(一)

(a) By choosing T-complete set：$\{ \frac{cos(\theta)}{r}, \frac{cos(2\theta)}{r^2}, \frac{sin(\theta)}{r}, \frac{sin(2\theta)}{r^2}, ln(r) \}$

$$\theta_i = \left\{ 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5} \right\}$$

$$u(r,\theta) = \sum_{n=1}^{2} A_n \frac{cos(n\theta)}{r^n} + \sum_{n=1}^{2} B_n \frac{sin(n\theta)}{r^n} + C_1\ ln(r)$$

Build $[K]\{a\} = \{b\}$

$$[K] = \begin{pmatrix} 1. & 1. & 0. & 0. & 0. \\ 0.309017 & -0.809017 & 0.951057 & 0.587785 & 0. \\ -0.809017 & 0.309017 & 0.587785 & -0.951057 & 0. \\ -0.809017 & 0.309017 & -0.587785 & 0.951057 & 0. \\ 0.309017 & -0.809017 & -0.951057 & -0.587785 & 0. \end{pmatrix} \qquad \{b\} = \begin{pmatrix} 1. \\ -0.809017 \\ 0.309017 \\ 0.309017 \\ -0.809017 \end{pmatrix}$$

$$\{a\} = \{A_1, A_2, B_1, B_2, C_1\}^T$$

Because $[K]$ is singular, we solve $\{a\}$ by using Gaussian Elimination Method.

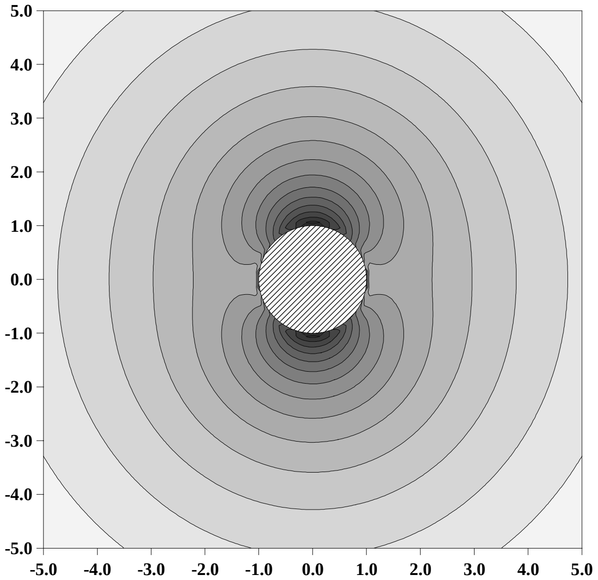
The augmented matrix is $\begin{pmatrix} 1. & 1. & 0. & 0. & 0. & 1. \\ 0.309017 & -0.809017 & 0.951057 & 0.587785 & 0. & -0.809017 \\ -0.809017 & 0.309017 & 0.587785 & -0.951057 & 0. & 0.309017 \\ -0.809017 & 0.309017 & -0.587785 & 0.951057 & 0. & 0.309017 \\ 0.309017 & -0.809017 & -0.951057 & -0.587785 & 0. & -0.809017 \end{pmatrix}$

Finally, we get $\begin{pmatrix} 1. & 1. & 0. & 0. & 0. & 1. \\ 0. & -1.11803 & 0.951057 & 0.587785 & 0. & -1.11803 \\ 0. & 0. & -0.812299 & -0.502029 & 0. & 0. \\ 0. & 0. & 0. & -14.6946 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. \end{pmatrix}$

$$\{A_1, A_2, B_1, B_2, C_1\}^T = \{0, 1, 0, 0, arbitrary\}^T$$

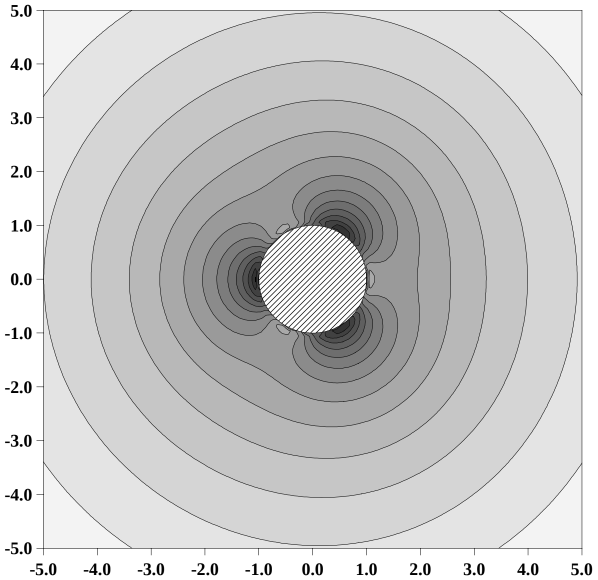
The Trefftz solution $u(r,\theta) = \frac{cos(2\theta)}{r^2} + C_1\ ln(r)$

