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國立臺灣海洋大學河海工程學系 2002 碩士班工程數學參考解答

1.

$\frac{1}{s(s^2+1)}$ 可表爲 $= f(s)g(s)$, 其中 $f(s)=\frac{1}{s}$ 及 $g(s)=\frac{1}{s^2+1}$

$\because \mathcal{L}^{-1}\{\frac{1}{s}\}=1,\ \mathcal{L}^{-1}\{\frac{1}{s^2+1}\}=\sin(t)$

$\mathcal{L}^{-1}\{f(s)g(s)\}=\mathcal{L}^{-1}\{f(s)\}*\mathcal{L}^{-1}\{g(s)\}$

$$\begin{aligned}\mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\} &= \mathcal{L}^{-1}\{f(s)g(s)\}\\ &= \mathcal{L}^{-1}\{f(s)\}*\mathcal{L}^{-1}\{g(s)\}\\ &= (1)*(\sin(t))\\ &= \int_0^t 1\cdot\sin(t-x)dx\\ &= \int_0^t(\sin(x))(1)dx\end{aligned}$$

$$\therefore \mathcal{L}^{-1}\{\frac{1}{s(s^2+1)}\}=\int_0^t\sin(x)dx=-\cos(x)|_{x=0}^t=1-\cos(t)$$

2.

$x_p(t)=\frac{A_0}{2}+\sum_{k=1}^{\infty}(A_k\cos(kwt)+B_k\sin(kwt))$ (1) 之 Fourier Series

當微分方程式具 $x''+ax'+bx=g(t)$ (2) 之形式可求得 A_k 及 B_k 之通解.

(2)式乘以 $\cos(kwt)$, 再由 $t=0$ 積分至 $t=\frac{2\pi}{w}$, 則

$$\int_0^{\frac{2\pi}{w}}(x_p''+ax_p'+bx_p)\cos(kwt)dt=\int_0^{\frac{2\pi}{w}}g(t)\cos(kwt)dt \qquad (3)$$

$\because g(t)$ 爲 Fourier Cosine 係數:

$$\begin{aligned}a_k &= \frac{w}{\pi}\int_0^{\frac{2\pi}{w}}g(t)\cos(kwt)dt\\ b_k &= \frac{w}{\pi}\int_0^{\frac{2\pi}{w}}g(t)\sin(kwt)dt\end{aligned} \Rightarrow \begin{aligned}\int_0^{\frac{2\pi}{w}}g(t)\cos(kwt)dt &= \frac{\pi}{w}a_k\\ \int_0^{\frac{2\pi}{w}}g(t)\sin(kwt)dt &= \frac{\pi}{w}b_k\end{aligned} \qquad (4)$$

(3) 式左側, 先求 $\int_0^{\frac{2\pi}{w}}x_p''\cos(kwt)dt$ (5)

令 $u=\cos(kwt),\ dv=x_p''dt$, 則 (5) 式變爲

$$x_p'\cos(kwt)|_0^{\frac{2\pi}{w}}+kw\int_0^{\frac{2\pi}{w}}x_p'\sin(kwt)dt \qquad (6)$$

(6) 式之第一項爲0, $\therefore$ 令 $u=\sin(kwt),\ dv=x_p'dt$, (6) 式變成

$$kw[x_p\sin(kwt)]|_0^{\frac{2\pi}{w}}-kw\int_0^{\frac{2\pi}{w}}x_p\cos(kwt)dt \qquad (7)$$

(7) 之第一項爲 0, $\therefore$

$$-k^2w^2\int_0^{\frac{2\pi}{w}} x_p\cos(kwt)dt = -k^2w^2\frac{\pi}{w}A_k \qquad (8)$$

同理,

$$\int_0^{\frac{2\pi}{w}} ax_p'\cos(kwt)dt = akw\frac{\pi}{w}B_k \qquad (9)$$

$$A_k = \int_0^{\frac{2\pi}{w}} x_p\cos(kwt)dt$$
$$B_k = \int_0^{\frac{2\pi}{w}} x_p\sin(kwt)dt$$

$$\int_0^{\frac{2\pi}{w}} bx_p\cos(kwt)dt = \frac{\pi}{w}bA_k \qquad (10)$$

(8)(9)(10) 式代入 (3) 式, 及利用 (4) 式

$\frac{\pi}{w}(-k^2w^2A_k + akwB_k + bA_k) = \frac{\pi}{w}A_k$ 或 $(b-k^2w^2)A_k + akwB_k = a_k$ (11)

