1])$ with $g(0) = 0$, solve

$$T\varphi = g$$

Remark

(IP) has a solution $\varphi \in C([0,1])$ if and only if $g \in C^1([0,1])$.

Assume $g^\delta \in C([0,1])$ with $\|g^\delta - g\|_\infty \leq \delta, \quad 0 < \delta < 1$.
Define $g_n^\delta(x) := g(x) + \delta \sin \frac{nx}{\delta}, \quad x \in [0,1]$
We have $(g_n^\delta)'(x) := g'(x) + n \cos \frac{nx}{\delta}, \quad x \in [0,1]$
It holds $\|(g_n^\delta)' - g'\|_\infty = n$.

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Example 6 (Backward Heat Conduction)

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x}$$

in $D := [0, 1] \times [0, T]$ with

$$(BC :) \quad u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq T$$

$$(IC :) \quad u(x, 0) = \varphi(x), \quad x \in [0, 1]$$

Backward Heat Conduction II

Direct Problem :
Given the initial temperature $\varphi \in L^2([0,1])$,
find the final temperature $f := u(\cdot, T)$

$$u(x,t) = \sqrt{2}\sum_{n=1}^{\infty} \varphi_n e^{-\pi^2 n^2 t} \sin(n\pi x)$$

Inverse Problem :
Given the final temperature f, find the initial temperature φ.

$$(T_H\varphi)(x) := \int 2\sum_{n=1}^{\infty} \left(e^{-\pi^2 n^2 T} \sin(n\pi x)\sin(n\pi y)\right) \varphi(y)dy$$

$$T_H\varphi = f$$

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$$T_H\varphi = f$$

Example 7 (Inverse Obstacle Scattering Problem)

Direct Problem :

Given an impenetrable, smooth, bounded obstacle D.

Find the far field pattern u_∞ of the scattered field u^s satisfying the following conditions :

1. (Helmholtz equation) $\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus D$
2. (Dirichlet BC) $u = 0$ on ∂D.
3. (Sommerfeld Radiation Condition, SRC)

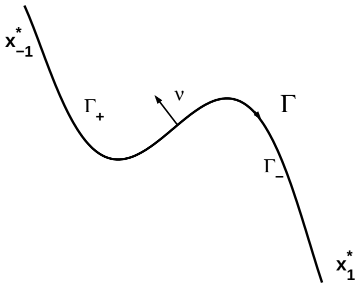
$$\lim_{r\to\infty} r^{(n-1)/2}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad \text{uniformly for all } \hat{x}$$

where the total field u is the superposition of the unknown scattered field u^s and the known incident field $u^i(x) = e^{ik<x,d>}$ with incident direction $d \in S^{n-1}$.

Inverse Problem :

Find D from the knowledge of u_∞

Mark Kac
Can One Hear the Shape of a Drum?
The American Mathematical Monthly, Vol. 73, No. 4, Part 2: Papers in Analysis. (Apr., 1966), pp. 1-23.



$$\Gamma_0 := \Gamma \setminus \{x^*_{-1}, x^*_1\}$$

Direct Impedance Problem

Definition 6 (Direct Impedance Problem, DP)

Find: $u \in C^2(\mathbb{R}^2 \setminus \Gamma)$

1. $\Delta u + k^2 u = 0$ in $\mathbb{R}^2 \setminus \Gamma, k > 0$.
2. (IBC) For $\lambda \in C^{0,\alpha}(\Gamma)$ with $Re(\lambda) \geq 0$ and $f_\pm \in C^{0,\alpha}(\Gamma)$:

$$\boxed{\frac{\partial u_\pm}{\partial \nu} \pm ik\lambda u_\pm = f_\pm \quad \text{auf } \Gamma_0} \tag{13}$$

3. (SRC) $\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial \nu} - iku\right) = 0, \qquad r := |x|, \hat{x} := \frac{x}{|x|}$

Green's Theorem

Lemma 12

Assume that u is a solution to the homogen impedance BVP for the open arc. Then for R large enough, we have

1. *gradu* $\in L^2(B_R)$
2. *Green's theorem*

$$\int_{B_R} |\mathrm{grad}\, u(y)|^2 dy - k^2 \int_{B_R} |u(y)|^2 dy$$
$$= \int_{\partial B_R} u(y) \frac{\partial \bar{u}(y)}{\partial \nu} ds(y) - ik \int_\Gamma \bar{\lambda}(y) \left(|u_+(y)|^2 + |u_-(y)|^2 \right) ds(y). \tag{14}$$

Rellich's Lemma

Theorem 13 (Rellich)

Assume that $u \in C^2(D_+)$ *is a solution of the Helmholtz equation with* $k > 0$. *If*

$$\lim_{r \to \infty} \int_{|x|=r} |u(x)|^2 ds(x) = 0$$

then $u = 0$ *in* D_+.

