HEAT SOURCE IN INFINITE PLANE WITH ELLIPTIC RIGID INCLUSION AND HOLE

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ABSTRACT: The plane thermoelastic problems of a stationary heat source in an infinite plane with an elliptic rigid inclusion and an elliptic hole are analyzed under thermally adiabatic and isothermal boundary conditions. The problems of an elliptic rigid inclusion are derived for the following cases: (1) the case that there are rigid-body displacement and rotation; and (2) the case that there is no rigid-body displacement or rotation. To analyze these problems, the following three fundamental solutions are derived: Problem A, in which a point heat source exists within an infinite domain; Problem B, in which the inclusion has a small amount of rotation; and Problem C, in which the inclusion is subjected to concentrated loads. Two cases can be obtained by superimposing these fundamental solutions. For the hole problem, the fundamental solution (Green's function) is also derived. In analysis, the complex stress functions, the mapping function, and the thermal dislocation method are used. The complex stress functions are obtained as a closed form. For analytic examples, the stress distributions are shown under thermally adiabatic and isothermal boundary conditions. For the crack problem, the stress intensity factors are shown for the location of the heat source.

INTRODUCTION

The steady-state, plane thermoelastic problem of an infinite plane with a point heat source is important in engineering, and a number of technical articles have been dedicated to this subject. For instance, solutions for a point heat source in a strip (Nowacki 1962) and a semiinfinite plate (Parkus 1968) have been shown. Fukui et al. (1970) treated analytically a problem of an infinite plane having a circular hole, where heat flux generates from a heat source flowing out by the boundary of the hole, on which the isothermal boundary condition is prescribed. Fukui et al. (1974) also obtained a solution for an infinite plane with a circular inclusion, in which heat flux generates from a heat source, passes through the inclusion, and flows into a sink. Zhang and Hasebe (1993) solved the problem of an adiabatic crack contained in an infinite plane that is thermally deformed in a temperature field caused by a single heat source. Employing a basic solution for a point heat source in an infinite plane without a hole, Nisitani et al. (1991) derived a solution for an infinite plane with a circular hole under uniform heat flow, by summing up the basic solutions for heat sources of various intensities, distributed over the boundary in such a manner that the boundary condition on the hole is satisfied (the body force method). Furthermore, the problem of two nearby cracks in an infinite plane, under a prescribed crack surface temperature (temperature boundary condition) and assuming continuous distributions of edge and quasi-Volterra dislocations on the crack positions in order to model the crack and heat sources distributed on it, was solved by Sekine (1979, 1987).

In problems of displacement and mixed boundary conditions, the stress state differs, depending on whether there are rigid-body displacement and rotation of the inclusion. In general, rigid-body displacement and rotations must be considered in the model of inclusion. On the other hand, when a clamped column and the rigid reinforcement of an elliptic hole are modeled, reaction force and reaction moment on the boundary from the support serve to suppress the rigid-body displacement and rotation. For instance, in a problem in which there is no rigid-body displacement of the inclusion, the final solution must include such a portion, which is the solution to a case where concentrated loads are applied on the inclusion to cancel its rigid-body displacement.

This article develops fundamental solutions for the following three problems: Problem A, in which a point heat source exists within an infinite plane containing an elliptic rigid inclusion with rigid-body displacement; Problem B, in which the elliptic rigid inclusion has a small amount of rigid-body rotation; and Problem C, in which concentrated loads are applied on the elliptic rigid inclusion. By superposing these solutions, the solution for the problem of a point heat source existing within an infinite plane containing an elliptic rigid inclusion with rigid-body displacement and rotation can be obtained. The mapping function and complex stress functions are used in the analysis. Employing the solution of complex stress functions for Problem A, stress distributions for zero heat flow (thermally adiabatic) and prescribed temperature (isothermal) boundary conditions are worked out and shown graphically in analytic examples.

Also, an infinite plane with an elliptic hole or a crack is treated under thermally adiabatic or isothermal boundary conditions. Complex stress functions are derived, and the thermal stress intensity factors at the tip of the crack are presented in graphs.

The general solutions developed herein for heat sources can be used in engineering to model thermal processes in welding and in the boundary element method as fundamental solutions (Green's functions) for thermal problems involving inclusions and cracks. Indeed, the application of the boundary element method is continuously extending due to the development of various fundamental solutions for thermal problems.

MAPPING FUNCTION

As shown in Fig. 1, the infinite plane with an elliptic hole in the z-plane can be mapped into the infinite region outside of a unit circle in the ζ-plane by the following mapping function:

$$z = \omega(\zeta) = E_0\zeta + \frac{E_1}{\zeta}, \quad E_0 = \frac{(a + b)}{2}, \quad E_1 = \frac{(a - b)}{2} \qquad (1)$$

where a and b = length of the semiaxes of the ellipse on the x and y axes, respectively. If $a = b$, the ellipse becomes a circle; for $b = 0$, it becomes a crack.

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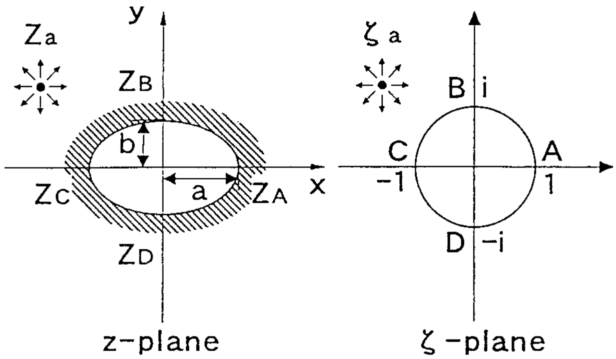

FIG. 1. Elliptic Hole and Heat Source in Arbitrary Point and Sink Source at Infinity

TEMPERATURE ANALYSIS

Basic Formulas for Temperature and Heat Flux

In two-dimensional, steady-state heat conduction, the temperature function $\Theta(x, y)$, referring to an (x, y) rectangular coordinate system, should satisfy the Laplace equation. Let us introduce an analytic function $\Omega(z)$ of complex variable $z = x + iy$:

$$\Omega(z) = \Theta(x, y) + iT(x, y) \tag{2}$$

where $T(x, y)$ = harmonic conjugate function of Θ. Using $\Omega(z)$ as well as its complex conjugate, the temperature field Θ can be expressed as

$$\Theta(z, \bar{z}) = \frac{1}{2}\{\Omega(z) + \overline{\Omega(z)}\} \tag{3}$$

Denoting the thermal conductivity of the material as k, q_x, and q_y, the heat flux in the x and y directions, respectively, can be written as

$$q_x - iq_y = -k\left\{\frac{\partial\Theta}{\partial x} - i\frac{\partial\Theta}{\partial y}\right\} = -k\Omega'(z) \tag{4}$$

Making use of the mapping function $\omega(\zeta)$, relations (3) and (4) may be rewritten in the following form (Hasebe et al. 1988):

$$\Omega(z) = \Omega[\omega(\zeta)] \equiv \Psi(\zeta); \quad \Theta(\zeta, \bar{\zeta}) = \frac{1}{2}[\Omega(\zeta) + \overline{\Omega(\zeta)}] \tag{5a,b}$$

$$q_x - iq_y = -k\frac{\Omega'(\zeta)}{\omega'(\zeta)} \tag{6}$$

$$q_r - iq_\theta = \frac{\zeta\omega'(\zeta)}{|\zeta\omega'(\zeta)|}(q_x - iq_y) \tag{7}$$

where q_r and q_θ denote the components of heat flux in the normal and tangential directions, respectively, in an orthogonal curvilinear coordinate system given by the mapping function.

