Theoretical and Experimental Study of Electromagnetic Scattering by Two Identical Conducting Cylinders*

R. V. Row
Cruft Laboratory, Harvard University, Cambridge, Massachusetts
(Received December 7, 1953; revised July 6, 1954)

The scattering of a cylindrical TEM wave by two parallel, identical conducting circular cylinders is developed as a special case of a theoretical analysis which treats the scattering by an arbitrary array of cylinders. Only the case with the incident E-vector parallel to the axes of the cylinders is considered, and attention is focused on the mutual effects present when the cylinder diameter and spacing are comparable to a wavelength.

The approximations made in the theory are tested experimentally using 3 cm microwaves in a parallel plate region. Significant departures from the results of the independent scattering hypothesis as predicted by the theory have been confirmed experimentally.

1. INTRODUCTION

Within recent years a number of papers[1–8] have been published dealing with various theoretical and experimental aspects of the diffraction of electromagnetic waves by a planar grating of parallel conducting wires. In the main, the results of the theoretical analyses have been directed towards obtaining information about gratings of elements whose size and spacing are either small compared with a wavelength, or spacings large enough to allow mutual effects between elements to be neglected. Several workers[4,6,7–9] have formulated their particular problems for quite general element size and spacings but have later specialized the results to the aforementioned cases. This paper reports a preliminary investigation into the effect of mutual coupling in a planar grating of parallel identical conducting cylinders where the cylinder diameters and spacing are comparable to a wavelength. In view of the interest that this problem has generated recently, it is interesting at this point to take a brief look at the work that has been done on it in the past.

It is interesting to note that about 60 years ago the development of reliable spark oscillators which radiated barely useful amounts of energy in the decimeter wavelength range stimulated interest in the problem of diffraction of electromagnetic waves by the planar grating. J. J. Thompson[10] and Rayleigh[11] both worked on limited aspects of the problem and H. Lamb[12] applied the technique of conjugate functions to the successful calculation of the transmission coefficient of gratings of closely spaced small cylinders or thin strips. His results were confirmed experimentally by Schaefer and Laugwitz.[13] Ignatowsky[9] was the first to develop a general theory of scattering from an infinite planar grating of identical elements using a formal solution of Maxwell's field equations satisfying the appropriate boundary conditions. The periodic nature of the boundary conditions allowed the field to be represented in a series of propagating and evanescent plane waves. Ignatowsky's work received little attention until quite recently when the mode or waveguide type of expansion has been used in connection with a variational principle to calculate the mode coefficients. Here again results have been presented only for small cylinders.

Concurrently with Ignatowsky, Zaviska[14] developed an analysis of diffraction from an arbitrary array of parallel cylinders by expanding the scattered field in a series of Hankel functions representing cylindrical waves radiating from each cylinder. The results of this analysis were applied to scattering by two small cylinders. However, the method of analysis is interesting and is essentially the method employed in this paper. It has an advantage in the directness with which various approximations may be introduced at the end of the rigorous analysis.

Wessel's theory is restricted to small wires from the outset, and as such, requires no futher mention here except to say that his results have been recently confirmed experimentally by Esau, Ahrens, and Kebbel.[2]

Twersky's[7] novel grating analysis by the use of "multiply scattered" waves is essentially an iteration process based on systematically improving the results of the independent scattering hypothesis. The method has not been applied in the case of large cylinders so far as this writer is aware. The problem discussed here is

* Research supported by Office of Naval Research contract.

1 W. Wessel, Hochfrequenztechnik und Electroakustik **54**, No. 2, 62–69 (1939).
2 Esau, Ahrens, and Kebbel, Hochfrequenztechnik und Electroakustik **54**, No. 2, 113–115 (1939).
3 N. Marcuvitz, *Waveguide Handbook* (Massachusetts Institute of Technology Radiation Laboratory Series), Vol. 10, pp. 184.
4 J. W. Miles, Quart. Appl. Math. **7**, 45 (1949).
5 W. Franz, Z. angew. Phys. **1**, 9 416–423 (1949).
6 J. Schmoys, Report EM-18, Math. Research Group, Washington Square College of Arts and Sciences, New York University (1951).
7 V. Twersky. Report EM-34, Math. Research Group, Washington Square College of Arts and Sciences, New York University (1951); also, V. Twersky, J. Appl. Phys. **23**, 1099 (1952).
8 W. E. Groves, Antenna Laboratory Report, Issue 179, Series 7, University of California (1952).
9 W. von Ignatowsky, Ann. Physik **44**, 369 (1914).
10 J. J. Thompson, *Recent Researches in Electricity and Magnetism* (Oxford University Press, New York, 1893), p. 425.
11 Lord Rayleigh, Proc. Roy. Soc. (London), **79**, 399 (1907).
12 H. Lamb, Proc. Roy. Soc. (London) **27**, 523–544 (1898).
13 C. L. Schaefer and M. Laugwitz, Ann. Physik **21**, 587–594 (1906).
14 F. Zaviska, Ann. Physik **40**, 1023 (1913).

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

first set up for scattering of an incident cylindrical TEM wave (with the electric vector oriented parallel to the cylinder axes) by an arbitrary configuration of parallel cylinders, and later specialized to the case of a planar grating of identical cylinders. Owing to the complexity of the results the problem is reduced a step further by considering only two identical cylinders. The choice of a cylindrical incident wave, rather than the conventional plane wave excitation, is to facilitate comparison of the theoretical results with measurements made on the diffraction of 3.2 cm microwaves by two cylinders in a parallel plate region.

2. OUTLINE OF THE THEORY

In the following only scalar scattering by circular cylinders is considered since this results in a mathematically tractable problem; although from Lamb's[12] work on the grating of closely spaced small wires or strips it is expected that for small scattering elements the precise form of their boundary is secondary in determining their scattered field in directions away from the source of radiation. The theory assumes a current distribution on the surface of each *perfectly* conducting cylinder; the total field is then calculated through the use of one of Green's theorems. Application of the boundary conditions gives a series of integral equations for the current on each cylinder which takes into account arbitrary excitation and coupling between all the elements. The unknown surface current on each cylinder is then expanded in a complex Fourier series whose coefficients may be evaluated using the usual orthogonality property of the trigonometric functions. The resulting system of linear algebraic equations in the unknown coefficients may be written as an infinite matrix equation. The problem then remaining is to solve this system of linear algebraic equations. Various methods of numerical solution may be used, depending on the number of terms and accuracy required in the final result. For small cylinders the terms off the principal diagonal are small, and the meaning of the term "small" may be evaluated readily in estimating the importance of higher mode currents contributing to the scattered field. The results obtained at this point in the analysis are similar in form to those of Zaviska[14] who started by assuming a spectrum of scattered cylindrical waves and determining the spectral amplitudes from a consideration of the boundary conditions. There is also a formal analogy to the results of Twersky's[7] multiple order scattering analysis.

