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國立臺灣海洋大學河海工程學系 2001 工程數學 (三) 第二次作業小考參考解答

兩階線性微分方程系統性解法

$$x^2y''(x) - 4xy' - 6y = -6 \tag{1}$$

(a).Sol.

$$\begin{aligned}
&y = x^n \\
&y' = nx^{(n-1)} \\
&y'' = n(n-1)x^{(n-2)} \\
&n(n-1)x^n - 4nx^n - 6x^n = 0 \\
&(n^2 - n - 4n - 6)x^n = 0 \\
&n = \ -1 \ or\ 6
\end{aligned}$$

(b).Sol.

$$\begin{aligned}
&y_1(x) = \frac{1}{x} \\
&y_2(x) = y_1(x)u(x) = \frac{1}{x}u(x) \\
&y_2' = -\frac{1}{x^2}u + \frac{1}{x}u' \\
&y_2'' = \frac{2}{x^3}u - \frac{2}{x^2}u' + \frac{1}{x}u''
\end{aligned}$$

代回原式:

$$\begin{aligned}
&xu'' - 6u' = 0 \\
&u = \frac{1}{7}x^7 \\
\therefore\, &y_2 = y_1u = \frac{1}{x}x^7 = x^6
\end{aligned}$$

(c). Sol.

$$\begin{cases} y_1v_1' + y_2v_2' = 0 \\ y_1'v_1' + y_2'v_2' = -\frac{6}{x^2} \end{cases}$$

代入$y_1 = \frac{1}{x}, y_2 = x^6$

$$\begin{cases} \quad v_1' + x^7v_2' = 0 & (1) \\ -v_1' + 6x^7v_2' = -6 & (2) \end{cases}$$

$$\begin{aligned}
&v_2 = \frac{1}{7}x^{-6}, v_1 = \frac{6}{7}x \\
\therefore\, &y_p = y_1v_1 + y_2v_2 = \frac{1}{x}\,\frac{6}{7}x + x^6\frac{1}{7x^6} = 1
\end{aligned}$$

(d). Sol.

$$
\begin{aligned}
&x = e^t,\ dx = e^t dt,\ y(x) = y(e^t) = Y(t)\\
&\frac{d^2}{dx^2}(y(x)) = \frac{d^2}{dx^2}(Y(t)) = \frac{d}{dt}(\frac{d}{dt}(Y(t))\frac{dt}{dx})\frac{dt}{dx} = -e^{-2t}Y'(t) + e^{-2t}Y''(t)\\
\because\, & x^2y'' - 4xy' - 6y = -6\\
&e^{2t}(-e^{-2t}Y'(t) + e^{-2t}Y''(t)) - 4e^tY'(t)e^{-t} - 6Y(t) = -6\\
&Y''(t) - 5Y'(t) - 6Y(t) = -6\\
&Y_1(t) = e^{-t},\ Y_2(t) = e^{6t}
\end{aligned}
$$

同上述方法 (b), 可得$Y_p(t)$:

$$\therefore Y_p(t) = c_1e^{-t} + c_2e^{6t}$$

(e). Sol.

$$
\begin{aligned}
&\mathcal{L}(y(x)) = Y(s),\ \mathcal{L}(y'(x)) = sY(s) - y(0)\\
&\mathcal{L}(xy'(x)) = -(Y(s) + sY'(s))\\
&\mathcal{L}(y''(x)) = s^2Y(s) - sy(0) - y'(0)\\
&s^2Y''(s) + 4sY'(s) + 2Y(s) + 4(Y(s) + sY'(s)) - 6Y(s) = \frac{-6}{s}\\
&s^2Y''(s) + 8sY'(s) = \frac{-6}{s} \qquad (3)
\end{aligned}
$$

將 (3) 再 Laplace 轉換

$$
\begin{aligned}
&v^2Y''(v) + 4vY'(v) + 2Y(v) + 8(-Y(v) - vY'(v) = -t\\
&v^2Y''(v) - 4vY'(v) - 6Y(v) = -6
\end{aligned}
$$

同理, 由 (a)(b) 可得$Y(v) = y(x) = 1$