Solutions for Heat Source at Arbitrary Point and Heat Sink at Infinity

If the heat flux q_n across the boundary is given, the heat flux boundary condition can be expressed by the following formula (Hasebe et al. 1988):

$$-\kappa\{\Omega(\sigma) - \overline{\Omega(\sigma)}\} = 2i\int q_n\, ds + \text{const} \tag{8}$$

where σ denotes ζ on the unit circle; and the integration in regards to s is carried out along the elliptical boundary.

On the other hand, the temperature boundary condition can be written from (5) as:

$$\Omega(\sigma) + \overline{\Omega(\sigma)} = 2\Theta(\sigma, \bar{\sigma}) \tag{9}$$

Without loss of generality, we consider boundary condition (8) in which $q_n = 0$, that is, when the boundary is thermally adiabatic, and boundary condition (9) with temperature $\Theta = 0$ (isothermal condition). In such situations, the boundary conditions can be expressed as follows:

$$\Omega(\sigma) - \Gamma\overline{\Omega(\sigma)} = \text{const} \tag{10a}$$

$$\text{thermally adiabatic condition: } \Gamma = 1 \tag{10b}$$

$$\text{isothermal condition: } \Gamma = -1 \tag{10c}$$

Next, we consider the case that the heat source is located at point z_a in the z-plane, as shown in Fig. 1, whereas the sink is at infinity. Point ζ_a in the ζ-plane corresponds to z_a in the z-plane. $\Omega(\zeta)$, the complex thermal function to be determined for the problem, can be put into the following form:

$$\Omega(\zeta) = \Omega_1(\zeta) + \Omega_2(\zeta), \quad \Omega_1(\zeta) = -\frac{M}{2\pi k}\log(\zeta - \zeta_a) \tag{11}$$

where M = intensity of the heat source; and $\Omega_1(\zeta)$ = temperature function for a point heat source in an infinite region. Substituting (11) into (10), multiplying both sides of the equation with $d\sigma/[2\pi i(\sigma - \zeta)]$, and carrying out the Cauchy integral on the unit circle, $\Omega_2(\zeta)$ can be obtained; subsequently $\Omega(\zeta)$ is expressed as

$$\Omega(\zeta) = -\frac{M}{2\pi k}\left\{\log(\zeta - \zeta_a) + \Gamma\log\left(\frac{\zeta - \zeta_a'}{\zeta}\right) + C\right\} \tag{12}$$

where $\zeta_a' \equiv 1/\bar{\zeta}_a$; and C = a constant, which should be fixed by the temperature at a referenced base point.

THERMAL STRESS ANALYSIS

Heat Source in Infinite Plane with Elliptic Rigid Inclusion

For an infinite plane with a rigid inclusion, rigid-body displacement and rigid-body rotation will take place in the inclusion if the geometry of inclusion and the external loads in the problem do not have symmetry. To develop solutions for the problem of a heat source in an infinite plane, containing an elliptic rigid inclusion with or without rigid-body displacement and rotation, we consider the following three problems (Fig. 2):

1. Problem A, in which a heat source exists within an infinite plane with an elliptic rigid inclusion. Stress functions for this problem are denoted by $\phi_A(\zeta)$ and $\psi_A(\zeta)$.
   Rigid-body displacements of the inclusion are allowable: $u_A \neq 0$; $v_A \neq 0$; $X_A = 0$; $Y_A = 0$. Rigid-body rotation of the inclusion is not allowed: $\varepsilon_A = 0$; $M_A \neq 0$.
2. Problem B, in which the elliptic rigid inclusion in the

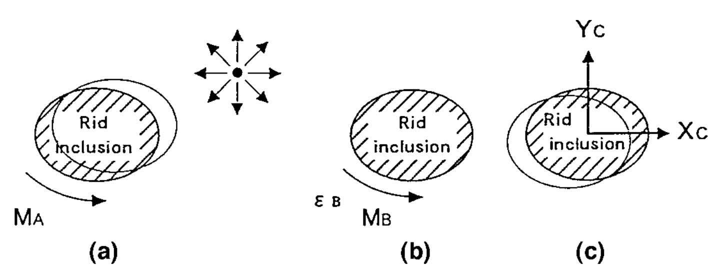

FIG. 2. Problems of Infinite Plane with Elliptic Rigid Inclusion: (a) Problem A, Rigid Inclusion Subject to Heat Source ($u_A \neq 0$, $v_A \neq 0$, $X_A = 0$, $Y_A = 0$, $\varepsilon_A = 0$, $M_A \neq 0$); (b) Problem B, Rigid Inclusion under Rotation ($u_B = 0$, $v_B = 0$, $X_B = 0$, $Y_B = 0$, $\varepsilon_B \neq 0$, $M_B \neq 0$); (c) Problem C, Rigid Inclusion under Concentrated Loads ($u_C \neq 0$, $v_C \neq 0$, $X_C \neq 0$, $Y_C \neq 0$, $\varepsilon_C = 0$, $M_C = 0$)

infinite plane rotates at a small angle. Stress functions for this problem are denoted by $\phi_B(\zeta)$ and $\psi_B(\zeta)$.

Rigid-body displacements of the inclusion are not allowed: $u_B = 0$; $v_B = 0$; $X_B = 0$; $Y_B = 0$. Rigid-body rotation of the inclusion is allowable: $\varepsilon_B \neq 0$; $M_B \neq 0$.

3. Problem C, in which concentrated loads apply on the elliptic rigid inclusion contained in the infinite plane. Stress functions for this problem are denoted by $\phi_C(\zeta)$ and $\psi_C(\zeta)$.

   Rigid-body displacements of the inclusion are allowable: $u_C \neq 0$; $v_C \neq 0$; $X_C \neq 0$; $Y_C \neq 0$. Rigid-body rotation of the inclusion is not allowed: $\varepsilon_C = 0$; $M_C = 0$. (These two equations are satisfied by the symmetry of the geometry of the ellipse.)

