1. $(1-x^2)y''(x) - 2xy'(x) + 2y(x) = 0$

(a)Using series form:

Let

$$y(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$y'(x) = \sum_{n=0}^{\infty} nC_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2}$$

代回原式,

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)C_n x^n - 2\sum_{n=1}^{\infty} C_n x^n + 2\sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2}x^n - \sum_{n=0}^{\infty} n(n-1)C_n x^n - 2\sum_{n=0}^{\infty} C_n x^n + 2\sum_{n=0}^{\infty} C_n x^n = 0$$

$$(n+2)(n+1)C_{n+2} = [n(n-1)+2n-2]C_n = (n+2)(n-1)C_n$$

$$C_{n+2} = \frac{n-1}{n+1}C_n$$

$$C_0 = C_0,\ C_1 = C_1$$

$$C_2 = -C_0,\ C_3 = 0,\ C_4 = \frac{-1}{3}C_0,\ C_6 = \frac{-1}{5}C_0 \cdots$$

則方程式之齊次解

$y(x) = C_1 x + C_0(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \cdots)$。

(b)Using Wronskian (make it to be a first order ODE):

原方程式可改寫爲

$$y''(x)-\frac{2x}{1-x^2}y'(x)+\frac{2}{1-x^2}y(x)=y''(x)+P(x)y'(x)+Q(x)y(x)=0$$

One complementary solution

$$y_1=x$$

又

$$W'+P(x)W=0$$

$$W=C\exp[-\int P(x)dx]=\frac{C}{x^2-1} \tag{1}$$

Wronskian

$$W=\begin{vmatrix} x & y \\ 1 & y' \end{vmatrix}=xy'-1y \tag{2}$$

由 (1)(2) 得

$$xy'-y=\frac{C}{x^2-1}$$

$$y'-\frac{1}{x}y=\frac{C}{x(x^2-1)}$$

解上式一階 ODE

$$\frac{1}{x}y=\frac{-C}{2}(\ln|1-x|-\ln|1+x|-\tfrac{1}{x})+C^*$$

將上式右邊之對數函數對 $x=0$ 做泰勒級數展開

$$=\frac{-C}{2}[(-\tfrac{x}{1}-\tfrac{x^2}{2}-\tfrac{x^3}{3}-\tfrac{x^4}{4}-\cdots)-(\tfrac{x}{1}-\tfrac{x^2}{2}+\tfrac{x^3}{3}-\tfrac{x^4}{4}+\cdots)]+C^*$$

$$=\frac{-C}{2}[(-\tfrac{2x}{1}-\tfrac{2x^3}{3}-\tfrac{2x^5}{5}-\tfrac{2x^7}{7}-\cdots+\tfrac{1}{x}]+C^*$$

$$y=C(1+x^2-\tfrac{x^4}{3}-\tfrac{x^6}{5}-\tfrac{x^8}{7}-\cdots)+C^*x=Cy_2+C^*y_1$$

the other complementary solution is $y_2=1-x^2-\frac{x^4}{3}-\frac{x^6}{5}-\frac{x^8}{7}-\cdots$。

(c)Using Variation of parameters:

原方程式可改寫為

$$y''(x) - \frac{2x}{1-x^2}y'(x) + \frac{2}{1-x^2}y(x) = y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

Assume $y_2(x) = u(x)y_1(x)$, where $y_1(x) = x$

then $u(x) = \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$

$\therefore \quad y_2(x) = x \int \frac{1}{x^2} e^{\int \frac{2x}{1-x^2}dx} dx$

$= x \int \frac{1}{x^2} e^{-\ln(1-x^2)} dx$

$= x \int \frac{1}{x^2} \frac{1}{1-x^2} dx$

$= x \int (\frac{1}{x^2} + \frac{1}{1-x^2}) dx$

$= x \int (\frac{1}{x^2} + \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}) dx$

$= x[-\frac{1}{x} - \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x)]$

又泰勒展開式

$\ln(1-x) = -1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \cdots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots$

$\therefore$

$y_2(x) = -1 + x^2 + \frac{x^4}{3} + \frac{x^6}{5} + \frac{x^8}{7} + \cdots$

則方程式之解

$y(x) = C_0(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \frac{x^8}{7} - \cdots) + C_1 x$