(2) 式乘上 $\sin(kwt)$, 再用以上計算步驟, 可得

$$-akwA_k + (b^2-k^2w^2)B_k = b_k \qquad (12)$$

式中 b_k 爲 $g(t)$ Fourier Series 係數

由 (11) 及 (12) 式,

$$A_k = \frac{(b-k^2w^2)a_k - akwb_k}{(b-k^2w^2)^2 + a^2k^2w^2} \qquad (13)$$
$$B_k = \frac{akwa_k + (b-k^2w^2)b_k}{(b-k^2w^2)^2 + a^2k^2w^2} \qquad (14)$$

已知 $-\pi < t < \pi$, $g(t) = t$, 其 Fourier Series 爲

$$g(t) \sim 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nt)}{n}$$

$\therefore w = 1$, $a_k = 0$ 對所有 k 均成立, 而 $b_k = \frac{2(-1)^{k+1}}{k}$

此外, $b = 2$, $a = 2$ 代入 (13) 及 (14) 式

$$A_0 = 0$$
$$A_1 = \frac{-2(2)}{(2-1)^2+2^2} = \frac{-4}{5}$$
$$B_1 = \frac{(2-1)2}{(2-1)^2+2^2} = \frac{2}{5}$$
$$A_2 = \frac{-2(2)(-2/2)}{(2-4)^2+2^2(2)^2} = \frac{1}{5}$$
$$B_2 = \frac{(2-4)(-2/2)}{(2-4)^2+2^2(2)^2} = \frac{1}{10}$$

$$\therefore x_p = -\frac{4}{5}\cos(t) + \frac{2}{5}\sin(t) + \frac{1}{5}\cos(2t) + \frac{1}{10}\sin(2t) + ...$$

3.(a)

$$\det(\mathrm{A}-\lambda\mathrm{I})=0 \Rightarrow \det\left\{\begin{bmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & -3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}=0$$

$$\Rightarrow (-3-\lambda)(\lambda-3)^2=0$$

用 A 代入 λ, 檢查 A 是否滿足上式

$$\text{i.e.} -(3\mathrm{I}+\mathrm{A})(\mathrm{A}-3\mathrm{I})^2$$

$$\because -(3I+A) = -\left\{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & -3 \end{bmatrix}\right\} = \begin{bmatrix} -7 & -1 & 0 \\ 1 & -5 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

$$(A-3I)^2 = \left\{\begin{bmatrix} 4 & 1 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right\}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -11 & -5 & 36 \end{bmatrix}$$

$$\therefore -(3I+A)(A-3I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

i.e. A 為它本身特徵方程式之零根

(b)

$$f(x,y,z)=4(x^2+y^2)-z^2$$

$$\nabla f = 8x\vec{i}+8y\vec{j}-2z\vec{k}$$

在 P 點, $\nabla f = 8\vec{i}-4\vec{k}$

$\therefore$ a unit normal vector of the cone at P is

$$\vec{n}=\frac{1}{|\nabla f|}\nabla f = \frac{2}{\sqrt{5}}\vec{i}-\frac{1}{\sqrt{5}}\vec{k}$$

(另一為 $-\vec{n}$)

4.

令 $y=\frac{dx}{dt}$, 則單擺位移後之微分方程式變為

$$\begin{aligned} \tfrac{dx}{dt} &= y \\ \tfrac{dy}{dt} &= -\sin(x) \end{aligned} \qquad (1)$$

(1) 具以下形式

$$\begin{aligned} \tfrac{dx}{dt} &= P(x,y) \\ \tfrac{dy}{dt} &= Q(x,y) \end{aligned}$$