Uniqueness

Theorem 14

The direct Impedance problem (6) has at most one solution.

Proof:

SRC + Green's theorem + Rellich's Lemma □

Uniqueness

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Solution Ansatz

$$u(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi_1(y) ds(y) + \int_\Gamma \Phi(x,y)\varphi_2(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma. \tag{15}$$

$$\begin{cases} 2\left(\boxed{\frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi_1(y) ds(y)} + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(x)} \varphi_2(y)\right) + ik\lambda(x)\varphi_1(x) \\ = f_-(x) + f_+(x) \\ \varphi_2(x) - 2ik\lambda(x)\left(\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi_1(y) ds(y) + \underline{\int_\Gamma \Phi(x,y)\varphi_2(y) ds(y)}\right) \\ = f_-(x) - f_+(x) \end{cases} \tag{16}$$

for $x \in \Gamma_0$.

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for $x \in \Gamma_0$.

Theorem 15

The solution substitution (15) *solves the impedance BVP provided that the densities* φ_1, φ_2 *solve the system of equations* (16).

- Maue's identity
  For $\varphi \in C^{1,\alpha}_{0,lok}(\Gamma)$, it holds for $x \in \Gamma_0$

$$\frac{\partial}{\partial \nu(x)} \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \varphi(y) ds(y)$$
$$= \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \vartheta(x)} \frac{\partial \varphi(y)}{\partial \vartheta(y)} ds(y) + k^2 \left\langle \nu(x), \int_\Gamma \Phi(x,y)\varphi(y)\nu(y) ds(y) \right\rangle . \quad (17)$$

- Parameterization: $x = z(s), y = z(t)$
- Cosine-substitution : $s = \cos(\sigma), t = \cos(\tau)$

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$$\begin{cases} 2\left(\boxed{\int_0^\pi \{ \frac{1}{2\pi} \frac{\sin\sigma}{\cos\tau - \cos\sigma} \psi_1'(\tau)} - K_1(\sigma,\tau)\psi_1(\tau) - K_2(\sigma,\tau)\psi_2(\tau) \} d\tau \right) \\ -ik\lambda(z(\cos(\sigma)))\psi_1(\sigma)\sin\sigma |z'(\cos\sigma))| = g_1(\sigma) \\ \\ \psi_2(\sigma) - 2ik\lambda(z(\cos(\sigma))) \int_0^\pi \{ K_3(\sigma,\tau)\psi_1(\tau) + \underline{K_4(\sigma,\tau)\psi_2(\tau)} \} d\tau \\ = g_2(\sigma), \end{cases} \tag{18}$$

for $\sigma \in [0, \pi]$.

Operator Equation

$$(T_0\psi)(\sigma) := -\frac{1}{\pi}\int_0^{\pi} \frac{\sin\sigma}{\cos\sigma - \cos\tau}\psi'(\tau)d\tau \tag{19}$$

$$X^{p,q,\alpha} := C_0^{p,\alpha}[0,\pi] \times C^{q,\alpha}[0,\pi], \quad p,q \in \mathbb{N}\cup\{0\}, \alpha \in (0,1)$$

$$L - A : X^{1,0,\alpha} \to X^{0,0,\alpha}$$

$$(L - A)\,\Psi = g \tag{20}$$

$$L := \begin{pmatrix} T_0 & 0 \\ 0 & I_2 \end{pmatrix}, \quad A \text{ is compact}$$

$$\Psi := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad g := \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

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Theorem 16

The operator $L - A : X^{1,0,\alpha} \to X^{0,0,\alpha}$ is injective. If Ψ is the solution of the parameterizied equation (20) *, then the solution substitution* (15) *solves the direct impedance problem (6).*

Proof:

Uniqueness of the BVP □

$\tilde{T} : C^{1,\alpha}_{odd}[0, 2\pi] \to C^{0,\alpha}_{odd}[0, 2\pi]$

$$\left(\tilde{T}\psi\right)(\sigma) := -\frac{1}{2\pi}\int_0^{2\pi} \cot\frac{\tau - \sigma}{2}\psi'(\tau)d\tau, \tag{21}$$

$$C^{k,\alpha}_{odd}[0, 2\pi] := \left\{\psi \in C^{k,\alpha}[0, 2\pi], \quad \psi \text{ odd}\right\}, \qquad k = 0, 1.$$

Lemma 17

$\psi \in C^{1,\alpha}_0[0, \pi]$. If $\tilde{\psi}$ is the odd extension of ψ to $[0, 2\pi]$, it holds :

$$T_0\psi = \tilde{T}\tilde{\psi}$$

Theorem 18

The operator L is invertible and has a bounded inverse.

