Given a casual function f (t) = e^{-t} cos[t] , t > 0, otherwise f (t) = 0

(1) Please find f_e (t) and f_o (t)

(2) Plot f_e (t) and f_o (t)

(3) Please find its Fourier transform

(4) Check its Hilbert transform pair using complex inegrals

Sol :

(1)

$$f(t) = \begin{cases} e^{-t}\cos[t] & t > 0 \\ 0 & t \le 0 \end{cases}$$

$$f(-t) = \begin{cases} 0 & t > 0 \\ e^{t}\cos[t] & t \le 0 \end{cases}$$

$$f_e = \frac{1}{2}(f(t) + f(-t)) = \begin{cases} \frac{1}{2} e^{-t}\cos[t] & t > 0 \\ \frac{1}{2} e^{t}\cos[t] & t \le 0 \end{cases}$$

$$f_o = \frac{1}{2}(f(t) + f(-t)) = \begin{cases} \frac{1}{2} e^{-t}\cos[t] & t > 0 \\ \frac{-1}{2} e^{t}\cos[t] & t \le 0 \end{cases}$$

(2)

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z1[t_] := If[t > 0, (E^-t * Cos[t])/2, (-E^t * Cos[t])/2]
z2[t_] := If[t > 0, (E^-t * Cos[t])/2, (E^t * Cos[t])/2]
z3[t_] := If[t > 0, E^-t * Cos[t], 0]
g1 = Plot[z1[t], {t, -5, 5} PlotStyle → {RGBColor[1, 0.7, 0], Thickness[0.008]},
 AxesLabel → {"t", "fo[t]"}, PlotRange → {{5, -5}, {-0.7, 0.7}}]
g2 = Plot[z2[t], {t, -5, 5}, PlotStyle → {RGBColor[1, 0, 1], Thickness[0.008]},
 AxesLabel → {"t", "fe[t]"}, PlotRange → {{5, -5}, {-0.7, 0.7}}]
g3 = Plot[z3[t], {t, -5, 5}, PlotStyle → {RGBColor[0, 1, 1], Thickness[0.008]},
 AxesLabel → {"t", "f[t]"}, PlotRange → {{5, -5}, {-0.2, 1.3}}]
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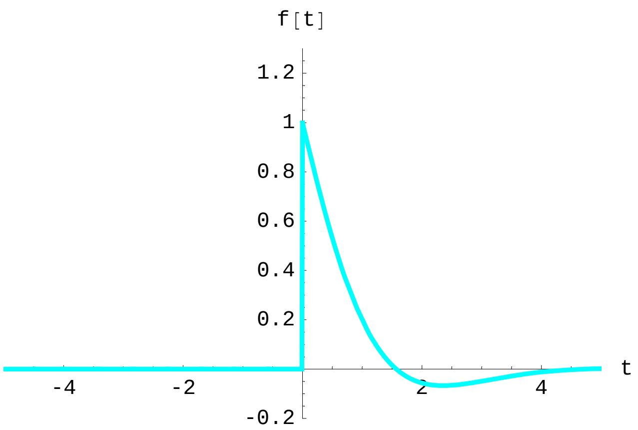
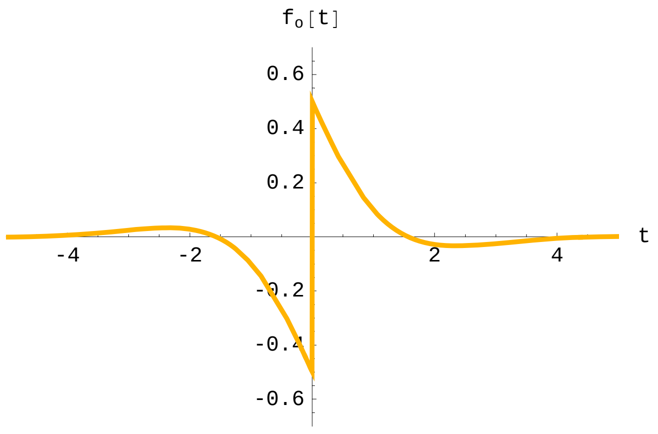
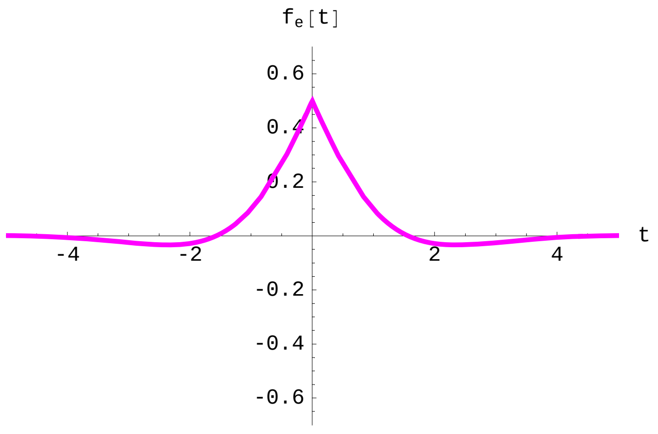






(3)

$$F_e(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_e * e^{-i\omega t})\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} \left(\frac{1}{2} e^{t} \cos[t] * e^{-i\omega t} \right) dt + \int_{0}^{\infty} \left(\frac{1}{2} e^{-t} \cos[t] * e^{-i\omega t} \right) dt \right)$$