In the above formulations, u_i and v_i ($i = A, B, C$) stand for the rigid-body displacements in the x and y directions, respectively; ε_i ($i = A, B, C$) denotes the rotation angle about the origin; X_i and Y_i ($i = A, B, C$) are the concentrated loads (the resultant forces), applied on the inclusion in the x and y directions, respectively; and M_i ($i = A, B, C$) is the resultant moment about the origin by the stresses on the boundary of the inclusion.

Rigid-Body Displacement and Rigid-Body Rotation Taking Place in Elliptic Rigid Inclusion

When rigid-body displacement and rigid-body rotation take place in the elliptic rigid inclusion, the solution can be obtained by adding the solution of Problem A to the solution of Problem B. The angle ε_B in Problem B is determined from the condition $M_A + M_B = 0$, that is, the sum of the resultant moments in Problems A and B vanishes. This situation fits the model that no constraints in rigid-body motion are fixed on the inclusion embedded in the infinite plane. Or, it applies to the model in which a reinforcing substance of an elliptic shape is inserted into the hole to strengthen the structure, or in which the boundary on the elliptic hole is reinforced by a rigid material.

No Rigid-Body Displacement and No Rigid-Body Rotation Taking Place in Elliptic Rigid Inclusion

The solution for the case that there is no rigid-body displacement or rigid-body rotation taking place in the elliptic rigid inclusion can be obtained by adding the solution of Problem A to the solution of Problem C. The magnitude of the concentrated loads applied on the inclusion, X_C and Y_C, are determined by the requirement of $u_A + u_C = 0$ and $v_A + v_C = 0$, that is, the sum of the rigid-body displacements in Problems A and C is zero. X_C and Y_C represent the resultant forces of the reactions from the inclusion, and they induce no rigid-body rotation. On the other hand, $M_A + M_C$ ($M_C = 0$ in this case) represents the reaction moment. This case fits the model in which a clamped column or an elliptic pipe passes through the basic plate.

Derivation of Complex Stress Functions of Displacement Boundary Value Problem

Problem A

Utilizing the complex stress functions $\phi(\zeta)$ and $\psi(\zeta)$, which are regular outside the unit circle, as well as the temperature function $\Omega(\zeta)$ just obtained, external traction and displacement boundary conditions can be written as (Muskhelishvili 1963):

$$\phi(\sigma) + \frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\phi'(\sigma)} + \overline{\psi(\sigma)} = i\int (p_x + ip_y)\,ds \quad (13)$$

$$\kappa\phi(\sigma) - \frac{\omega(\sigma)}{\omega'(\sigma)}\overline{\phi'(\sigma)} - \overline{\psi(\sigma)} + 2G\alpha'\int \Omega(\zeta)\omega'(\zeta)\,d\zeta = 2G(u + iv) \quad (14)$$

where p_x and p_y stand for the boundary tractions in the x and y directions, respectively; the integral in (13) is along the boundary; u and v = boundary displacements in the x and y directions, respectively; G = shear modulus of elasticity; for plane strain, $\alpha' = \alpha(1 + \nu)$, $\kappa = 3 - 4\nu$; and for generalized plane stress, $\alpha' = \alpha$, $\kappa = (3 - \nu)/(1 + \nu)$, where α = coefficient of linear expansion and ν = Poisson's ratio. The fourth term in the left-hand side of (14) stands for the displacements caused by the temperature $\Omega(\zeta)$.

The free thermal expansion displacements in the infinite plane without hole, brought about by a heat source at z_a, are given as follows:

$$u_0 + iv_0 = \alpha'\int \Omega_0(z)\,dz = -\frac{\alpha' M}{2\pi k}(z - z_a)\{\log(z - z_a) - 1\} \quad (15)$$

where $\Omega_0(z) = -(M/2\pi k)\log(z - z_a)$.

The stress functions corresponding to the heat source inducing the free expansion displacements in (15) can be determined on the grounds that the stresses as well as the displacements must recover to their original values, after around the heat source (point z_a) once. The result (Nisitani et al. 1991) is:

$$\phi_0(z) = \frac{\alpha MGR}{4\pi k}(z - z_a)\{\log(z - z_a) - 1\} \quad (16a)$$

$$\psi_0(z) = -\frac{\alpha MGR}{4\pi k}\overline{z_a}\log(z - z_a) \quad (16b)$$

where $R = (1 + \nu)/(1 - \nu)$ for plane strain; and $R = 1 + \nu$ for generalized plane stress.

For the infinite plane with a hole, the displacements induced by the temperature $\Omega(\zeta)$ in the fourth term of the left-hand side of (14) contain multivalued functions (logarithmic functions); therefore, after around the hole once, a mismatch in displacement (thermal dislocation) appears. Consequently, in order to satisfy the single-valuedness of the displacements, another function that can cancel the effect of these multivalued functions must be introduced. Invoking $\log\zeta$ as the function to cancel the mismatches in stress and displacements, the required complex stress functions, $\phi_A(\zeta)$ and $\psi_A(\zeta)$, can be written in the following form:

$$\phi_A(\zeta) = \phi_{A1}(\zeta) + \phi_{A2}(\zeta); \quad \psi_A(\zeta) = \psi_{A1}(\zeta) + \psi_{A2}(\zeta) \quad (17a,b)$$

$$\phi_{A1}(\zeta) = \frac{\alpha MGR}{4\pi k}[\{\omega(\zeta) - \omega(\zeta_a)\}\{\log(\zeta - \zeta_a) - 1\}] + A\log\zeta \quad (18a)$$

$$\psi_{A1}(\zeta) = -\frac{\alpha MGR}{4\pi k}[\overline{\omega(\zeta_a)}\log(\zeta - \zeta_a)] + B\log\zeta \quad (18b)$$

where the first terms of $\phi_{A1}(\zeta)$ and $\psi_{A1}(\zeta)$ in (18) are the stress functions corresponding to the heat source at z_a in (16), with ζ being substituted for z according to the mapping function. The second terms are stress functions that should be determined as follows: When (17) and (18) are inserted into (13) and (14), the mismatches in stress and displacement resulting from around the hole once must vanish. $\phi_{A2}(\zeta)$ and $\psi_{A2}(\zeta)$ are necessarily single-valued functions. Substituting (17) and (18) into the external traction boundary condition (13), from the requirement that the stresses must preserve single-valuedness after around the hole once, the following relation is obtained:

$$B = \bar{A} \tag{19}$$

Substituting (17) and (18) into the displacement boundary condition (14), and from the requirement that the mismatch in displacement resulting from around the hole once must vanish, the coefficient A is fixed as

$$A = \frac{\alpha MGR}{4\pi k}\left\{-\Gamma E_0\zeta'_a + \frac{E_1}{\zeta_a}\right\} \tag{20}$$