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Unique solvability of the BVP

Theorem 19

For every right hand side $g \in X^{0,0,\alpha}$, there exists one and only one solution $\Psi \in X^{1,0,\alpha}$ to the following integral equation

$$(L - A)\,\Psi = g$$

Theorem 20 (Unique solvability of the BVP)

The direct impedance problem is uniquely solvable.

Unique solvability of the BVP

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The direct impedance problem is uniquely solvable.

Numerical Method I - Interpolation

$n \in \mathbb{N}$
Interpolation nodes : $\sigma_j^{(n)} := \frac{j\pi}{n}, \quad j = 0, \dots, n$

$$T_{1,n} := \left\{ \phi \in C_0[0,\pi] | \, \phi(\sigma) = \sum_{k=1}^{n-1} a_k \sin k\sigma, \quad a_k \in \mathbb{C} \right\}$$

$$T_{2,n} := \left\{ \phi \in C[0,\pi] | \, \phi(\sigma) = \sum_{k=0}^{n} b_k \cos k\sigma, \quad b_k \in \mathbb{C} \right\}$$

The interpolation operator

$$\mathcal{P}_n : C_0[0,\pi] \times C[0,\pi] \to \mathcal{T}_n := T_{1,n} \times T_{2,n} \tag{22}$$

Numerical Method II - Collocation

Approximate integrals in Equation (20) through quadrature formulas :

$$(L_n - A_n)\,\Psi = g, \qquad \text{(semi-discrete)} \tag{23}$$

As a result of the following fact

$$\mathcal{P}_n L_n \Psi_n = L\Psi_n, \qquad \text{für } \Psi_n \in \mathcal{T}_n$$

we have

$$L\Psi_n - \mathcal{P}_n A_n \Psi_n = \mathcal{P}_n g, \qquad \text{(full-discrete)} \tag{24}$$

for $\Psi_n \in \mathcal{T}_n$.

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Error Analysis

Theorem 21

The approximate equation (24) *has a unique solution* Ψ_n *for* $n \in \mathbb{N}$ *large enough. Let* Ψ *denote the unique solution of the equation* (20). *We have the following error estimate*

$$\|\Psi_n - \Psi\|_{X^{1,0,\alpha}} \leq C\{\|\mathcal{P}_n g - g\|_{X^{0,0,\alpha}} + \|(\mathcal{P}_n A_n - A)\Psi\|_{X^{0,0,\alpha}}\}, \qquad (25)$$

with a constant C, *which depends only on* α.

Theorem 22

For all $\alpha \in (0,1)$ *and for all right hand side* $g \in X^{0,0,\gamma}$ *with* $\alpha < \gamma < 1$,*the sequence of the approximate solutions* (Ψ_n) *converges to the real solution. We have additionally the estimate :*

$$\|\Psi_n - \Psi\|_{X^{1,0,\alpha}} \leq C\frac{\log n}{n^{\gamma-\alpha}}\|\Psi\|_{X^{0,0,\gamma}} \qquad (26)$$

Error Analysis

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with a constant C*, which depends only on* α*.*

Theorem 22

For all $\alpha \in (0,1)$ *and for all right hand side* $g \in X^{0,0,\gamma}$ *with* $\alpha < \gamma < 1$*,the sequence of the approximate solutions* (Ψ_n) *converges to the real solution. We have additionally the estimate :*

$$\|\Psi_n - \Psi\|_{X^{1,0,\alpha}} \leq C\frac{\log n}{n^{\gamma-\alpha}}\|\Psi\|_{X^{0,0,\gamma}} \qquad (26)$$

$$k = 1, \quad \Gamma = (t, 0), \quad t \in [-1, 1] \qquad d = \frac{1}{\sqrt{2}}(1, 1)$$

n	$\mathrm{Re}u_\infty(d)$	$\mathrm{Im}u_\infty(d)$
4	0.2188930593	0.7048408180
8	0.2193498394	0.7059285113
16	0.2193498387	0.7059285133
32	0.2193498387	0.7059285133