此系統之臨界點爲 $P(x,y)=0,\ \ Q(x,y)=0$ 之點

$\therefore$ (1) 之臨界點爲 $(x_c,0)$, 其中 $x_c=\pm n\pi\ \ (n=0,1,2,...)$

$\therefore$ 系統 (1) 具有臨界點 $(n\pi,0)\ \ (n=0,1,2,...)$ 之無窮集合

消去 (1) 中之參數 dt

$$\frac{dy}{dx}=-\frac{\sin x}{y} \qquad (2)$$

(2) 式爲軌跡在 xy (或 $x\frac{dx}{dt}$) 相平面上之微分方程式

假設初始條件爲 $x(t_0)=x_0,\ \ y(t_0)=y_0$

則積分 (2) 式

$$\int_{y_0}^{y} y dy=-\int_{x_0}^{x}\sin(x)dx \qquad (3)$$

$$\because \int_{x_0}^{x} f(x)dx=\int_0^x f(x)dx-\int_0^{x_0} f(x)dx$$

$$\therefore \int_{y_0}^{y} y dt=-\int_0^x \sin(x)dx+\int_0^{x_0}\sin(x)dx$$

$$\therefore \frac{1}{2}y^2-\frac{1}{2}y_0^2=(\cos(x)-1)-(\cos(x_0)-1)$$

或 $\frac{1}{2}y^2+(1-\cos(x))=h \qquad (4)$

$h=\frac{1}{2}y_0^2-\cos(x_0)+1$ 爲一常數

$\therefore$ 此系統爲保守系統, i.e. 能量爲常數

(4) 式中, $\frac{1}{2}y^2$ 爲動能 $(y=\frac{dx}{dt}=V)$,

而 $V=\int_0^x \sin(x)dx=1-\cos(x)$ 爲位能

由 (4) 中解 y,

$$y=\pm\sqrt{2(h+\cos(x)-1)} \qquad (5)$$

先畫 $Y=V(x)$ 及 $Y=h$ 於 xy 平面, 圖 (a)

再畫 xy 相平面圖, 圖 (b)

可由臨界點 $(n\pi,0)$ 與系統之位能函數的關係考慮其特性及穩定性,

當 $F(x)=-\sin(x)=0$ 時可得臨界點之 x 座標

由 $V(x)=1-\cos(x),\ \ -V(x)=F(x)$

但在 x_c 處, $V'(x)=0\rightarrow$ 在 x_c 處有相對極大, 極小或水平反曲點

由 $V(x)=1-\cos(x)$, 可得 $V'(x)=\sin(x),\quad V''(x)=\cos(x)$

而 $V'(x)=0$ at $x=n\pi\ \ (n=0,\pm1,\pm2,...)$

若 n 爲偶數, $V''(n\pi)>0$,

若 n 爲奇數, $V'(n\pi)<0$

$\therefore V(x)$ 在 $x=2n\pi$ 處有相對極小值

$V(x)$ 在 $x=(2n+1)\pi$ 處有相對極大值

而 $V(x)$ 爲相對極小之臨界點爲一中心點且是穩定的

而 $V(x)$ 爲相對極大之臨界點爲一馬鞍點且爲不穩定的

$\therefore$ 臨界點:

$(2n\pi, 0)$ $(n = 0, \pm 1, \pm 2, ...)$ 爲中心點且爲穩定的

$((2n+1)\pi, 0)$ $(n = 0, \pm 1, \pm 2, ...)$ 爲馬鞍點且爲不穩定的

再由圖 (b), 能階小於 $h = 2$ 者, 軌跡爲封閉的

$h > 2$ 者, 軌跡爲開放的

總能量 $h = 2$ 將來回運動 $(h < 2)$ 及圓周運動 $(h > 2)$

$\rightarrow$ 其中擺錘由其平衡位置開始繞固定點作圓周運動。

5.

Green Theorem:

$$\int P dx + \int Q dy = \int\int (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx dy$$

Green function:

$$\frac{d^2 G(x,s)}{dx^2} = \delta(x - s), \ \ 0 < x < l$$

$$G(0, s) = G(l, s) = 0$$

6.

$$\begin{array}{ll} A^T = A & \text{(real matrix)} \\ A^T = \bar{A} & \text{(Hermitian matrix)} \end{array}$$

Their eigenvalues are real.

7.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$$

Method 1:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial t} dt$$

$$\left\{ \begin{array}{rl} \frac{dx}{d\tau} = & 1 \\ \frac{dt}{d\tau} = & 1 \\ \frac{du}{d\tau} = & 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{rl} x(0,s) = & s \\ t(0,s) = & 0 \\ u(0,s) = & \sin(s) \end{array} \right.$$

$$u(x, t) = \sin(x + t)$$

特徵線: $x + t = k$, 波速: $c = -1$, 方向: 向左

Method 2:

$$\mathcal{L}\{\frac{\partial u}{\partial t}\} = sU(x,s) - u(x,0) = sU(x,s) - \sin(x)$$

$$\mathcal{L}\{\frac{\partial u}{\partial x}\} = U'(x,s)$$

$$U'(x,s) = sU(x,s) - \sin(x), \ \ U'(x,s) = \frac{\partial U(x,s)}{\partial x}$$

$$U'(x,s) - sU(x,s) = -\sin(x)$$

$$U(x,s) = \frac{-1}{s^2+1}[s\sin(x) - \cos(x)]$$

$$\mathcal{L}^{-1}\{U(x,s)\} = u(x,t) = \cos(t)\,\sin(x) + \sin(t)\cos(x) = \sin(x+t)$$