Next, the function $\phi_{A2}(\zeta)$ is sought. Substituting (17) and (18) into the displacement boundary condition (14) and letting $u = v = 0$ without loss of generality, after some arrangement, we obtain

$$\frac{\alpha MGR}{4\pi k}\{\omega(\sigma) - \omega(\zeta_a)\}\log(\sigma - \zeta_a) + \frac{\alpha MGR}{4\pi k} \cdot \left[\{\omega(\sigma) - \omega(\zeta_a)\}\log\left(\frac{1}{\sigma} - \bar{\zeta}_a\right) - \omega(\sigma)\right] + \kappa\phi_{A2}(\sigma) + \frac{\omega(\sigma)}{\overline{\omega'}(1/\sigma)}\left[\frac{\alpha MGR}{4\pi k}\left\{\overline{E_0} - \frac{\overline{E_1}}{\bar{\bar{\zeta}}_a}\sigma\right\} + \bar{A}\sigma + \overline{\phi'_{A2}}\left(\frac{1}{\sigma}\right)\right] + \overline{\psi_{A2}}\left(\frac{1}{\sigma}\right) - \frac{\alpha MGR}{4\pi k}\kappa\{\omega(\sigma) - \omega(\zeta_a)\} - F(\sigma) = 0 \tag{21a}$$

$$F(\sigma) = \frac{\alpha MGR}{4\pi k}(1 + \kappa)\left[E_0\left\{\Gamma\sigma\log\left(\frac{\sigma}{\sigma - \zeta'_a}\right) + \sigma - \Gamma\zeta'_a\log\left(\frac{\sigma}{\sigma - \zeta'_a}\right)\right\} + \Gamma\frac{E_1}{\sigma} + \Gamma\frac{E_1}{\sigma}\log\left(\frac{\sigma}{\sigma - \zeta'_a}\right) - \Gamma\frac{E_1}{\zeta'_a}\left\{\log\left(\frac{\sigma}{\sigma - \zeta'_a}\right)\right\}\right] - C\frac{\alpha MGR}{4\pi k} \cdot (1 + \kappa)\left\{E_0\sigma + \frac{E_1}{\sigma}\right\} \tag{21b}$$

Multiplication of both sides of (21a) by the factor $d\sigma/[2\pi i(\sigma - \zeta)]$ and integration of the Cauchy type along the unit circle yields the result of $\phi_{A2}(\zeta)$.

The stress function $\phi_A(\zeta)$ is eventually given by the following equation:

$$\phi_A(\zeta) = \frac{\alpha MGR}{4\pi k}[\{\omega(\zeta) - \omega(\zeta_a)\}\log(\zeta - \zeta_a) - \{\omega(\zeta) - \omega(\zeta_a)\}] + A\log\zeta + \frac{\alpha MGR}{4\pi k}\frac{E_1}{\zeta} + \frac{1}{\kappa}\frac{\alpha MGR}{4\pi k}\frac{E_1}{\zeta}\log(-\zeta_a) + \frac{1}{\kappa}\frac{\alpha MGR}{4\pi k}\left[\{\omega(\zeta) - \omega(\zeta_a)\}\log \cdot \left(\frac{1}{\zeta} - \bar{\zeta}_a\right) - E_0\zeta\log(-\bar{\zeta}_a) - \frac{E_1}{\zeta}\right] + \frac{1}{\kappa}\frac{\alpha MGR}{4\pi k}\frac{E_1}{\zeta} - \frac{1}{\kappa}\frac{1}{2\pi i}\int\frac{F(\sigma)}{\sigma - \zeta}d\sigma + \text{const} \tag{22a}$$

$$\frac{1}{2\pi i}\int\frac{F(\sigma)}{\sigma - \zeta} = \frac{\alpha MGR}{4\pi k}(1 + \kappa)\left[E_0\left\{\zeta\log\left(\frac{\zeta}{\zeta - \zeta'_a}\right) + \zeta'_a\log\left(\frac{\zeta}{\zeta - \zeta'_a}\right)\right\} + \Gamma\frac{E_1}{\zeta} + \Gamma\frac{E_1}{\zeta}\log\left(\frac{\zeta}{\zeta - \zeta'_a}\right) - \Gamma\frac{E_1}{\zeta'_a}\left\{\log\left(\frac{\zeta}{\zeta - \zeta'_a}\right)\right\}\right] - C\frac{\alpha MGR}{4\pi k}(1 + \kappa)\frac{E_1}{\zeta} \tag{22b}$$

Multiplication of both sides of the conjugate form of (21) by the factor $d\sigma/[2\pi i(\sigma - \zeta)]$ and integration of the Cauchy type along the unit circle yields $\psi_{A2}(\zeta)$. Substituting the solution $\psi_{A2}(\zeta)$ and (18b) into (17), $\psi_A(\zeta)$ is obtained as

$$\psi_A(\zeta) = -\frac{\alpha MGR}{4\pi k}\{\overline{\omega(\zeta_a)}\log(\zeta - \zeta_a)\} - \frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi'(\zeta) + B\log\zeta + \frac{\alpha MGR}{4\pi k}\left[\overline{E_1}\zeta\log(-\bar{\zeta}_a) - \left\{\bar{\omega}\left(\frac{1}{\zeta}\right) - \overline{\omega(\zeta_a)}\right\}\log\left(\frac{1}{\zeta} - \bar{\zeta}_a\right)\right] - \frac{\alpha MGR}{4\pi k}\left\{\frac{\overline{E_0}}{\zeta}\log(-\zeta_a) - \frac{\overline{E_0}}{\zeta}\right\} + \frac{\alpha MGR}{4\pi k}\bar{\omega}\left(\frac{1}{\zeta}\right)\{\log(\zeta - \zeta_a) - 1\} + \frac{\alpha MGR}{4\pi k}\overline{E_1}\zeta - \kappa\frac{\alpha MGR}{4\pi k}\frac{\overline{E_0}}{\zeta} + \frac{1}{2\pi i}\int\frac{\bar{F}(1/\sigma)}{\sigma - \zeta}d\sigma + \text{const} \tag{23a}$$

where

$$\frac{1}{2\pi i}\int\frac{\bar{F}(1/\sigma)}{\sigma - \zeta}d\sigma = \frac{\alpha MGR}{4\pi k}(1 + \kappa)\frac{\overline{E_0}}{\zeta} - C\frac{\alpha MGR}{4\pi k}(1 + \kappa)\frac{\overline{E_0}}{\zeta} \tag{23b}$$

Moreover, since on the boundary exists a portion on which the displacements vanish, analytic continuation on the displacement boundary provides another approach to obtain the stress function $\psi_A(\zeta)$. The result (Hasebe et al. 1988) is

$$\psi_A(\zeta) = \kappa\overline{\phi_A}\left(\frac{1}{\zeta}\right) - \frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi'_A(\zeta) + \overline{\left\{2G\alpha'\int\Omega(\zeta)\omega'(\zeta)\,d\zeta\right\}}_{\bar{\zeta}=1/\zeta} \tag{24}$$

Substituting the solution for the heat source and that for the heat sink (which can be obtained by the substitution of ζ_b for ζ_a and $-M$ for M in the heat source solution) into (22) and (23), respectively, and summing up the obtained solutions, we obtain the solution for the case of a heat source and a heat sink.