$$\lambda = 0$$

n	$\mathrm{Re}u_\infty(d)$	$\mathrm{Im}u_\infty(d)$
4	-0.2785431331	0.6380378058
8	-0.2783708059	0.6394323083
16	-0.2783435149	0.6394631981
32	-0.2783418024	0.6394651450
64	-0.2783416956	0.6394652672
128	-0.2783416890	0.6394652748

$$\lambda = 0.5$$

n	$\mathrm{Re}u_\infty(d)$	$\mathrm{Im}u_\infty(d)$
4	-0.2706934161	0.3769022310
8	-0.2770395614	0.3847161552
16	-0.2770367776	0.3847136679
32	-0.2770367755	0.3847136690
64	-0.2770367755	0.3847136690

$$\lambda = (1 - t^2)^2$$

$$k = 5, \quad \Gamma = (t, 0), \quad t \in [-1, 1]$$

n	$\mathrm{Re} u_\infty(d)$	$\mathrm{Im} u_\infty(d)$
4	-1.2655090731	1.1733478171
8	-0.8232049220	1.4463599064
16	-0.8191108671	1.4477138852
32	-0.8191108727	1.4477138906
64	-0.8191108727	1.4477138906

$$\lambda = 0$$

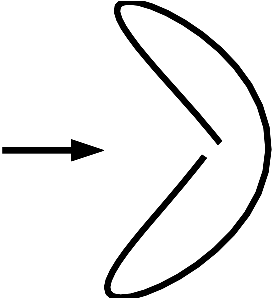
n	$\mathrm{Re} u_\infty(d)$	$\mathrm{Im} u_\infty(d)$
4	-1.1057449551	1.0073845548
8	-1.0321181482	1.1305992223
16	-1.0318551380	1.1306605046
32	-1.0318486423	1.1306713800
64	-1.0318482505	1.1306720631
128	-1.0318482263	1.1306721059

$$\lambda = 0.5$$

n	$\mathrm{Re} u_\infty(d)$	$\mathrm{Im} u_\infty(d)$
4	-0.8563180981	0.6392250142
8	-0.7649731169	0.8765565147
16	-0.7675574178	0.8710482182
32	-0.7675574890	0.8710481333
64	-0.7675574889	0.8710481334
128	-0.7675574889	0.8710481334

$$\lambda = (1 - t^2)^2$$

$$k = 1, d = (1, 0)$$
$$\Gamma = (\cos(5t) + 0.1t, 2\sin(3t) - 0.2t), t \in [-1, 1]$$



n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$
16	-1.0270849962	1.1833403947
32	-0.9471404714	1.2008539862
64	-0.9477179020	1.2005689586
128	-0.9477177121	1.2005691135
256	-0.9477177121	1.2005691135

$$\lambda = 0$$

n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$
16	-1.1208195318	1.1805107232
32	-1.0877388284	1.1946305225
64	-1.0879938741	1.1943662399
128	-1.0879939475	1.1943663130
256	-1.0879939559	1.1943663160

$$\lambda = 0.5$$

n	$\mathrm{Re}\, u_\infty(d)$	$\mathrm{Im}\, u_\infty(d)$
16	-1.3751723167	1.1685067430
32	-1.3415425975	1.1896861321
64	-1.3418197983	1.1893501329
128	-1.3418197160	1.1893501474
256	-1.3418197159	1.1893501474

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Inverse Impedance Problem

Definition 7 (Inverse Impedance Problem, IP)

Given : Γ, $u^i(x) := e^{ik<x,d>}$, u_∞,
find : λ

Theorem 23 (Uniqueness)

If u_1^s, u_2^s are solutions to the inverse impedance problem with impedance λ_1, λ_2 respectively and if the corresponding far field patterns $u_{1,\infty}, u_{2,\infty}$ are identical, then $\lambda_1 = \lambda_2$. In another word, the impedance is uniquely determined by the far field pattern.

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Numerical Method I - Information

$$\frac{\partial u_\pm}{\partial \nu} \pm ik\lambda u_\pm = 0 \quad \text{on } \Gamma_0 \tag{27}$$

$$u^s(x) := \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi_1(y) ds(y) + \int_\Gamma \Phi(x,y)\psi_2(y) ds(y), x \in \mathbb{R}^2 \setminus \Gamma. \tag{28}$$

$$u_\infty(\hat{x}) = C_1 \int_\Gamma < \nu(y), \hat{x} > e^{-ik<\hat{x},y>} \psi_1(y) ds(y) + C_2 \int_\Gamma e^{-ik<\hat{x},y>} \psi_2(y) ds \tag{29}$$

Far field operator

$$F : L^2(\Gamma) \times L^2(\Gamma) \to L^2(\Omega), \Psi = (\psi_1, \psi_2) \mapsto u_\infty$$

Theorem 24

F is injective and has dense range.