This theory is readily specialized to the case of a plane wave incident on an infinite planar grating of small wires. If the effects of higher order current modes are neglected, this result becomes identical with that of Wessel[1] who considered a uniform current distribution on the surface of the wires in his analysis.

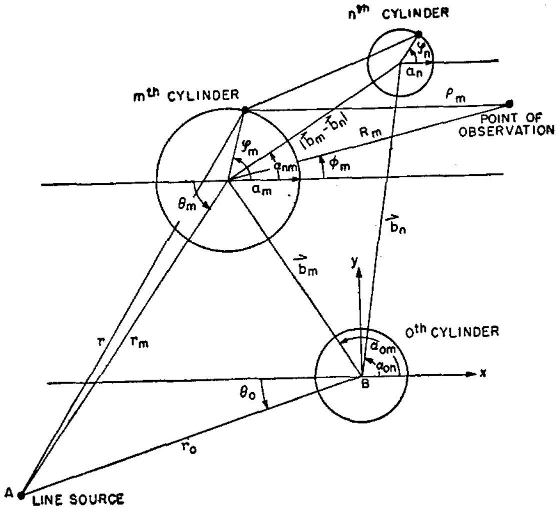

FIG. 1. Geometry for scattering from an arbitrary array of cylinders.

3. GENERAL THEORY (ARBITRARY CONFIGURATION OF PARALLEL CYLINDERS)

Figure 1 shows the general arrangement of line source and scattering cylinders. The axes of the cylinders and line source are all parallel to the z-axis so that all relevant electromagnetic field quantities may be derived from the single scalar quantity E_z, the electric field intensity in the z-direction, for convenience written as $\psi(x,y)$. With the customary time dependence $\exp(-i\omega t)$ suppressed throughout, and $k=2\pi/\lambda$, $\psi(x,y)$ satisfies

$$(\nabla_{x,y}^2+k^2)\psi(x,y)=0,$$

subject to the appropriate boundary conditions and the radiation condition at infinity.

If a Green's function $G(x,y;x',y')$ is defined as a solution of the inhomogeneous wave equation

$$(\nabla_{x,y}+k^2)G(x,y;x',y')=-\delta(x-x')\delta(y-y'),$$

and substituted into Green's scalar identity

$$\int(\psi\nabla^2G-G\nabla^2\psi)d\sigma'=\oint\left(\psi\frac{\partial G}{\partial n}-G\frac{\partial\psi}{\partial n}\right)dC',$$

the result

$$\psi(x,y)=\oint\left(G\frac{\partial\psi}{\partial n}-\psi\frac{\partial G}{\partial n}\right)dC',$$

is readily found, where the line integral is taken over a closed contour containing the source and all the cylinders. By imposing the boundary condition $\psi=0$ on the surface of each cylinder and making the convenient definition

$$\left.\frac{\partial\psi}{\partial n}\right|_{\text{on } n\text{th cylinder}}=\frac{1}{2\pi a_n}I_n(\phi_n),$$

where $I_n(\phi_n)$ may be considered as the surface current

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

on the nth cylinder, the previous result can be reduced to

$$\psi(\mathbf{r})=\psi^{\mathrm{inc}}(\mathbf{r})+\frac{1}{2\pi}\sum_n\int_0^{2\pi}I_n(\phi_n)G(\mathbf{r},\mathbf{r}')d\phi_n, \quad (1)$$

where $\psi^{\mathrm{inc}}(\mathbf{r})$ is the field that would exist at the point $\mathbf{r}$ if no scattering obstacles were present.

Application of the boundary condition $\psi(\mathbf{r})=0$ where r is on the surface of each cylinder and use of the appropriate two dimensional Green's function leads to the following integral equation for the surface current on each cylinder,

$$\psi^{\mathrm{inc}}(\mathbf{r})\Big|_{\mathbf{r}\text{ on cylinder}}=-\frac{i}{8\pi}\sum_n\int_0^{2\pi}I_n(\phi_n)H_0^{(1)}(K|\mathbf{r}-\mathbf{r}'|)d\phi_n\Big|_{\mathbf{r},\mathbf{r}'\text{ on cylinders}} \quad (2)$$

The problem now is to find a set of $I_n(\phi_n)$ which satisfy this set of simultaneous integral equations. One method of solution is to expand the unknown function in a complete set of *ortho*-normal functions appropriate to the geometry of the particular problem and then to determine the resulting unknown coefficients. Following this method a natural choice here is to expand the surface current on each cylinder in the complex Fourier series

$$I_n(\phi_n)=\sum_{s=-\infty}^{\infty}a_{ns}\exp(is\phi_n).$$

By assuming the $I_n(\phi_n)$ to have a sufficiently regular behavior and using the orthogonal properties of the set of functions $\exp(is\phi_n)$, it is possible to reduce the problem of finding the a_{ns} to the solution of an infinite set of linear inhomogeneous simultaneous algebraic equations in which the a_{ns} are the unknowns.

If the mth integral equation is multiplied by $\exp(it\phi_m)$ (where t is any integer including zero), and both sides are integrated with respect to ϕ_m from 0 to 2π it follows that:

$$\gamma_{tm}=-\sum_n\sum_s K_{tmns}a_{ns} \quad (3)$$

where the definitions

$$\gamma_{tm}=\int_0^{2\pi}\psi^{\mathrm{inc}}(\mathbf{r})\Big|_{\mathbf{r}\text{ on }m\text{th cylinder}}\exp(-it\phi_m)d\phi_m,$$

and

$$K_{tmns}=\frac{i}{8\pi}\int_0^{2\pi}\int_0^{2\pi}\exp(is\phi_n)H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|)\Big|_{\substack{\mathbf{r}\text{ on }m\text{th cylinder}\\ \mathbf{r}'\text{ on }n\text{th cylinder}}}\exp(-it\phi_m)d\phi_n d\phi_m \quad (4)$$

have been used.