In the case of the isothermal condition, $\Gamma = -1$, if the heat source exists on the boundary ($\zeta_a = \sigma_a$, $\zeta'_a = \sigma_a$), the term of $\log(\zeta - \zeta_a)$ disappears in (22) and (23). Therefore, the heat source is not a singular point for stress and displacement.

The resultant moment M about the origin, formed by the stresses on the boundary, can be determined by taking $\zeta = \sigma$ on the unit circle and conducting a contour integration along the boundary. The result is expressed (Muskhelishvili 1963) as

$$M = \text{Re}\left[\oint\psi(\sigma)\omega'(\sigma)\,d\sigma - \omega(\sigma)\psi(\sigma) - \omega(\sigma)\bar{\omega}\left(\frac{1}{\sigma}\right)\frac{\phi'(\sigma)}{\omega'(\sigma)}\right] \tag{25}$$

When (24) is substituted into (25), the resultant moment M_A for the case of the heat source turns out to be

$$M_A = \text{Re}\left[\kappa\oint\omega(\sigma)\overline{\phi'_A}\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma^2} - \oint\overline{\omega(\sigma)}\phi'_A(\sigma)\,d\sigma + 2G\alpha'\oint\omega(\sigma)\overline{\omega'}\left(\frac{1}{\sigma}\right)\bar{\Omega}\left(\frac{1}{\sigma}\right)\frac{d\sigma}{\sigma^2}\right] \tag{26}$$

The rigid-body displacements u_A and v_A, the relative displacements between the heat source location and the rigid inclusion, are obtained by using (22) and (23), i.e., after putting $\sigma = \zeta$ in the displacement expression (14), evaluating the values of displacements at point $\zeta = \zeta_a$ (heat source location) and, for instance, at point $\zeta = 1$ (a point on the boundary), then working out the difference between the two results.

Problem B

In this subsection, the complex stress functions are derived for the case when the rigid inclusion rotates at a small angle ε in the counterclockwise direction [Fig. 2(b)] (Muskhelishvili 1963).

Letting $\phi_B(\zeta)$ and $\psi_B(\zeta)$ be the required complex stress functions and assuming $u = -\varepsilon_B y$, $v = \varepsilon_B x$ on the boundary, the displacement boundary conditions can be expressed by (14), excluding the fourth term in its left-hand side, as follows:

$$\kappa\phi_B(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi_B'(\sigma)} - \overline{\psi_B(\sigma)} = 2G(u + iv) = 2G\varepsilon_B i\omega(\sigma) \quad (27)$$

Multiplication of both sides of (27) by the factor $d\sigma/[2\pi i(\sigma - \zeta)]$ and the integral in the Cauchy type yields $\phi_B(\zeta)$ in the following form:

$$\phi_B(\zeta) = \frac{2G\varepsilon_B i}{\kappa}\frac{E_1}{\zeta} + \text{const} \quad (28)$$

As for $\psi_B(\zeta)$, it can be worked out either by multiplying the conjugate form of (27) by the factor $d\sigma/[2\pi i(\sigma - \zeta)]$ and then implementing an integral of the Cauchy type on the unit circle, or by analytic continuation on the displacement boundary to obtain

$$\psi_B(\zeta) = \kappa\overline{\phi_B}\left(\frac{1}{\zeta}\right) - \frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_B'(\zeta) + 2G\varepsilon_B i\bar{\omega}\left(\frac{1}{\zeta}\right) \quad (29)$$

Taking $\zeta = \sigma$ on the boundary and substituting (28) and (29) into (25), M_B, the resultant moment, is obtained as

$$M_B = 4\pi G\varepsilon_b\left\{E_0\overline{E_0} + \frac{E_1\overline{E_1}}{\kappa}\right\} \quad (30)$$

The angle ε_B at which the rigid inclusion rotates can be determined by the condition $M_A + M_B = 0$; that is, the sum of the resultant moments, M_A in (26) and M_B in (30), is zero. The stress functions then become $\phi_A(\zeta) + \phi_B(\zeta)$ and $\psi_A(\zeta) + \psi_B(\zeta)$.

Problem C

In this subsection, the complex stress functions are derived for the case when X_C and Y_C, the concentrated loads in x and y directions, apply on the elliptic rigid inclusion with rigid-body displacement and without rigid-body rotation [Fig. 2(c)]. Denoting $\phi_C(\zeta)$ and $\psi_C(\zeta)$ the required complex stress functions and employing $\phi_{C1}(\zeta)$ and $\psi_{C1}(\zeta)$, the stress functions for the concentrated loads applying in an infinite plane without a hole, we obtain

$$\phi_C(\zeta) = \phi_{C1}(\zeta) + \phi_{C2}(\zeta), \quad \psi_C(\zeta) = \psi_{C1}(\zeta) + \psi_{C2}(\zeta) \quad (31)$$

$$\phi_{C1}(\zeta) = D\log\zeta, \quad \Psi_{C1}(\zeta) = -\kappa\bar{D}\log\zeta, \quad D = -\frac{X_C + iY_C}{2\pi(1 + \kappa)} \quad (32)$$

Using (14) without the fourth term in its left-hand side, the displacement boundary condition can be expressed as

$$\kappa\phi_C(\sigma) - \frac{\omega(\sigma)}{\overline{\omega'(\sigma)}}\overline{\phi_C'(\sigma)} - \overline{\psi_C(\sigma)} = 2G(u + iv) \quad (33)$$

Substituting (31) and (32) into (33), taking $u = v = 0$ on the boundary without loss of generality, multiplying both sides of the resulted equation by the factor $d\sigma/[2\pi i(\sigma - \zeta)]$, and implementing an integration of the Cauchy type along the unit circle, $\phi_{C2}(\zeta)$ is obtained. Finally, $\phi_C(\zeta)$ can be determined from (31) to yield

$$\phi_C(\zeta) = D\log\zeta + \text{const} \quad (34)$$

Since there is a portion of the boundary on which the displacements vanish ($u = v = 0$), another stress function $\psi_C(\zeta)$ can be determined by analytic continuation on the displacement boundary. The result is

$$\psi_C(\zeta) = \kappa\overline{\phi_C}\left(\frac{1}{\zeta}\right) - \frac{\bar{\omega}(1/\zeta)}{\omega'(\zeta)}\phi_C'(\zeta) \quad (35)$$

Taking $\zeta = \sigma$ on the boundary and substituting (34) and (35) into (25), the resultant moment M_C about the origin by the stresses on the boundary is obtained. In the present case of a rigid inclusion of elliptic shape, $M_C = 0$ due to the symmetry in geometry. If the geometry is unsymmetric, in general, M_C is not zero.

