$$(1-x^2)y'' - xy' + m^2 y = 0$$

$$y'' - \frac{x}{1-x^2}y' + \frac{m^2}{1-x^2}y = 0$$

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得知 $xp(x) = \frac{x^2}{1-x^2}$　$x^2 q(x) = \frac{x^2 m^2}{1-x^2}$　**所以 0 點為常點**

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

令 $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

代回原方程得

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} + m^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + m^2\sum_{n=0}^{\infty} a_n x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n = 0$$

$$(2a_2 + m^2 a_0)x^0 + (6a_3 + m^2 a_1 - a_1)x + \sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + m^2 a_n\right]x^n - \sum_{n=2}^{\infty} n^2 a_n x^n = 0$$

$$\sum_{n=2}^{\infty}\left[(n+2)(n+1)a_{n+2} + (m^2 - n^2)a_n\right]x^n + (2a_2 + m^2 a_0)x^0 + (6a_3 + m^2 a_1 - a_1)x = 0$$

$$\therefore a_{n+2} = \frac{n^2 - m^2}{(n+2)(n+1)}a_n$$

且 $2a_2 + m^2 a_0 = 0$

$6a_3 + (m^2 - 1)a_1 = 0$

當

$n = 0$　$a_2 = \frac{-m^2}{2}a_0$

$n = 1$　$a_3 = \frac{1-m^2}{6}a_1$

$n = 2$　$a_4 = \frac{4-m^2}{12}a_2 = \frac{m^2(m^2-4)}{24}a_0$

$n = 3$　$a_5 = \frac{9-m^2}{20}a_3 = \frac{(9-m^2)(1-m^2)}{120}a_1$

$$\therefore y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots\dots\dots$$

$$= a_0 + a_1 x - \frac{m^2}{2} a_0 x^2 + \frac{(1-m^2)}{6} a_1 x^3 + \dots\dots\dots$$

$$= a_0 (1 - \frac{m^2}{2} x^2 + \frac{m^2(m^2-4)}{24} x^4 + \dots.) + a_1 (x + \frac{(1-m^2)}{6} x^3 + \frac{(9-m^2)(1-m^2)}{120} x^5 + \dots)$$

為所求

討論

$m = 0 \quad y(x) = a_0 + a_1 (x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \dots)$

$m = 1 \quad y(x) = a_0 (1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 + \dots) + a_1 x$

$m = 2 \quad y(x) = a_0 (1 - 2x^2) + a_1 (x - \frac{1}{2} x^3 + \frac{1}{8} x^5 + \dots)$

$m = 3 \quad y(x) = a_0 (1 - \frac{9}{2} x^2 + \frac{15}{8} x^4 + \dots) + a_1 (x - \frac{4}{3} x^3)$