The incident field characteristic of a uniform line source may be represented as

$$\psi^{\mathrm{inc}}(\mathbf{r})=AH_0^{(1)}(k|\mathbf{r}|),$$

where A is a complex constant. With this choice of excitation the expressions for γ_{tm} and K_{tmns} as evaluated in the Appendix are

$$\gamma_{tm}=2\pi AJ_t(ka_m)H_t^{(1)}(k|\mathbf{r}_m|)\exp[-it(\theta_m+\pi)]$$

$$K_{tmns}=\frac{\pi i}{2}J_t(ka_m)\begin{cases}H_s^{(1)}(ka_m)\delta_{st}\Big|_{\mathbf{r},\mathbf{r}'\text{ on }m\text{th cylinder}}\\ J_s(ka_n)H_{t-s}^{(1)}(k|\mathbf{b}_m-\mathbf{b}_n|)\exp(it\alpha_{nm}+is\alpha_{nm})\Big|_{\mathbf{r},\mathbf{r}'\text{ on different cylinders}}\end{cases} \quad (5)$$

For any given values of the parameters k, a_n, θ_m and $|\mathbf{b}_m-\mathbf{b}_n|$; γ_{tm} and K_{tmns} may be evaluated using existing tables of the Bessel and Neumann functions.

Finally, the total field may be calculated from formula (1),

$$\psi^{\mathrm{tot}}(x,y)=\psi^{\mathrm{inc}}(x,y)+\frac{i}{4}\sum\sum a_{ns}J_s(ka_n)H_s^{(1)}(kR_n)\exp(is\phi_n). \quad (6)$$

Here the reader familiar with Twersky's[7] multiple scattering analysis will notice a formal analogy between Eq. (6) and his formula for the scattered field.

At this point it is convenient to show the connection between the analysis of this paper and the work of Wessel[1] on diffraction of a plane incident wave by a planar grating of small wires.

If the constant A is so chosen as to make the exciting field a plane wave incident from the direction θ_0 and the array is taken as a planar grating of identical cylinders then $b_m=mb$,

$$\alpha_{0m}=\begin{cases}\alpha_0 & \text{for } m>0\\ \alpha_0\pm\pi & \text{for } m<0\end{cases}$$

and

$$a_m=a,$$

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

and from symmetry it is apparent that all the cylinders have the same current distribution except for the phase factor $\exp(ikmb\cos(\theta_0-\alpha_0))$ relative to the zeroth cylinder. Thus,

$$a_{ms}=\exp(ikmb\cos(\theta_0-\alpha_0))a'_{0s}$$

where a'_{0s} is the sth Fourier coefficient on the zeroth cylinder in the array. It is only necessary to calculate the current on the zeroth cylinder, thus for $m=0$ the system of Eq. (1) becomes

$$4i\exp\left(-it\left(\theta_0-\frac{\pi}{2}\right)\right)=H_t^{(1)}(ka)a_{0t}'+\exp(-it\alpha_0)\sum_{n\neq 0}\sum_{s}a_{0s}'\exp(iknb\cos(\theta_0-\alpha_0)+is\alpha_0)J_s(ka)H_{t-s}^{(1)}(k|n|b). \quad (7)$$

As t ranges through all positive and negative integers, an infinite number of linear simultaneous equations in the unknown a'_{0s} is generated. In principle this set of equations could be solved, but practically it is not feasible to solve for all the unknowns. For small ka and $b\gg a$, it may be seen from the behavior of the Bessel and Hankel functions involved that the dominant terms in the right-hand side of the previous expression are those with $t=s=0$. Thus Eq. (7) becomes

$$4i=\{H_0^{(1)}(ka)+\sum_{\substack{n=-\infty\\ \neq 0}}^{\infty}J_0(ka)H_0^{(1)}(k|n|b)\times\exp[inkb\cos(\theta_0-\alpha_0)]\}a_{00}'.$$

This is the same equation as obtained by Wessel for the current on the zeroth cylinder. Tables of the series $\sum_1^\infty J_0(nkb)$ and $\sum_1^\infty Y_0(nkb)$ corresponding to normal incidence have been computed by Ignatowsky[9] and Wessel.[1]

As far as coupling effects and their dependence on cylinder radius and spacing are concerned, a finite number of mode coefficients could be computed with a large amount of labor for the case of an infinite planar grating using Eq. (7). However, when it comes to comparing the theoretical results for the scattered field to experimental results, it is not feasible to use plane-wave excitation, and the system of equations for the mode coefficients for a line source excitation and the infinite grating are exceedingly complex. Hence, it is expedient to consider the simplest configuration for which mutual coupling effects may be calculated with a reasonable amount of labor. For these reasons the problem of scattering of a cylindrical wave by two identical cylinders has been chosen as an example to test the general theory.

4. TWO IDENTICAL CYLINDERS (NORMAL INCIDENCE)

Figure 2 shows the geometrical arrangement of source and cylinders for scattering of a cylindrical incident wave by two identical cylinders.

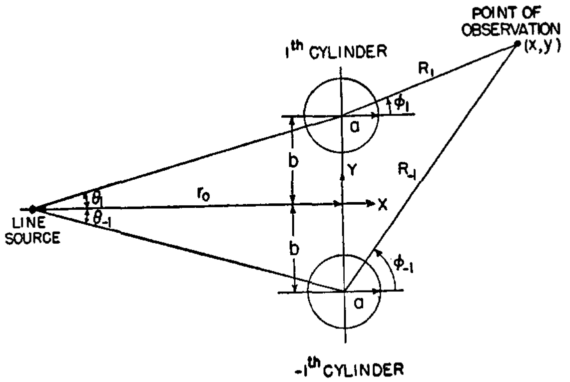

FIG. 2. Geometry for scattering from two identical cylinders (normal incidence).

The system of Eq. (3) for the unknowns $a_{\pm 1,s}$ (for $m=-1$) may be written

$$H^{(1)}(ka)a_{-1,t}+\exp\left(-\frac{it\pi}{2}\right)\{J_0(ka)H_t^{(1)}(2kb)a_{1,0}+\sum_{s=1}^{\infty}\exp(is\pi/2)J_s(ka)[H_{t+s}(2kb)a_{1,-s}+H_{t-s}^{(1)}(2kb)a_{1,s}]\}=4iH_t^{(1)}[k(r_0^2+b^2)^{\frac{1}{2}}]\times\exp[-it(\pi+\theta_{-1})]. \quad (8)$$

It is evident that

$$I_1(\phi_1)=I_{-1}(-\phi_1),$$

where $I_1(\phi_1)$ and $I_{-1}(-\phi_1)$ are the surface currents at mirror image points on the upper and lower cylinders. If the Fourier representation for the surface currents is combined with the above symmetry requirement, the following identity is readily established,

$$a_{1,s}=a_{-1,-s},$$

for all s.

