(b)Using Wronskian (make it to be a first order ODE):

原方程式可改寫爲

$$y''(x) - \frac{2}{4x}y'(x) + \frac{1}{4x}y(x) = y''(x) + P(x)y'(x) + Q(x)y(x) = 0$$

One complementary solution

$$y_1 = \cos\sqrt{x}$$

又

$$W' + P(x)W = 0$$

$$W = C\exp[-\int P(x)dx] = \frac{C}{\sqrt{x}} \qquad (1)$$

Wronskian

$$W = \begin{vmatrix} \cos\sqrt{x} & y \\ -\frac{\sin\sqrt{x}}{2\sqrt{x}} & y' \end{vmatrix} = (\cos\sqrt{x})y' + \frac{\sin\sqrt{x}}{2\sqrt{x}}y \qquad (2)$$

由 (1)(2) 得

$$(\cos\sqrt{x})y' + \frac{\sin\sqrt{x}}{2\sqrt{x}}y = \frac{C}{\sqrt{x}}$$

$$y' + \frac{\tan\sqrt{x}}{2\sqrt{x}}y = \frac{C}{\sqrt{x}\cos\sqrt{x}}$$

解上式一階 ODE

$$\frac{1}{\cos\sqrt{x}}y = C^*\tan\sqrt{x} + C^{**}$$

$$y = C^*\sin\sqrt{x} + C^{**}\cos\sqrt{x}$$

the other complementary solution is

$$y_2 = \sin\sqrt{x}$$

比較 (a)(b) , 結果皆相同

Solve another complementary solution for the ODE $4xy''(x)+2y'(x)+y(x)=0$
by using

(a)Variation of parameters

(b)Wronskian

(a)Using the variation of parameters:

原方程式可改寫爲

$$y''(x)+\frac{2}{4x}y'(x)+\frac{1}{4x}y(x)=y''(x)+P(x)y'(x)+Q(x)y(x)=0$$

Assume $\quad y_2(x){=}y_1(x)u(x),\;$ where $\quad y_1(x)=\cos\sqrt{x}$

then $\quad u(x){=}\int \frac{1}{y_1^2}e^{-\int p(x)dx}dx$

$$={\int \frac{1}{\cos^2\sqrt{x}}e^{\int \frac{-1}{2x}dx}dx}$$

$$={\int \frac{1}{\cos^2\sqrt{x}}\frac{1}{\sqrt{x}}dx}$$

$$={\int 2\sec^2\sqrt{x}\;d\sqrt{x}}$$

$$=2\tan\sqrt{x}+C$$

$$y_2(x){=}\cos\sqrt{x}(2\tan\sqrt{x}+C)$$

則方程式之解

$$y(x){=}C_1\cos\sqrt{x}+C_2\sin\sqrt{x}$$

$$y(x){=}C_1y_1+C_2y_2$$

所以

$$y_2(x){=}\sin\sqrt{x}$$