Alternative series and Stokes' transformation

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original function decomposition:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n cos(\frac{n\pi t}{l}) + b_n sin(\frac{n\pi t}{l})\}$$

derivative of original function decomposition:

$$f'(t) = \sum_{n=1}^{\infty}\{a_n(\frac{-n\pi}{l})sin(\frac{n\pi t}{l}) + b_n(\frac{n\pi}{l})cos(\frac{n\pi t}{l})\} = \frac{1}{2}a'_0 + \sum_{n=1}^{\infty}\{a'_n cos(\frac{n\pi t}{l}) + b'_n sin(\frac{n\pi t}{l})\}$$

where J_k is the jump value at t_k for the function $f(t)$.

relation of a_n, b_n, a'_n and b'_n:

$$a_n = -\frac{1}{n\pi}\sum_{k=1}^{m} J_k sin(\frac{n\pi t_k}{l}) - \frac{l}{n\pi}b'_n, \ n \neq 0 \quad (1)$$

$$b_n = \frac{1}{n\pi}\sum_{k=1}^{m} J_k cos(\frac{n\pi t_k}{p}) + \frac{l}{n\pi}a'_n, \ n \neq 0 \quad (2)$$

Termwise differentiation is only permissible if J_k is zero.

Stokes' transformation:

$$u(x) = \begin{cases} p, & x = 0 \\ \sum_{n=0}^{\infty} b_n sin(n\pi x/l), & 0 < x < l \\ q, & x = l \end{cases} \quad (3)$$

Assume

$$u'(x) = \sum_{n=0}^{\infty} a'_n cos(n\pi x/l), \ 0 \leq x \leq l \quad (4)$$

$$\begin{aligned} a'_n &= \frac{2}{l}\int_0^l u'(x)cos(n\pi x/l)dx \\ &= \frac{2}{l}[u(x)cos(n\pi x/l)]_0^l + \frac{2n\pi}{l^2}\int_0^l u(x)sin(n\pi x/l)dx \\ &= \frac{2}{l}[(-1)^n q - p] + n\pi b_n/l, \end{aligned} \quad (5)$$

and as $n = 0$,

$$a'_0 = \frac{-1}{l}[p - q] \quad (6)$$

Defining

$$r_n \equiv \begin{cases} \frac{-1}{l}[p-q], & n = 0 \\ \frac{2}{l}[(-1)^n q - p], & n = 1, 2, \cdots \end{cases} \quad (7)$$

we have

$$a'_n = r_n + n\pi b_n/l, \ \ n \geq 0 \quad (8)$$

Proof:

$J_0 = 2p, \;\; at \;\; x = 0$

$J_l = 2q, \;\; at \;\; x = -l$