The system of equations relating the unknowns obtained from Eq. (8) by using this statement of symmetry and allowing t to range through all positive and negative integers may be conveniently summarized in matrix form as,

$$K\cdot a=4i\lambda$$

where K is a square (infinite) matrix, whose elements are the coefficients of the $a_{\pm 1,s}$, and $4i\lambda$ is the infinite matrix of elements on the right-hand side of Eq. (8).

The rigorous solution of the problem requires the solution of an infinite matrix equation. Except for certain special cases (in this problem corresponding to no coupling, i.e., $kb\to\infty$), the solution to such an infinite matrix equation is not immediately obvious.

The best that can be done by way of a solution is to solve by numerical means, a finite number of the equations represented by Eq. (8). Putting aside any discussion of the rigorous justification of this procedure,

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

and granting that the approximation is good, how many, and which equations should be chosen for solution? Since the problem of scattering by two cylinders must reduce to that of scattering by a single cylinder in the limit of very large spacing between cylinders, the first question is most readily answered by using the simple well-known solution to the problem of the scattering of a plane wave by an isolated cylinder. The symmetry of the diagonal elements of the K matrix about the element corresponding to $t=s=0$ points to an obvious choice of equations as those with $t=0, \pm1, \pm2, \cdots, \pm n$, and unknowns with s ranging from $-n$ to $+n$. The scattered field (for the same orientation of the incident plane wave as in the two cylinder problem) from an isolated cylinder may be put in the form

$$\psi^{\text{scatt}}(r,\phi) = -\sum_{s=0}^{\infty} \epsilon_s i^{s+1} \sin\delta_s(ka) \times \exp[-i\delta_s)(ka)]H_s^{(1)}(kr)\cos s\phi$$

where

$$H_s^{(1)}(ka) = -iC_s(ka)\exp[i\delta_s(ka)]$$

and

$$\epsilon_s = \begin{cases} 1, & s=0 \\ 2, & s\neq 0 \end{cases}.$$

In the far zone the amplitude of $H_s^{(1)}(kr)$ changes slowly with increasing s, and thus the change of amplitude of each term corresponding to a change in index s is essentially proportional to $|\sin\delta_s(ka)\cos s\phi|$. The term $\cos s\phi$ is one at most and from the tables for $\delta_s(ka)$ it is seen that $\sin\delta_s(ka)$ tends to zero with increasing s (for s greater than a certain integer). Thus, by reference to these tables, it is possible to pick out the greatest integer s for which any significant contribution will be made to the summation for the scattered field. This maximum integer may then be used as a guide in deciding upon the number of equations to be solved in the two cylinder problem. Thus, for $ka=3.0(2a/\lambda\sim1)$,

s	0	1	2	3	4	5	6	7
$\lvert\sin\delta_s(Ka)\rvert$	0.5680	0.7222	0.9496	0.4977	0.1426	0.02251	0.002094	0.000175

and $s=6$ is the largest index giving an appreciable term in the field summation. Hence in the two cylinder problem at least 6 modes must be solved for, corresponding to $t=0, \pm1, \pm2, \pm3, \cdots \pm6$. There is an enormous amount of labor involved in solving such a system of equations with complex coefficients, however, for a spacing of one wavelength between centers, such a system of equations has been solved exactly for the case $Ka=1.253$, and approximately for $Ka=2.0$ and 2.5, for t ranging between values determined by the above procedure.

Methods of solving such systems of equations on a desk calculator are well known,† but except for $ka<1.3$ the labor required to solve such a system by hand methods limits the usefulness of the theory. However, the solution of a "block" out of the matrix equation yields results in excellent agreement with the measurements to be discussed later.

In view of the computational difficulties encountered in solving a large number of linear algebraic equations with complex coefficients, it is desirable to have a simple approximation to the solutions for the unknowns a_1'$_s$ so that some of the major characteristics of the scattering by two cylinders may be more readily seen. The most obvious approximation, and the one to be discussed here, is suggested by the fact that the diagonal elements in the K matrix increase without bound with increasing t, and at the same time the off-diagonal terms tend to zero. Hence it seems reasonable as a first approximation to neglect all off-diagonal terms. The success of this approximation will be discussed later in connection with the experimental measurements.

The final expression for the scattered field using the diagonal approximation is readily found to be

$$\psi^{\text{scatt}}(R,\phi) = -\sum_{n=\pm1}\sum_{s=0}^{\infty}(-)^s \frac{J_s(ka)H_s^{(1)}[k(r_0^2+b^2)^{\frac{1}{2}}]H_s^{(1)}(kR_n)\exp[is(\phi_n-\theta_1)]}{H_s^{(1)}(ka)+J_s(ka)H_{2s}^{(1)}(2kb)}. \quad (9)$$

In the limit as kb and $kr_0\to\infty$ this expression (except for a constant) is identical with Seitz's[15] expression for the field scattered by an isolated conducting cylinder

The contribution of each mode in Eq. (9) depends principally on the quantity

$$\alpha_s(ka,kb) = \frac{J_s(ka)}{H_s^{(1)}(ka)+J_s(ka)H_{2s}^{(1)}(2kb)}. \quad (10)$$

As s increases, and for $ka, kb\gg1$ and $ka, kb<s$ the approximations for the Bessel and Hankel[16] functions yield for Eq. (11), the following asymptotic form,

$$\alpha_s(ka,kb) \simeq \frac{i}{\left(\frac{2t}{ka}\right)^{2t} + \left(\frac{1}{2\pi s}\right)^{\frac{1}{2}}\left(\frac{2s}{kb}\right)^{2s}},$$

† For the *exact* solution of 22 (real) linear equations Crout's method [P. D. Crout, Am. Inst. Elec. Engrs. **60**, (1941)] requires 5634 machine operations including a check column. For 1 percent accuracy ($ka=1.253$, $b/\lambda=1.0$) the Gauss-Seidel iteration method [See Whittaker and Robinson, *Calculus of Observations* (Blackie and Son Limited, London, England), fourth edition], requires 4 iterations totaling 2134 operations.

[15] W. Seitz, Ann. Physik **16**, 746 (1905), and Ann. Physik **19**, 544 (1908).

[16] G. N. Watson, *Bessel Functions* (Macmillan Company, New York, 1948), second edition, pp. 243, 198.