Suppose $C_1 = 1$



Trefftz solution (5 sets)

The exact solution $u(r,\theta) = \dfrac{cos(3\theta)}{r^3} + ln(r),\ \mathrm{r} \geq 1$



Exact solution (r=1)

(b) By choosing T-complete set：

$$\left\{ \frac{cos(\theta)}{r}, \frac{cos(2\theta)}{r^2}, \frac{cos(3\theta)}{r^3}, \frac{cos(4\theta)}{r^4}, \frac{sin(\theta)}{r}, \frac{sin(2\theta)}{r^2}, \frac{sin(3\theta)}{r^3}, \frac{sin(4\theta)}{r^4}, ln(r) \right\}$$

$$\theta_i = \left\{ 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{12\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9} \right\}$$

$$u(r,\theta) = \sum_{n=1}^{4} A_n \frac{cos(n\theta)}{r^n} + \sum_{n=1}^{4} B_n \frac{sin(n\theta)}{r^n} + C_1 \; ln(r)$$

Build $[K]\{a\} = \{b\}$

$$[K] = \begin{pmatrix} 1. & 1. & 1. & 1. & 0. & 0. & 0. & 0. & 0. \\ 0.766044 & 0.173648 & -0.5 & -0.939693 & 0.642788 & 0.984808 & 0.866025 & 0.34202 & 0. \\ 0.173648 & -0.939693 & -0.5 & 0.766044 & 0.984808 & 0.34202 & -0.866025 & -0.642788 & 0. \\ -0.5 & -0.5 & 1. & -0.5 & 0.866025 & -0.866025 & 0. & 0.866025 & 0. \\ -0.939693 & 0.766044 & -0.5 & 0.173648 & 0.34202 & -0.642788 & 0.866025 & -0.984808 & 0. \\ -0.939693 & 0.766044 & -0.5 & 0.173648 & -0.34202 & 0.642788 & -0.866025 & 0.984808 & 0. \\ -0.5 & -0.5 & 1. & -0.5 & -0.866025 & 0.866025 & 0. & -0.866025 & 0. \\ 0.173648 & -0.939693 & -0.5 & 0.766044 & -0.984808 & -0.34202 & 0.866025 & 0.642788 & 0. \\ 0.766044 & 0.173648 & -0.5 & -0.939693 & -0.642788 & -0.984808 & -0.866025 & -0.34202 & 0. \end{pmatrix}$$

$$\{a\} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1\}^T \qquad \{b\} = \begin{pmatrix} 1. \\ -0.5 \\ -0.5 \\ 1. \\ -0.5 \\ -0.5 \\ 1. \\ -0.5 \\ -0.5 \end{pmatrix}$$

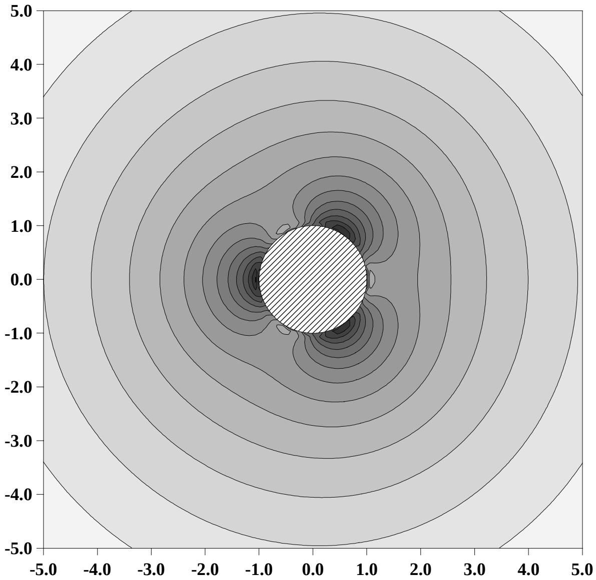
Because $[K]$ is singular, we solve $\{a\}$ by using Gaussian Elimination Method.