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Numerical Method II - Procedure

- F is compact ! Solving $F\Psi = u_\infty$ is ill-posed. Regularization needed.

$$(\alpha I + F^* F)\Psi_\alpha = F^* u_\infty, \quad \alpha > 0, \tag{30}$$

- Solve λ from the boundary condition,

$$\lambda = \frac{\psi_2}{2iku} \tag{31}$$

- Least square method

$$\lambda = \sum_{m=1}^{M} a_m \varphi_m \tag{32}$$

Determine the coefficients a_m

$$\min : \sum_{n=1}^{N} \left| \psi_2(\sigma_n) - 2iku(\sigma_n) \sum_{m=1}^{M} a_m \varphi_m(\sigma_n) \right|^2 \tag{33}$$

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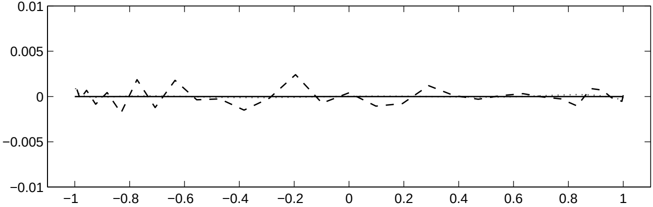
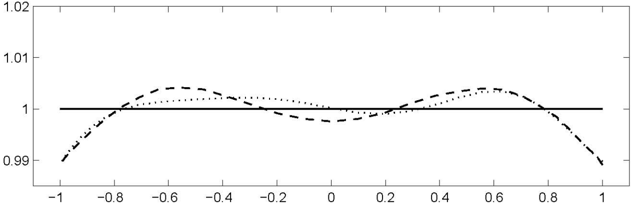
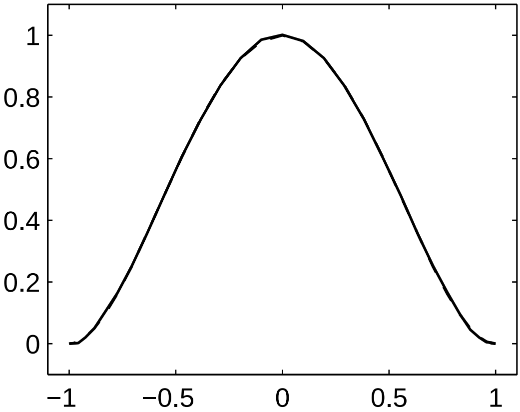
$$k = 1, \quad \Gamma = (t, 0), \quad t \in [-1, 1], \qquad d = (0, 1)$$
$$\lambda \ \underline{\qquad\qquad} \quad \lambda_{32} - - \quad \lambda_{*} \cdots\cdot$$



$\lambda = 0, \quad \alpha = 1.8\,E^{-14}, \quad \|\lambda_{32} - \lambda\|_2 = 1.3\,E^{-3}$

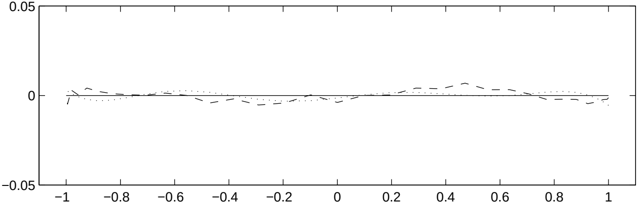


$\lambda = 1, \quad \alpha = 1.0\,E^{-12}, \quad \|\lambda_{32} - \lambda\|_2 = 8.6\,E^{-3}$

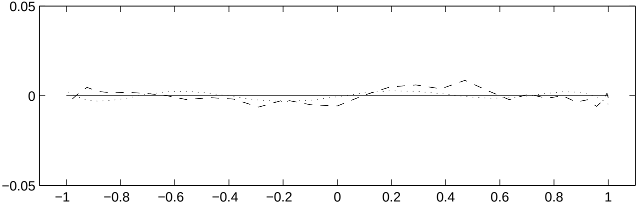


$\lambda = (1 - t^2)^2, \quad \alpha = 3.1\,E^{-14}\ \|\lambda_{32} - \lambda\|_2 = 3.8\,E^{-3}$