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

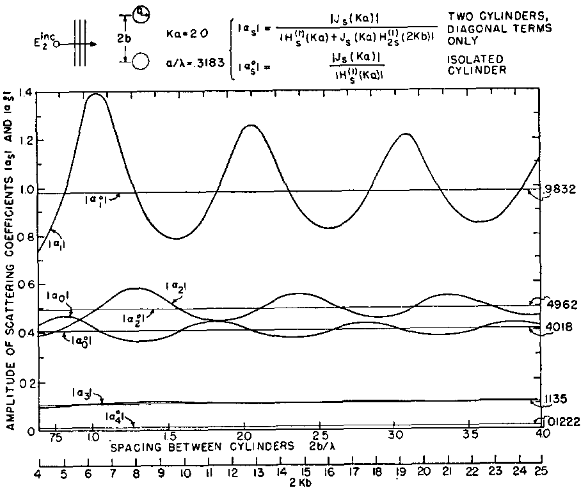

FIG. 3. Scattering coefficient $|\alpha_s|$ and $|\alpha_s^0|$ for two coupled cylinders (diagonal terms only), and an isolated cylinder as a function of spacing between centers for $Ka=2.0$ and normal incidence.

where the right-hand term in the denominator contains the coupling effect. From this last expression it is apparent that since $b \geqslant a$ that the two cylinders influence each other to a decreasing extent as the mode index s increases. This behavior is most strikingly demonstrated by the graphical plot in Fig. 3 of $|\alpha_s|$ computed from Eq. (10) as a function of spacing for $ka=2.0$. In addition $|\alpha_s^0|$, the corresponding quantity for an isolated cylinder is plotted for comparison along with $|\alpha_s|$. From this curve it is apparent that mutual effects tend to diminish slowly with increasing separation.

To compare the predictions of the diagonal approximation to measurable field quantities Eq. (9) has been used to calculate the scattered field. For a large separation between source and cylinders the following representation is used for the Hankel function $H_s^{(1)}(k(r_0^2+b^2)^{\frac{1}{2}})$

$$H_s^{(1)}(ku) \sim \left(\frac{2}{\pi ku}\right)^{\frac{1}{2}} \exp\left[i\left(ku-\frac{\pi s}{2}-\frac{\pi}{4}\right)\right] \times\left[S_s^{(1)}(ku)+0\left(\frac{1}{ku}\right)^p\right]$$

where $u=(r_0^2+b^2)^{\frac{1}{2}}$ and

$$S_s^{(1)}(ku)=\sum_{m=0}^{p-1}\frac{(-1)^m\Gamma(s+m+\frac{1}{2})}{(2iku)^m m!\Gamma(s-m+\frac{1}{2})}.$$

Since in practice $kr \gg s$ and $k=2\pi/\lambda$, the incident field at $x=x$ and $y=0$ is

$$\psi^{\text{inc}}(x,0) \simeq \exp(-i\pi/4)/\pi((r_0+x)/\lambda)^{\frac{1}{2}}\cdot\exp[ik(r_0+x)].$$

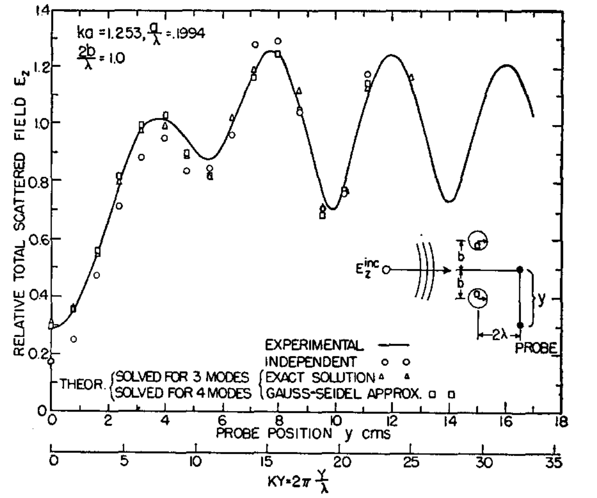

FIG. 5. Experimentally measured diffracted field from two cylinders and comparison with various theoretical approximations for $Ka=1.253$, as a function of probe position. (One wavelength between centers.)

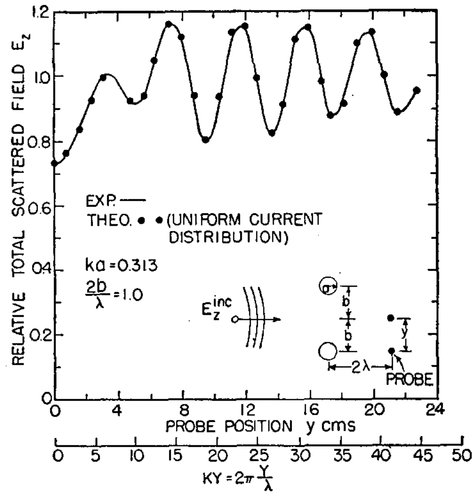

FIG. 4. Experimental and theoretical results of diffraction by two cylinders as a function of probe position for $ka=0.313$ and spaced 1.0 wavelength between centers.

Using the definition

$$\psi^{\text{tot}}=\psi^{\text{inc}}+\psi^{\text{scatt}},$$

and dividing this expression by the incident field at the reference point $x=x$, $y=0$ gives for the normalized total field.

$$\psi^{\text{tot}}{}_{\text{norm}}(x,y)=E_z=1-[(r_0+x)/(r_0^2+b^2)^{\frac{1}{2}}]^{\frac{1}{2}}$$
$$\times\exp[ik(r_0^2+b^2)^{\frac{1}{2}}-ik(r_0+x)]\times\sum_{n=\pm1}\sum_{s=-\infty}^{\infty}$$
$$\times\exp(\pi is/2)\{\alpha_s(ka,kb)S_s^{(1)}[k(r_0^2+b^2)^{\frac{1}{2}}]H_s^{(1)}(kR_n)$$
$$\times\exp[is(\phi_n-\theta_1)]\}.$$

This formula has been used in calculating the theoretical results labeled, diagonal terms only in Figs. 7, 9, 10, and 11.

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

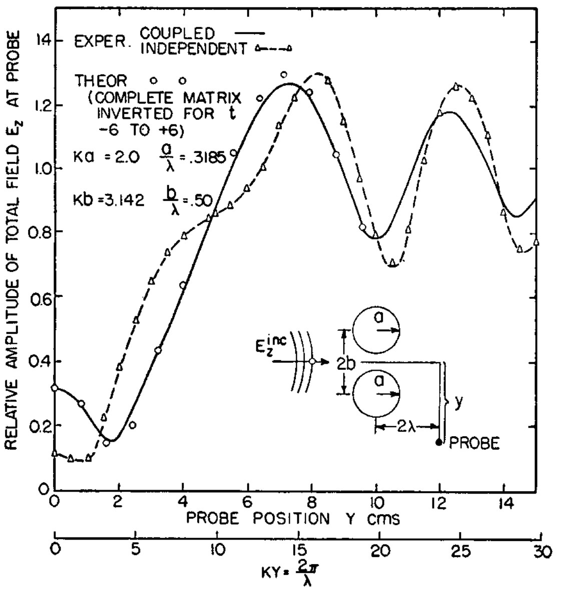

FIG. 6. Theoretical and experimental diffraction curves for scattering from two identical closely coupled cylinders, radius $a=0.3185\lambda$ and spacing between centers $=1.0\lambda$.