The rigid-body displacements u_C and v_C, the relative displacements between the heat source location and the rigid inclusion, can be obtained using (33), that is, after putting $\sigma = \zeta$ in (33), evaluating the values of displacement at point $\zeta = \zeta_a$ and, for instance, at point $\zeta = 1$ (a point on the boundary), then working out the difference between the two results. The magnitude of the concentrated loads applied on the inclusion, X_C and Y_C, can be determined by the conditions $u_A + u_C = 0$ and $v_A + v_C = 0$. X_C and Y_C signify the reaction forces applied on the rigid body. The stress functions then become $\phi_A(\zeta) + \phi_C(\zeta)$ and $\psi_A(\zeta) + \psi_C(\zeta)$.

Complex Stress Functions of External Boundary Value Problem

A heat source of intensity M is at point z_a and a sink is lying at infinity (Fig. 1). The complex stress functions for this case are derived in the following.

The complex stress functions to be determined for the problem are expressed as

$$\phi(\zeta) = \phi_{A1}(\zeta) + \phi_2(\zeta), \quad \psi(\zeta) = \psi_{A1}(\zeta) + \psi_2(\zeta) \quad (36)$$

where $\phi_{A1}(\zeta)$ and $\psi_{A1}(\zeta)$ are expressed by (18), and A and B in (18) are the same as (19) and (20), respectively.

Substituting (36) into the external traction boundary condition (13), noting that $p_x = p_y = 0$ and $\bar{\sigma} = 1/\sigma$ on the boundary, multiplying both sides of the resulting equation with $d\sigma/[2\pi i(\sigma - \zeta)]$, and carrying out the integral of Cauchy on the contour of the unit circle yield $\phi_2(\zeta)$. Finally, stress function $\phi(\zeta)$ turns out to be

$$\begin{aligned}\phi(\zeta) = {} & A\log\zeta + \frac{\alpha MGR}{4\pi k}[\{\omega(\zeta) - \omega(\zeta_a)\}\log(\zeta - \zeta_a) \\ & - \{\omega(\zeta) - \omega(\zeta_a)\}] - \frac{\alpha MGR}{4\pi k}\left[\{\omega(\zeta) - \omega(\zeta_a)\}\log\left(\frac{1}{\zeta} - \overline{\zeta_a}\right)\right. \\ & \left. - E_0\zeta\log(-\overline{\zeta_a}) + \frac{E_1}{\zeta}\log(-\zeta_a) - \frac{E_1}{\zeta}\right]\end{aligned} \quad (37)$$

where the constant term has been omitted.

Moreover, since the boundary is traction free, by analytic continuation across the boundary, the following expression can be obtained (Muskhelishivili 1963):

$$\psi(\zeta) = -\overline{\phi(1/\zeta)} - \frac{\overline{\omega(1/\zeta)}}{\omega'(\zeta)}\phi'(\zeta) \quad (38)$$

Replacing ζ_a and M in (37) with ζ_b and $-M$, respectively, the solution of $\phi(\zeta)$ for the heat sink is obtained. Adding up the solutions for the heat source and the heat sink, the solution is obtained for the case when both the source and the sink exist in the infinite plane with the elliptic hole.

Furthermore, by replacing ζ_a in (37) with σ_a, $\zeta_a' = \sigma_a$ and using the relation $\overline{\sigma_a} = 1/\sigma_a$, the solution $\phi(\zeta)$ for the case when the heat source is located on the boundary can be found as

$$\phi(\zeta) = A \log \zeta + \frac{\alpha MGR}{4\pi k} [\{\omega(\zeta) - \omega(\sigma_a)\}\log \zeta - E_0\zeta] \quad (39)$$

In (39), it can be noted that the point $\zeta = \sigma_a$, at which the heat source appears, is not a singular point for stress and displacement, because there is no displacement constraint around this point. Meanwhile, since the boundary is traction free, the other stress function, $\psi(\zeta)$, can be expressed by (38).

ANALYTIC EXAMPLES

Stress Distributions

Analytic examples for $\kappa = 2$ (corresponding to $\nu = 0.25$ for plane strain and 1/3 for generalized plane stress) and a heat source at point $z_a(-2a, 0)$ are treated. Dimensionless stress distributions for the isothermal boundary ($\Theta = 0$) and the thermally adiabatic boundary ($q_a = 0$; no heat flows across the boundary) are shown in Figs. 3, 4, and 5, respectively. In Figs. 4 and 5 of the thermally adiabatic case, the referenced base temperatures are taken from points *A* and *C*, respectively, for the stress distributions. The stress distributions shown in the figures are those on the elliptic boundary and on the *x* axis. On the elliptic boundary, the normal, tangential, and shearing stresses are denoted by σ_r, σ_θ, and $\tau_{r\theta}$, respectively. On the *x* axis, the axial stresses in the *x* and *y* directions are denoted by σ_x and σ_y, respectively, and the shearing stress is denoted by τ_{xy}. Numerical values of the axial stresses are taken to be positive for tensile and negative for compressive stress. Shearing stress $\tau_{r\theta}$ is regarded as positive in the counterclockwise direction.

For both the isothermal and the thermally adiabatic cases, compressive stresses of extremely large magnitudes set up around the heat source due to the thermal expansion. On the *x* axis, because of the symmetry in the problem shearing stress, τ_{xy} is zero. Since there is no effect of thermal expansion on the isothermal boundary $\Theta = 0$, the relation between the normal and tangential stresses $\sigma_\theta/\sigma_r = (3 - \kappa)/(1 + \kappa)$ holds on the boundary (Hasebe 1979). Furthermore, for the thermally adiabatic case, the stress distributions shown in Figs. 4 and 5 will be different if different referenced base points for the temperature are selected. This is because different base points bring about different values of constant *C* in (12). This con-

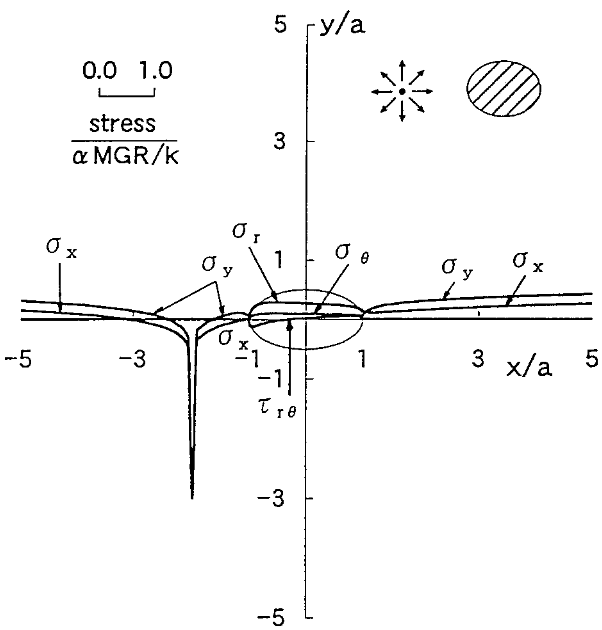

FIG. 3. Stress Distributions on *x* Axis and Elliptic Inclusion [z_a(−2*a*, 0), Isothermal Condition, Zero Temperature ($\Theta = 0$) on Boundary]