Finally, we get

$$\begin{pmatrix} 1. & 1. & 1. & 1. & 0. & 0. & 0. & 0. & 0. & 1. \\ 0. & 1.125 & 2.4043 & 3.23931 & -1.2207 & -1.87022 & -1.64464 & -0.649519 & 0. & 2.4043 \\ 0. & 0. & -4.90985 & -5.81715 & 3.35853 & 3.35853 & 5.14557 & 5.74074\times10^{-16} & 0. & -4.90985 \\ 0. & 0. & 0. & -19.8385 & 21.1211 & 1.78645 & 17.5482 & 9.66731 & 0. & 0. \\ 0. & 0. & 0. & 0. & 54.826 & -3.51861 & 16.7443 & -8.11183 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 3988.81 & 1301.76 & -4893. & 0. & 0. \\ 0. & 9.3441\times10^{-10} & -1.10019\times10^{-10} & 1.35425\times10^{-9} & 0. & 0. & -7.2076\times10^{6} & 8.63575\times10^{6} & 0. & -1.10019\times10^{-10} \\ 0. & 0.00106584 & -0.0000205228 & 0.00114416 & 0.000992586 & 0. & 0. & 8.41965\times10^{12} & 0. & -0.0000205228 \\ 0. & 0.000145478 & 0.000363694 & 0.00152751 & -0.000992586 & 0. & 0. & 0. & 0. & 0.000363694 \end{pmatrix}$$

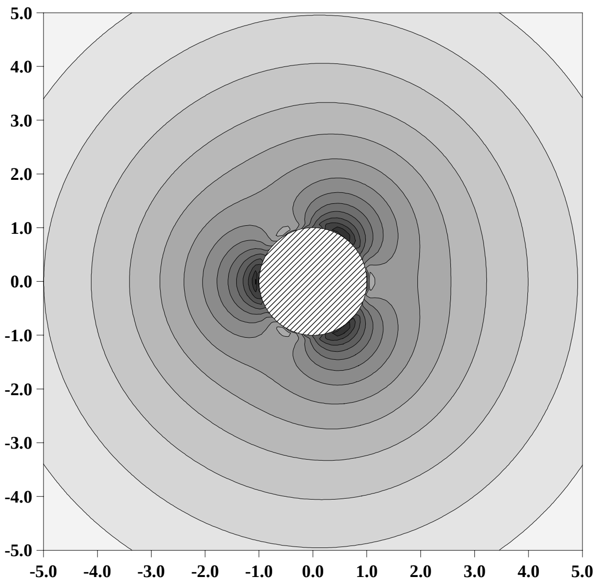
$$\{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1\}^T = \{0, 0, 1, 0, 0, 0, 0, 0, arbitrary\}^T$$

Suppose $C_1 = 1$



Trefftz solution (9 sets)

The exact solution $u(r,\theta) = \dfrac{cos(3\theta)}{r^3} + ln(r),\ \mathrm{r} \geq 1$



Exact solution (r=1)

(二)

(a) By choosing T-complete set：$\{ \frac{cos(\theta)}{r}, \frac{cos(2\theta)}{r^2}, \frac{sin(\theta)}{r}, \frac{sin(2\theta)}{r^2}, ln(r) \}$

$$\theta_i = \left\{ 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5} \right\}$$

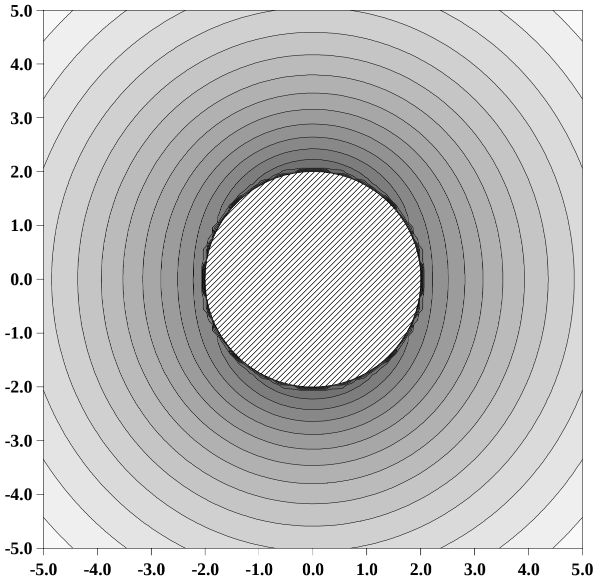
$$u(r,\theta) = \sum_{n=1}^{2} A_n \frac{cos(n\theta)}{r^n} + \sum_{n=1}^{2} B_n \frac{sin(n\theta)}{r^n} + C_1 \; ln(r)$$