The factor $[(r+x)/(r_0^2+b^2)^{\frac{1}{2}}]^{\frac{1}{2}}$ represents an amplitude correction owing to the use of a line source in place of a plane wave, and the factor $ik(r_0^2+b^2)^{\frac{1}{2}}$ in the exponent plays the role of a phase correction factor. Inside the summation the term $S_s^{(1)}[k(r_0^2+b^2)^{\frac{1}{2}}]$ corrects for the line source excitation for each mode.

5. EXPERIMENTAL RESULTS

The parallel plate region and associated field-probing equipment described by the author in an earlier paper[17]

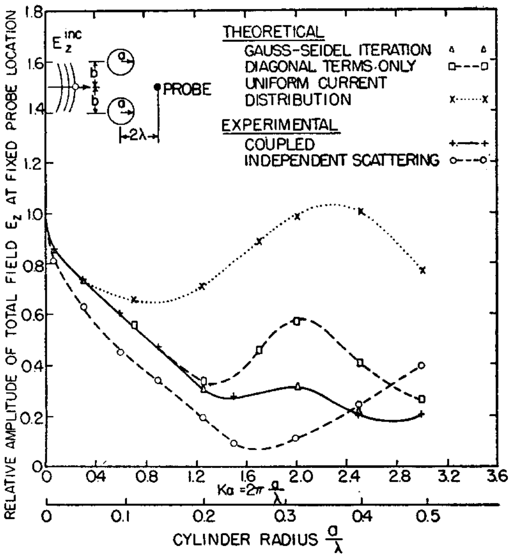

FIG. 7. Amplitude of field scattered by two identical cylinders, for fixed probe location and spacing $2b/\lambda=1.0$ as a function of radius of cylinders (line source).

[17] R. Row, J. Appl. Phys. 24, 12 (1953).

have been used to measure the electric field scattered by two identical highly conducting cylinders with a uniform line source for the primary excitation (at normal incidence) as sketched in Fig. 2.

These measurements are designed as a check on the validity of the approximations made in the theory developed in the preceding sections, and to compare the predictions of the independent scattering hypothesis with the field distributions actually measured. The results labeled "independent" scattering were determined in the following way. $E_z^{\text{inc}}(\mathbf{r})$ is the incident electric field at a point $\mathbf{r}$ and $E_{z_1}^{\text{tot}}(\mathbf{r})$ is the total electric field with cylinder 1 in place, then

$$E_{z_1}^{\text{scatt}}(\mathbf{r})=E_{z_1}^{\text{tot}}(\mathbf{r})-E_z^{\text{inc}}(\mathbf{r}).$$

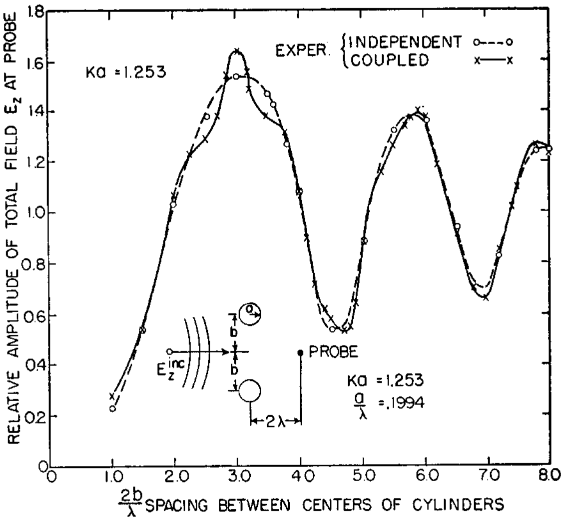

FIG. 8. Experimentally measured independent and coupled scattering from two identical cylinders with $Ka=1.253$ as a function of spacing. Probe position fixed.

Similarly, if cylinder 1 is removed and cylinder -1 is put in place:

$$E_{z_{-1}}^{\text{scatt}}(\mathbf{r})=E_{-1}^{\text{tot}}(\mathbf{r})-E_z^{\text{inc}}(\mathbf{r}).$$

According to the independent scattering hypothesis the total field with both cylinders in place is given by

$$\begin{aligned}E_z^{\text{tot}}(\mathbf{r})&=E_z^{\text{inc}}(\mathbf{r})+E_{z1}^{\text{scatt}}(\mathbf{r})+E_{z_{-1}}^{\text{scatt}}(\mathbf{r})\\&=E_{z_{-1}}^{\text{tot}}(\mathbf{r})+E_{z_{-1}}^{\text{tot}}(\mathbf{r})-E_z^{\text{inc}}(\mathbf{r}).\end{aligned}$$

The actual results presented required a measurement of amplitude and phase of the total scattered fields and the incident field. Brass cylinders $\frac{1}{2}$ inch thick and machined to the required diameters were used as the scatterers in the parallel plate region. In all cases the experimental results are reproducible to within 2 percent.

Figures 4 through 6 show the total field as measured by a probe moving along a line parallel to, and two wavelengths from (on the side away from the source), the line joining the centers of the cylinders; for a fixed center-to-center spacing of one wavelength and equal to 0.05λ, 0.20λ, and 0.318λ. Figure 7 shows the total

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

field at the point $x=2\lambda$, $y=0$ for a fixed center spacing of 1.0 wavelength as an increase from 0.0 to 0.477 wavelength. Figures 8 through 11 show the field at the same point as a function of spacing for a equal to 0.20λ, 0.24λ, 0.318λ, and 0.477λ.

In all cases the corresponding theoretical quantities are also shown on the same graph for comparison.

6. CONCLUSIONS

A study of Figures 5 through 11 shows that over a fairly large range or radii and spacings the independent scattering hypothesis may be used to predict large scale trends in the results. Thus for the probe fixed, and a constant spacing of 1.0 wavelength between centers (see Fig. 7), the trend in the measured field as the radius increases is closely predicted by the independent scattering hypothesis for ka ranging from zero to approximately 1.5.