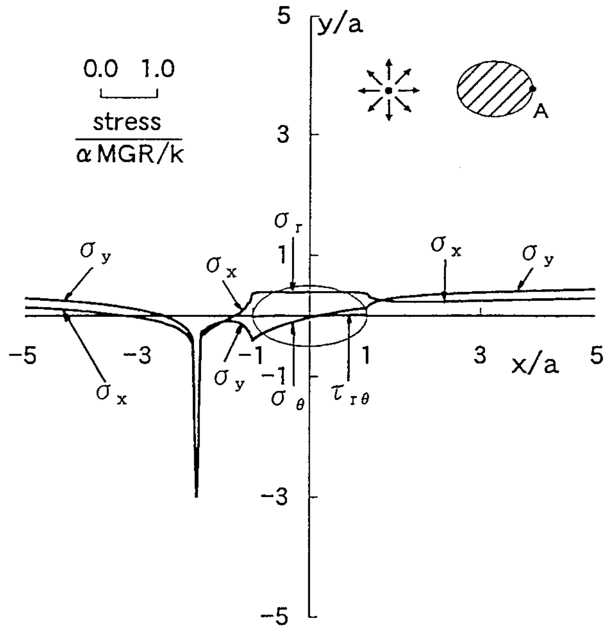

FIG. 4. Stress Distributions on *x* Axis and Elliptic Inclusion [z_a(−2*a*, 0), Adiabatic Condition, Basic Point of Temperature A]

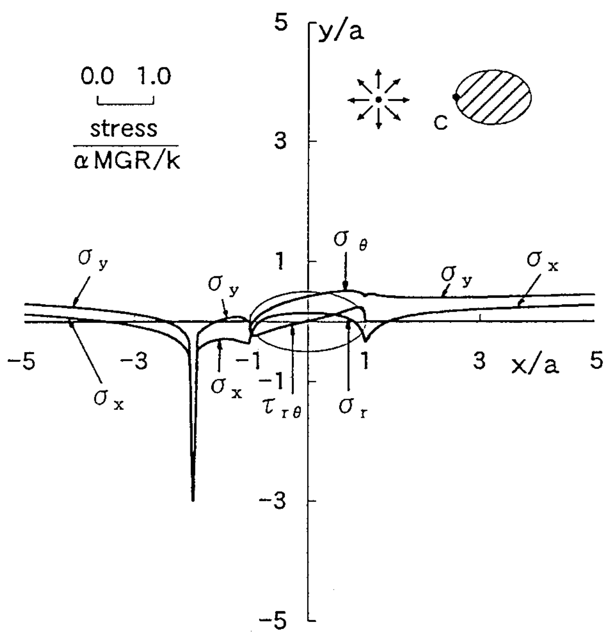

FIG. 5. Stress Distributions on *x* Axis and Elliptic Inclusion [z_a(−2*a*, 0), Adiabatic Condition, Basic Point of Temperature C]

stant is contained in the stress functions [see (22*b*)]; therefore, different stress functions are resulted.

Taking $\Gamma = 1$ (the thermally adiabatic condition) in the analytic examples, dimensionless stress distribution for the elliptic hole are shown in Fig. 6.

Stress Intensity Factors

The stress intensity factors at crack tips are calculated by the following formula, given by the complex stress function $\phi(\zeta)$ and the mapping function $\omega(\zeta)(b/a = 0)$ (Hasebe et al. 1988):

$$K_I - iK_{II} = 2\sqrt{\pi}e^{-i(\lambda/2)} \frac{\phi'(\zeta_0)}{\sqrt{\omega''(\zeta_0)}} \quad (40)$$

where λ = angle between the crack and the *x* axis; and ζ_0 denotes a point on the unit circle mapped from the crack tip.

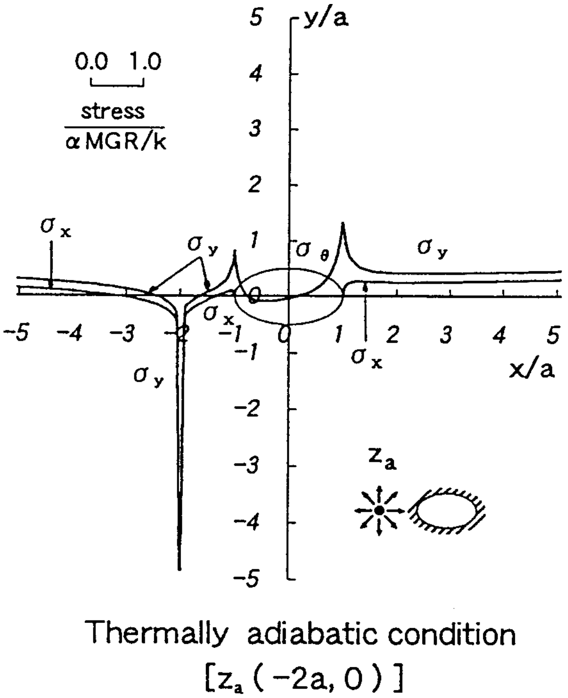

FIG. 6. Stress Distributions on *x* Axis and Elliptic Hole

In the present case, for crack tip z_A, we have $\zeta_0 = 1$ and $\lambda = 0$; whereas $\zeta_0 = -1$ and $\lambda = \pi$ hold for the other tip, z_B. The dimensionless stress intensity factors are given by the following expression:

$$F_I + iF_{II} = \frac{k}{\alpha MGR} \frac{K_I + iK_{II}}{\sqrt{\pi a}} \quad (41)$$

In the analytic examples, when both the heat source and the sink are located on the *x* axis, the stress intensity factors are shown in Fig. 7, while in Figs. 8 and 9, the results for a heat source on the *x* or *y* axes and a heat sink at infinity are demonstrated.

Heat Source and Sink Located on *x* Axis

In Fig. 7, under the condition that the locations of the heat source and the sink are fixed at the origin and the point (5*a*, 0) on the *x* axis, respectively, with the crack translating on the *x* axis, the values of the dimensionless stress intensity factors at crack tip z_A are examined. The abscissa of the midpoint of the crack, *L*, is used as a parameter to indicate the position of the crack, and it is made dimensionless by division by *a*, the half length of the crack. The resulted parameter, L/a, varies within the range $-20 \leqq L/a \leqq 20$ in Fig. 7. Moreover, it is assumed that once the position of the heat source (sink) falls on the crack, the source (sink) is located on the upper surface of the crack, where $y = 0^+$. For $L/a = -1.0$ or 4.0, the crack tip z_A coincides with the heat source at point z_a or the heat sink at point z_b; in these cases, due to the direct effect of the source or the sink, the stress intensity factor F_I attains a minimum (compressive mode) or a maximum (tensile mode), at z_a or z_b, respectively. For $0 \leqq L/a \leqq 5$, that is, when the crack lying between the heat source and the sink, F_I, changes from negative (compressive mode) to positive (tensile mode), the crack translates from the compressive zone effected by the heat source at z_a towards the tensile zone brought about by the heat sink at z_b. If the crack moves away from z_a and z_b, the effect of the heat source and the heat sink weakens, and, as a result, F_I converges towards zero.