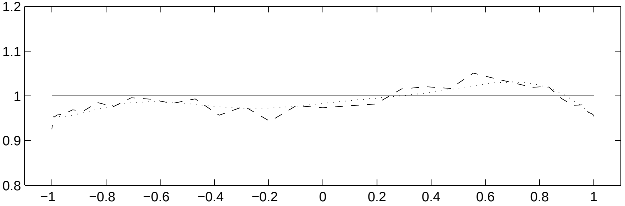
$$k = 1, \Gamma = (\cos(t), \sin(t)), \qquad t \in [-1, 1]$$
$$f_i^{\delta} := (1 + ((-1)^i + \tfrac{2}{5})\delta) f_i.$$



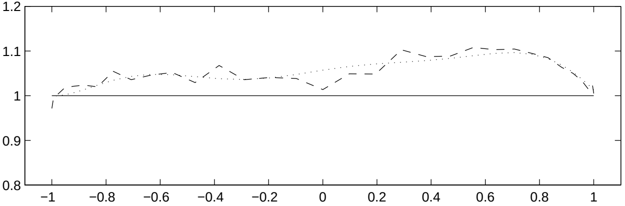
$$\lambda = 0, \quad \alpha = 4\,E^{-15}, \quad \|\lambda_{32} - \lambda\|_2 = 6.7\,E^{-3}$$



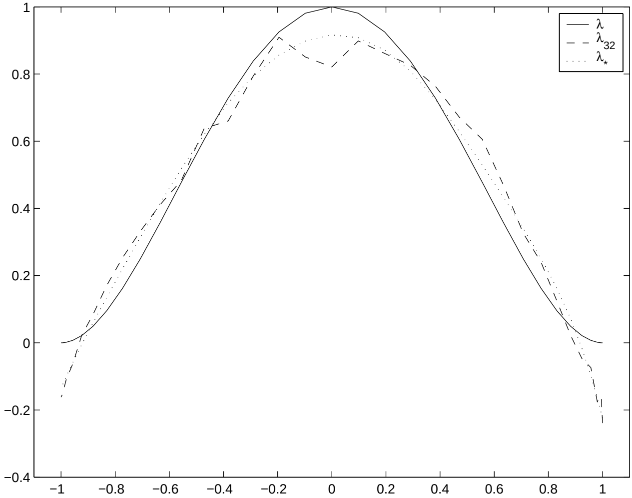
$$\lambda = 0, \quad \delta = 0.1, \quad \alpha = 4\,E^{-15} \|\lambda_{32} - \lambda\|_2 = 7.2\,E^{-3}$$



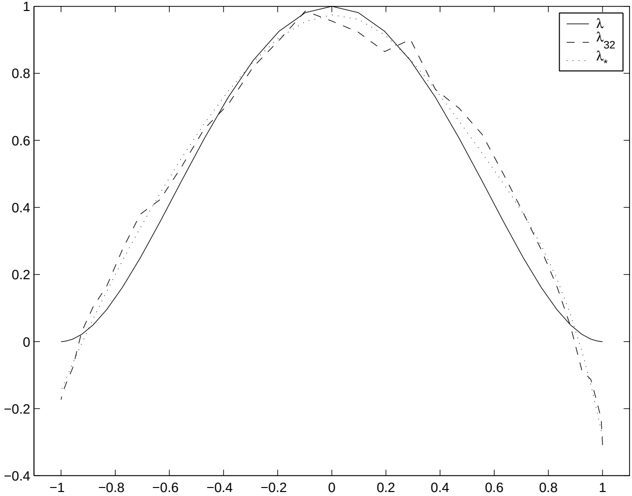
$$\lambda = 1, \quad \alpha = 8.0\,E^{-15}, \quad \|\lambda_{32} - \lambda\|_2 = 6.1\,E^{-2}$$



$$\lambda = 1, \quad \delta = 0.1, \quad \alpha = 8.0\,E^{-15} \|\lambda_{32} - \lambda\|_2 = 6.5\,E^{-2}$$



$$\lambda = (1 - t^2)^2, \quad \alpha = 2\,E^{-15} \|\lambda_{32} - \lambda\|_2 = 1.3\,E^{-1}$$



$$\lambda = (1 - t^2)^2, \quad \delta = 0.1, \quad \alpha = 3\,E^{-15} \; \|\lambda_{32} - \lambda\|_2 = 1.4\,E^{-1}$$