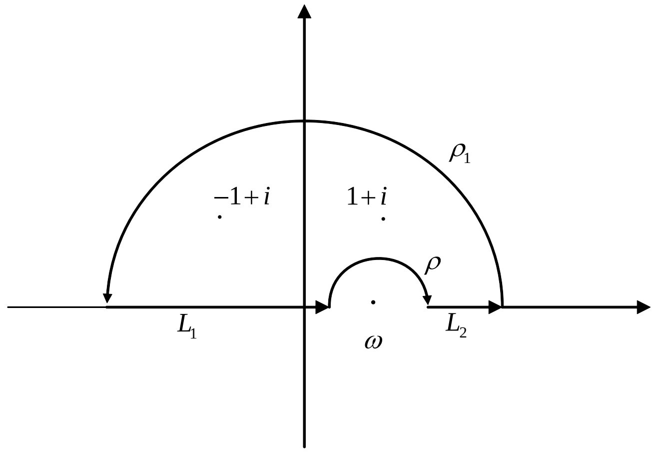
$$= \frac{1}{\sqrt{2\pi}} \frac{2+\omega^2}{4+\omega^4}$$

$$-i F_o(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_o * e^{-i\omega t})\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} \left(\frac{-1}{2} e^{t} \cos[t] * e^{-i\omega t} \right) dt + \int_{0}^{\infty} \left(\frac{1}{2} e^{-t} \cos[t] * e^{-i\omega t} \right) dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-i\,\omega^3}{4+\omega^4}$$

(4)



$$F_e(\omega) * \sqrt{\frac{2}{\pi}}\left(\frac{-i}{\omega}\right) = \int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\ \frac{2+\tau^2}{4+\tau^4}\right)\sqrt{\frac{2}{\pi}}\left(\frac{-i}{\omega-\tau}\right)d\tau$$

$$= \frac{-i}{\pi}\int_{-\infty}^{\infty}\left(\frac{2+\tau^2}{4+\tau^4}\right)\left(\frac{1}{\omega-\tau}\right)d\tau$$

$$\int_{-\infty}^{\infty}\left(\frac{2+\tau^2}{4+\tau^4}\right)\left(\frac{1}{\omega-\tau}\right)d\tau = 2\pi i\,(\text{Res}\,(1+i)+\text{Res}\,(-1+i))$$

$$= \int_{\rho_1}+\int_{L_1}+\int_{L_2}+\int_{\rho}\left(\frac{2+\tau^2}{4+\tau^4}\right)\left(\frac{1}{\omega-\tau}\right)d\tau = \int_{\rho_1}+\int_{L_1}+\int_{L_2}+\int_{\rho} f(\tau)\,d\tau$$

$$= \int_{L_1}+\int_{L_2}+\int_{\rho} f(\tau)\,d\tau$$

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n (\tau-\omega)^n$$

$$\int_{\rho} f(\tau)\,d\tau = \int_{\rho}\sum_{n=-\infty}^{\infty} a_n(\tau-\omega)^n\,d\tau = \int_{\rho} a_{-1}(\tau-\omega)^{-1}\,d\tau$$

set $\tau = \omega + \varepsilon e^{i t}$, $d\tau = i\,\varepsilon\,e^{i t}\,dt$

$$\int_{\rho} f(\tau)\,d\tau = \int_{\pi}^{0} a_{-1}\frac{1}{(\omega+\varepsilon e^{i t}-\omega)}\,i\,\varepsilon\,e^{i t}\,dt = a_{-1}\,i\int_{\pi}^{0} dt$$

$$= -\pi\,i\,a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i}\oint\left(\frac{2+\tau^2}{4+\tau^4}\right)\left(\frac{1}{\omega-\tau}\right)d\tau$$

$$= \lim_{\tau\to\omega}\frac{-(\tau^2+2)}{\left(\tau-\sqrt{2}\,e^{\frac{\pi}{4}}\right)\left(\tau-\sqrt{2}\,e^{\frac{3\pi}{4}}\right)\left(\tau-\sqrt{2}\,e^{\frac{5\pi}{4}}\right)\left(\tau-\sqrt{2}\,e^{\frac{7\pi}{4}}\right)}$$

$$= -\frac{\omega^2+2}{\omega^4+4}$$

$$\int_{\rho} f(\tau)\,d\tau = \pi\,i\,\frac{\omega^2+2}{\omega^4+4}$$

$$\text{Res}\,(1+i) = -\frac{i}{(4+4i)-4\omega}$$

$$\text{Res}\,(-1+i) = \frac{i}{(4-4i)+4\omega}$$

$$\int_{L_1} + \int_{L_2} f(\tau)\, d\tau = 2\pi i \left(\frac{i}{(4 + 4 i) - 4\omega} - \frac{i}{(4 - 4 i) + 4\omega} \right) - \pi i \frac{\omega^2 + 2}{\omega^4 + 4}$$

$$= \frac{-\pi\,\omega^3}{4 + \omega^4}$$

$$\frac{-i}{\pi} \int_{-\infty}^{\infty} \left(\frac{2 + \tau^2}{4 + \tau^4} \right) \left(\frac{1}{\omega - \tau} \right) d\tau = \frac{-i\,\omega^3}{4 + \omega^4}$$