The other stress intensity factor, F_{II}, becomes different from zero when the heat source (sink) lies on the upper surface of the crack and attains a maximum (minimum) once the midpoint of the crack and the position of heat source (sink) coincide. Due to the symmetry about the *x* axis, F_{II} vanishes when the heat source (sink) is outside the crack.

Heat Source on *x* or *y* Axis and Heat Sink at Infinity

Assuming that the heat source z_a is on the *x* ($x_a/a \geqq 0$, $y_a/a = 0$) or *y* axis ($x_a/a = 0$, $y_a/a \geqq 0$), values of the dimensionless stress intensity factor for tips z_A and z_C of the crack are shown in Figs. 8 and 9, respectively. In Figs. 8(a) and 9(a), the value x_a/a is taken on the horizontal axis, whereas in Figs. 8(b) and 9(b), y_a/a is taken. In these figures, the case that the heat source lies on the upper surface of the crack where $y_a/a = 0^+$ is also considered and demonstrated.

Stress Intensity Factors at Point z_a (Fig. 8)

When the heat source lies on the *x* axis, F_I becomes zero at a point near $x_a/a = 0.5$ and attains a minimum (compressive mode) under the action of the heat source at $x_a/a = 1.0$, the crack tip. After that, with the increase of x_a/a, the heat source moves forward and F_I increases monotonically. This phenom-

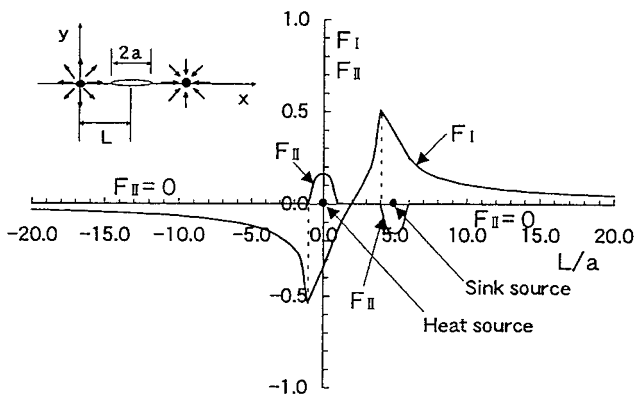

FIG. 7. Nondimensional Stress Intensity Factors at Point z_A

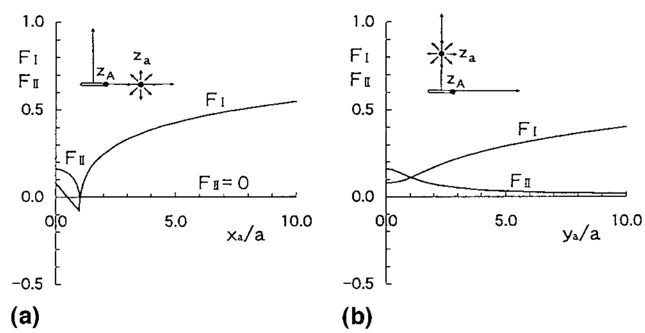

FIG. 8. Nondimensional Stress Intensity Factors at Point z_A: (a) Heat Source on *x* Axis; (b) Heat Source on *y* Axis

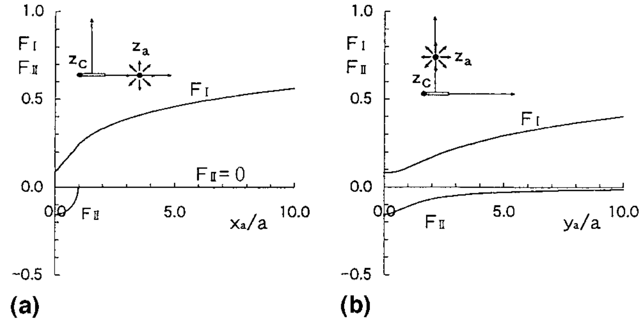

FIG. 9. Nondimensional Stress Intensity Factors at Point z_c: (a) Heat Source on *x* Axis; (b) Heat Source on *y* Axis

enon can be explained as follows: The relative temperature at the crack tip for the base temperature becomes lower, when the heat source is moving away from the tip. This temperature induces an excessive amount of shrinkage deformation, which increases the stress intensity factor of tensile mode. The other stress intensity factor, F_{II}, vanishes for $x_a/a \geqq 1.0$, due to the symmetry about the x axis.

When the heat source lies on the y axis, based on the reason given above, F_I increases monotonically as y_a/a increases. However, since for a remote heat source, the unsymmetry of the temperature field about the x axis weakens, F_{II} converges to zero for increasing y_a/a.

Stress Intensity Factors at Point z_C *(Fig. 9)*

Based on the reason given above, F_I increases monotonically as x_a/a or y_a/a increases. If the heat source is on the x axis and $x_a/a \geqq 1.0$, F_{II} vanishes due to symmetry, and when the heat source lies on the y axis, F_{II} is negative; otherwise, it behaves like F_{II} as demonstrated in Fig. 8(b).

CONCLUSIONS

Complex stress functions for an infinite plane containing an elliptic rigid inclusion, hole, and crack are developed for a point heat source. For the inclusion problem, the relative rigid displacement (Problem C) and the rotation (Problem B) were considered. The general solution of the inclusion problem can be obtained by superposition. The resultant moments about the origin formed by the stresses on the boundary are also worked out. All of the general solutions are closed form without integral terms. Stress distributions for the isothermal and thermally adiabatic boundary conditions were worked out in the analytic examples and were demonstrated by the figures. The stress distributions are different, due to the location of the base temperature adopted in the inclusion problem.

In the case of the isothermal condition ($\Gamma = -1$) of the inclusion problem, the heat source on the boundary is not a singular point for stress and displacement. When the heat source lies on the boundary of the hole, it is also not a singular point for stress and displacement, because there is no displacement constraint around the point.

Furthermore, for an infinite plane with a crack ($b/a = 0$), values of the stress intensity factors have been shown graphically, and the effects of the position of the heat source on them have been examined and discussed. The values of the intensity factor Mode I become larger and larger when the heat source moves away from the crack tip, because the temperature at the crack tip becomes lower and lower. However the values of the stress intensity factor Mode II become smaller and smaller, because the field of temperature around the crack tip becomes uniform.

The solution of the heat source can be used as the fundamental solution of the boundary element method in the thermal problem.

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