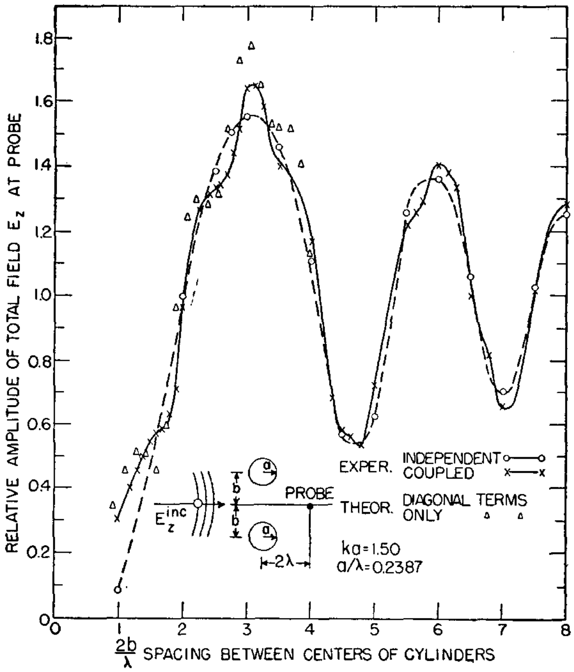

FIG. 9. Experimentally measured independent and coupled scattering from two identical cylinders with $ka=1.50$ as a function of spacing. Probe position fixed.

In all the measurements taken there is practically no detailed agreement between the independent scattering data and the results obtained with both cylinders present. As is to be expected the larger the radius, and the smaller the spacing, the poorer becomes the detailed agreement between the predictions of the simple independent scattering hypothesis and the corresponding measurements.

Attention will now be focused on the results of the theory developed in Sec. 4.

For small cylinders only the zeroth mode or uniform current mode is of significance in calculating the scat-

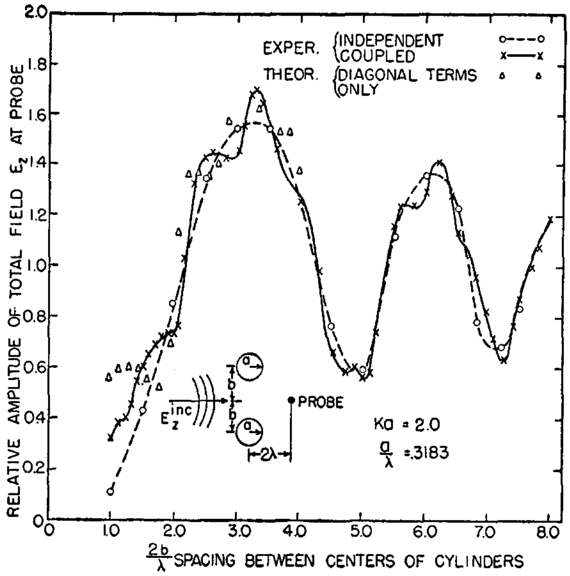

FIG. 10. Experimentally measured independent and coupled scattering from two identical cylinders with $Ka=2.0$, as a function of spacing. Probe position fixed.

tered field. The range of radii and spacing over which this mode alone is sufficient is determined primarily by the radius,‡ since here this parameter determines the number of modes required.

Thus, with a fixed probe and a constant spacing of 1.0 wavelength between centers, Fig. 8 shows that the uniform current distribution gives excellent agreement with experiment for ka less than 0.3. The next step is

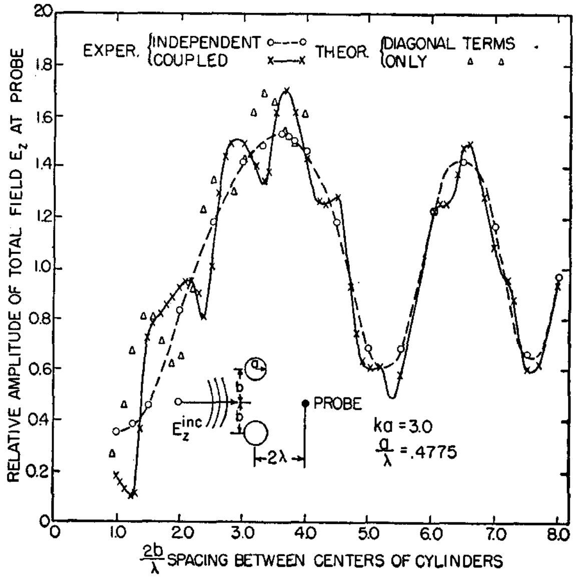

FIG. 11. Experimentally measured independent and coupled scattering from two identical cylinders with $Ka=3.0$ as a function of spacing. Probe position fixed.

‡ For cylinder diameters and spacings much less than a wavelength, the problem can be handled by the electrostatic approximation much as Lamb (see reference 12) treated the problem of a planar grating of small wires.

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

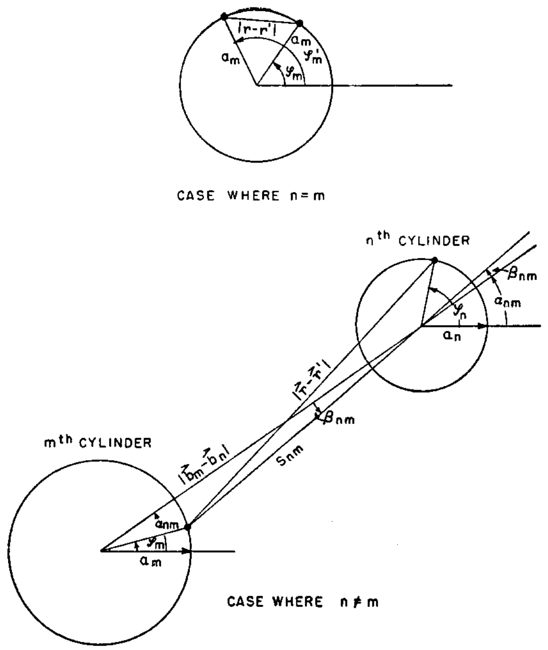

FIG. 12. Geometry for calculation of K_{tmns}.

to keep all the modes required for good convergence in the problem of diffraction by a single isolated cylinder. Owing to the amount of computing required, the effect of solving a finite number of equations for a fewer or greater number of modes has not been thoroughly investigated. However in one case, that for ka equal to 1.253 with a spacing of 1.0 wavelength (see Fig. 6), a finite "block" from the matrix equation was solved keeping at first three modes, and then four modes. Both cases lead to results in about equally close agreement with experiment. In all cases where the number of modes solved for was the same as required by the single cylinder problem, the detailed agreement between theory and experiment is excellent.