Build $[K]\{a\} = \{b\}$

$$[K] = \begin{pmatrix} 0.5 & 0.25 & 0. & 0. & 0.693147 \\ 0.154508 & -0.202254 & 0.475528 & 0.146946 & 0.693147 \\ -0.404508 & 0.0772542 & 0.293893 & -0.237764 & 0.693147 \\ -0.404508 & 0.0772542 & -0.293893 & 0.237764 & 0.693147 \\ 0.154508 & -0.202254 & -0.475528 & -0.146946 & 0.693147 \end{pmatrix} \qquad \{b\} = \begin{pmatrix} 0.818147 \\ 0.59202 \\ 0.731774 \\ 0.731774 \\ 0.59202 \end{pmatrix}$$

Finally, we get $\{A_1, A_2, B_1, B_2, C_1\}^T = \{0, 0.5, 0, 0, 1\}^T$

The Trefftz solution $u(r,\theta) = \frac{1}{2}\frac{cos(2\theta)}{r^2} + ln(r)$



Trefftz solution (5 sets)

(b) By choosing T-complete set：

$$\left\{ \frac{cos(\theta)}{r}, \frac{cos(2\theta)}{r^2}, \frac{cos(3\theta)}{r^3}, \frac{cos(4\theta)}{r^4}, \frac{sin(\theta)}{r}, \frac{sin(2\theta)}{r^2}, \frac{sin(3\theta)}{r^3}, \frac{sin(4\theta)}{r^4}, ln(r) \right\}$$

$$\theta_i = \left\{ 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{12\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9} \right\}$$

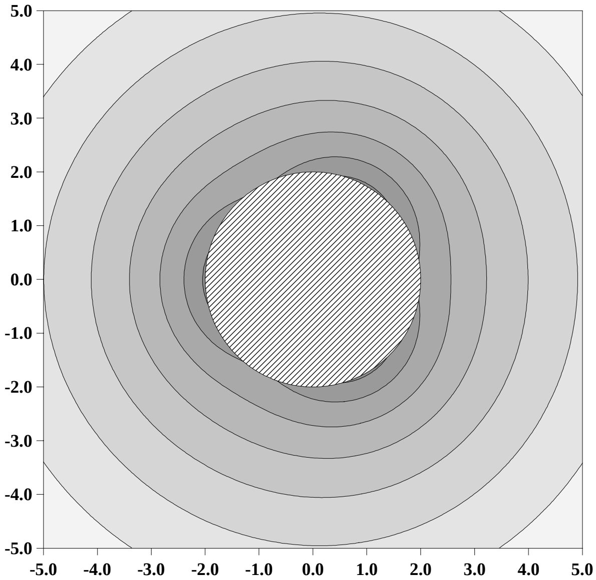
$$u(r,\theta) = \sum_{n=1}^{4} A_n \frac{cos(n\theta)}{r^n} + \sum_{n=1}^{4} B_n \frac{sin(n\theta)}{r^n} + C_1\ ln(r)$$