比較 (a)(b)(c)，結果皆相同。

2. $-x^2y''(x) - 2xy'(x) + N(N+1)y(x) = 0$

$$\underline{(a)\ Using\ \ y_n(x) = \sum_{n=0}^{\infty} a_n x^n\ \ to\ solve:}$$

$$y_n'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$y_n''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

代回原式,

$$-x^2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + N(N+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}\{-n(n-1) - 2n + N(N+1)\}a_n x^n = 0$$

$$n^2 + n - N(N+1) = 0$$

$$n = N, -(N+1)$$

$$y_N(x) = a_N x^N$$

只能得一解。

$$\underline{(b)\ Using\ \ y_n(x) = \sum_{n=0}^{\infty} a_n x^{-n}\ \ to\ solve:}$$

$$y_n'(x) = \sum_{n=0}^{\infty} -n a_n x^{-n-1}$$

$$y_n''(x) = \sum_{n=0}^{\infty} -n(-n-1) a_n x^{-n-2}$$

代回原式,

$$-x^2\sum_{n=0}^{\infty}-n(-n-1)a_n x^{-n-2}-2x\sum_{n=0}^{\infty}-na_n x^{-n-1}+N(N+1)\sum_{n=0}^{\infty}a_n x^{-n}=0$$

$$\sum_{n=0}^{\infty}\{-n(n+1)+2n+N(N+1)\}a_n x^{-n}=0$$

$$n^2-n-N(N+1)=0$$

$$n=N+1,-N$$

$$y_{(N+1)}(x)=a_{(N+1)}x^{-(N+1)}$$

只能得一解。

$$\underline{(c)\ Using\ \ y_n(x)=\sum_{n=-\infty}^{\infty}a_n x^n\ \ to\ solve:}$$

$$y_n'(x)=\sum_{n=-\infty}^{\infty}na_n x^{n-1}$$

$$y_n''(x)=\sum_{n=-\infty}^{\infty}n(n-1)a_n x^{n-2}$$

代回原式,

$$-x^2\sum_{n=-\infty}^{\infty}n(n-1)a_n x^{n-2}-2x\sum_{n=-\infty}^{\infty}na_n x^{n-1}+N(N+1)\sum_{n=-\infty}^{\infty}a_n x^n=0$$

$$\sum_{n=-\infty}^{\infty}\{-n(n-1)-2n+N(N+1)\}a_n x^n=0$$

$$n^2+n-N(N+1)=0$$

$$n=N,-(N+1)$$

$$y_N(x)=a_N x^N$$

$$y_{-(N+1)}(x)=a_{-(N+1)}x^{-(N+1)}$$

可以得兩解。

$$(d)\ Using\ \ y_n(x) = \sum_{n=0}^{\infty} a_n x^{n+r}\ \ to\ solve:$$

$$y_n'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y_n''(x) = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

代回原式

$$-x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - 2x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + N(N+1)\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\{-(n+r)(n+r-1) - 2(n+r) + N(N+1)\}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}\{(n+r)(n+r+1) - N(N+1)\}a_n x^{n+r} = 0$$

令

$$p(n) = (n+r)(n+r-1) + 2(n+r) - N(N+1)$$

$$\sum_{n=0}^{\infty}\{(n+r)(n+r+1) - N(N+1)\}a_n x^{n+r} = 0$$

其指標方程式及指標根

$$r(r-1) + 2r - N(N+1) = 0$$

$$r = N, -(N+1)$$

(i) $r{=}N$

$$p(n)a_n x^{n+N} = 0$$
$$p(n) = 0$$
$$n^2 + (2N+1)n + [(N^2+N) - (N^2+N)] = 0$$
$$n = 0, -(2N+1)$$

$$a_n = 0, \text{ otherwise } \ a_0 \neq 0$$

故得

$$y_0(x) = \sum_{n=0}^{\infty} a_0 x^N$$

只能得一解。

(ii) $r{=}{-}(N+1)$

$$p(n)a_n x^{n-(N+1)} = 0$$
$$p(n) = 0$$
$$n^2 + (-2N-1)n + [(N+1)^2 - (N+1) - N(N+1)] = 0$$

$$n = 0, 2N+1$$

$$a_n = 0, \text{ otherwise } \ a_0 \neq 0, \ a_{2N+1} \neq 0$$

故得

$$y_0(x) = \sum_{n=0}^{\infty} a_0 x^{-(N+1)}$$
$$y_{(2N+1)}(x) = \sum_{n=0}^{\infty} a_{(2N+1)} x^N$$

可以得兩解。