Without the use of large scale automatic computing machinery, it would be impractical to compute the solutions to the system of Eqs. (8) for any appreciable range of radii and spacings. Therefore, the diagonal approximation has. been used to compute the total field at a point equidistant from each cylinder and two wavelengths behind the line joining their centers (see Figs. 7, 9 through 11).

From these results it is apparent that the diagonal approximation yields a satisfactory approximation to the detailed shape of the experimental curves. In comparing the experimental data with the theory for constant radius there is noted a progressive shift in the peaks and valleys of the theoretical curves towards smaller spacings as the radius increases, and for ka equal to 3.0 the initial valley in the experimental curve is absent in the theory. These effects indicate the increasing need for considering interactions between modes of different order in the theory as the radius is increased; that is, the mode coefficients must be obtained as the solutions to a block from the matrix equation.

Computations based on the diagonal approximation involve very little additional labor above that required by the independent scattering hypothesis and give much better results than this latter hypothesis, over a limited range, of course. Consequently, this method might be considered in examining other problems where the independent scattering hypothesis does not yield satisfactory results.

The author wishes to thank Professor J. E. Storer for his illuminating discussions on the theory, and Miss M. Tynan, Miss E. Miller, and Mrs. M. Amith for their assistance with the computations. Mr. E. Roffey modified the parallel plate apparatus to allow measurements to be made of the scattering by two cylinders.

APPENDIX

Evaluation of the Integrals γ_m and K_{tmns}

With reference to Fig. 1

$$r\Big|_{\text{on } m\text{th cylinder}} = [r_m^2 + a_m^2 - 2r_m a_m \cos(\theta_m + \pi - \phi_m)]^{\frac{1}{2}},$$

and

$$r_m = [r_0^2 + b_m^2 + 2r_0 b_m \cos(\theta_0 - \alpha_{0m})]^{\frac{1}{2}}.$$

Using the addition theorem for cylinder functions,

$$H_0^{(1)}(Kr) = \sum_{l=-\infty}^{\infty} J_l(Ka_m) H_l^{(1)}(Kr_m) \exp[il(\theta_m + \pi - \phi_m)];$$

then

$$\gamma_{tm} = A \sum_l \int_0^{2\pi} J_l(Ka_m) H_l^{(1)}(Kr_m) \exp(il(\theta_m + \pi) - il\phi_m - it\phi_m) d\phi_m,$$

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp

where interchanging the order of summation and integration is assumed to be valid. The orthogonality relations for the trigonometric functions enable γ_{tm} to be evaluated explicitly, hence

$$\gamma_{tm}=2\pi A J_t(Ka_m)H_t^{(1)}(Kr_m)\exp[-it(\theta_m+\pi)].$$

It should be noted that for cylinders with centers above the line $A-B$ in Fig. 1, θ_m is to be taken as positive, and for those with centers below $A-B$, it is negative.

The computation of K_{tmns} must proceed in two steps. The first case is that in which the indices m and n are equal, corresponding to r and r' lying on the surface of the same cylinder.

With reference to Fig. 12 for the case $n=m$,

$$|\mathbf{r}-\mathbf{r}'|=a_m[1-\cos(\phi_m-\phi_m')]^{\frac{1}{2}}.$$

Using the above addition theorem for the cylinder functions gives us

$$K_{tmns}=\frac{i}{8\pi}\sum_l\int_0^{2\pi}\int_0^{2\pi}\exp(is\phi_m-it\phi_m)J_l(Ka_m)H_l^{(1)}(Ka_m)\exp[il(\phi_m-\phi_m')]d\phi_m d\phi_m'=\frac{\pi i}{2}J_t(Ka_m)H_t^{(1)}(Ka_m)\delta_{st},$$

where

$$\delta_{st}=\begin{cases}1 & \text{for } s=t\\ 0 & \text{for } s\neq t.\end{cases}$$

We turn now to the evaluation of K_{tmns} when $n\neq m$. From Fig. 12 it may be seen that

$$|\mathbf{r}-\mathbf{r}'|=(s_{nm}^2+\alpha_n^2-2s_{nm}a_n\cos[\pi+\alpha_{nm}-\phi_n+\beta_{nm}])^{\frac{1}{2}}.$$

Again, applying the same addition theorem,

$$K_{tmns}=\frac{i}{8\pi}\sum_l\int_0^{2\pi}\int_0^{2\pi}\exp(is\phi_n-it\phi_m)J_l(ka_m)H_l^{(1)}(ks_{nm})\exp[il(\pi+\alpha_{nm}-\phi_n-\beta_{n\beta})]d\phi_n d\phi_m.$$

Carrying out the ϕ_n integration, we get

$$K_{tmns}=\frac{i}{4}\int_0^{2\pi}\exp(-it\phi_m)J_s(ka_n)H_s^{(1)}(ks_{nm})\exp[is(\pi+\alpha_{nm}+\beta_{nm})]d\phi_m.$$

Both s_{nm} and β_{nm} are dependent on ϕ_m so that in order to perform the remaining integration, the more general addition theorem

$$H_s^{(1)}(Ks_{nm})\exp(is\beta_{nm})=\sum_{q=-\infty}^{\infty}J_q(Ka_m)H_{s+q}^{(1)}(K|\mathbf{b}_m-\mathbf{b}_n|)\exp[iq(\alpha_{nm}-\phi_m)]$$

must be used whence it may readily be shown that

$$K_{tmns}=\frac{\pi i}{2}J_t(ka_m)J_s(ka_n)H_{t-s}^{(1)}(k|\mathbf{b}_m-\mathbf{b}_n|)\exp(-it\alpha_{nm}+is\alpha_{nm}).$$

Summarizing,

$$\gamma_{tm}=2\pi A J_t(ka_m)H_t^{(1)}(kr_m)\exp[-it(\theta_m+\pi)]$$

$$K_{tmns}=\frac{\pi i}{2}J_t(ka_m)\begin{cases}H_s^{(1)}(ka_m)\delta_{st}, & n=m\\ J_s(ka_n)H_{t-s}^{(1)}(k|\mathbf{b}_m-\mathbf{b}_n|)\exp(-it\alpha_{nm}+is\alpha_{nm}), & n\neq m.\end{cases}$$

Downloaded 28 Apr 2008 to 140.121.146.143. Redistribution subject to AIP license or copyright; see http://jap.aip.org/jap/copyright.jsp