Build $[K]\{a\} = \{b\}$

$$[K] = \begin{pmatrix} 0.5 & 0.25 & 0.125 & 0.0625 & 0. & 0. & 0. & 0. & 0.693147 \\ 0.383022 & 0.043412 & -0.0625 & -0.0587308 & 0.321394 & 0.246202 & 0.108253 & 0.0213763 & 0.693147 \\ 0.0868241 & -0.234923 & -0.0625 & 0.0478778 & 0.492404 & 0.085505 & -0.108253 & -0.0401742 & 0.693147 \\ -0.25 & -0.125 & 0.125 & -0.03125 & 0.433013 & -0.216506 & 0. & 0.0541266 & 0.693147 \\ -0.469846 & 0.191511 & -0.0625 & 0.010853 & 0.17101 & -0.160697 & 0.108253 & -0.0615505 & 0.693147 \\ -0.469846 & 0.191511 & -0.0625 & 0.010853 & -0.17101 & 0.160697 & -0.108253 & 0.0615505 & 0.693147 \\ -0.25 & -0.125 & 0.125 & -0.03125 & -0.433013 & 0.216506 & 0. & -0.0541266 & 0.693147 \\ 0.0868241 & -0.234923 & -0.0625 & 0.0478778 & -0.492404 & -0.085505 & 0.108253 & 0.0401742 & 0.693147 \\ 0.383022 & 0.043412 & -0.0625 & -0.0587308 & -0.321394 & -0.246202 & -0.108253 & -0.0213763 & 0.693147 \end{pmatrix}$$

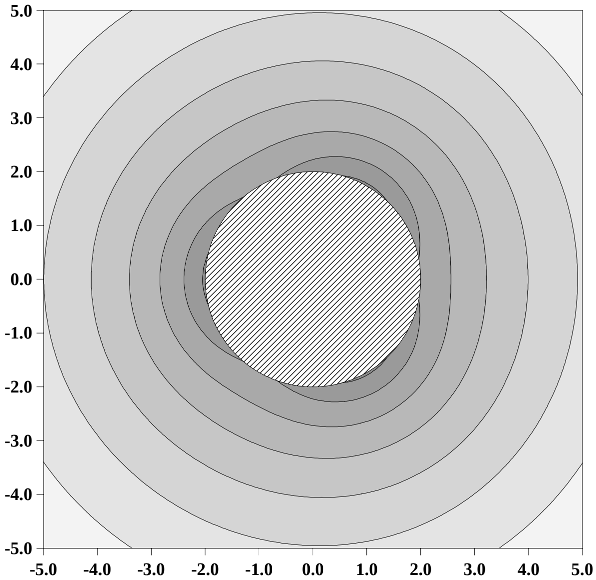
$$\{a\} = \{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1\}^T \qquad \{b\} = \begin{pmatrix} 0.818147 \\ 0.630647 \\ 0.630647 \\ 0.818147 \\ 0.630647 \\ 0.630647 \\ 0.818147 \\ 0.630647 \\ 0.630647 \end{pmatrix}$$

$$\{A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1\}^T = \{0,0,1,0,0,0,0,0,1\}^T$$



Trefftz solution (9 sets)

The exact solution $u(r,\theta) = \dfrac{cos(3\theta)}{r^3} + ln(r),\ \mathrm{r} \geq 2$



Exact solution (r=2)

討論：

(1) 由第一個 $r=1$ 的外域問題中，可知原本命題所提供之控制方程式與邊界條件所得到的解並不唯一，因此，可知不管利用什麼方法來求解此命題，其所求得的解也應該是不唯一的。

(2) 在 $r=1$ 的外域問題中，利用選點法所建立的 $[K]$ 為奇異方陣，這是因為在代入邊界條件時，$\ln(1)$ 會使得 $[K]$ 中每一個 C_1 的係數皆為 0，並且造成 $[K]$ 的 rank 下降，但是，透過高斯消去法我們可發現：當取 5 個基底函數來求解時，可得係數 C_1 為任意數且係數 A_2 為 1，而當取 9 個基底函數來求解時，亦可得係數 C_1 為任意數且係數 A_3 為 1。

(3) 在 $r=2$ 的外域問題中，我們可發現：利用選點法所建立的 $[K]$ 可以做 inverse，而當取 5 個基底函數來求解時，可得係數 C_1 為 1 且係數 A_2 為 0.5，而當取 9 個基底函數來求解時，可得係數 C_1 為 1 且係數 A_3 為 1。

(4) 從這兩個外域問題中，觀察其所求得的結果，我認為在使用 Trefftz 法時，若是所選擇的基底函數沒有缺項，可是項數卻取的不夠時，其數值上所反應出來的答案並不理想，而且答案會落在最接近真實答案的那一項上。可是如果項數取的足夠多時，答案就會蠻不錯的，該是那個係數有答案就是那個，一個也跑不掉。