班級：______________學號：_______________姓名：_______________

國立台灣海洋大學河海工程研究所 BEM 第 2 次作業解答(2004)

1. In the course, we have shown

$$\lim_{\varepsilon\to 0}\int_{-1}^{1}\frac{1}{x^2+\varepsilon^2}dx=-2$$

2. In calculating the hypersingular integral, we can determine the improper integral by using limiting process. The *M* kernel contains two parts, one is calculated, the other one is as follows:

$$\lim_{\varepsilon\to 0}\int_{a}^{b}\frac{2x^2}{(x^2+\varepsilon^2)^2}dx=0$$

Please determine the value using Symbolic software (Macsyma, Reduce, Mathematica, Maple……)

1.

(sol 1)

$$\int \frac{1}{x^2+y^2}\,dx$$

$$\frac{\mathrm{ArcTan}\left[\frac{x}{y}\right]}{y}$$

$$=\frac{\left(\frac{\pi}{2}-\mathrm{ArcTan}\left[\frac{y}{x}\right]\right)}{y}$$

when y → 0

$$\frac{\left(\frac{\pi}{2}-\mathrm{ArcTan}\left[\frac{y}{x}\right]\right)}{y}=\frac{\frac{\pi}{2}-\frac{y}{x}}{y}$$

$$\int_{-1}^{1}\frac{1}{x^2+y^2}\,dx,\ y\to 0$$

$$=-1-1=-2$$

(sol 2)

(Leibnitz rule)

$$\frac{d}{dx}\int_{a(x)}^{b(x)}f(x,t)dt=\int_{a(x)}^{b(x)}\frac{\partial\ f(x,t)}{\partial\ x}dt+f(x,b(x))b^{'}(x)-f(x,a(x))a^{'}(x)$$

$$\frac{d}{dx}\left\{C.P.V.\int_{a}^{c}\frac{-1}{(x-s)}ds\right\}$$

$$=\frac{d}{dx}\left\{\int_{a}^{x-\varepsilon}\frac{-1}{(x-s)}ds+\int_{x+\varepsilon}^{c}\frac{-1}{(x-s)}ds\right\}$$

$$=\frac{1}{a-x}-\frac{1}{c-x}$$

$$H.P.V.\int_{-1}^{1}\frac{1}{x^2}dx$$
$$=\lim_{\varepsilon\to 0}\int_{-1}^{1}\frac{1}{x^2+\varepsilon^2}dx$$
$$=\frac{d}{dx}\left\{C.P.V.\int_a^c\frac{-1}{x-s}ds\right\}$$
$$=\frac{1}{a-x}-\frac{1}{c-x}\quad(a=-1,\,c=1,\,x=0)$$
$$=-2$$

2.

(Sol)

Mathematica 5

$$\int\frac{2x^2}{(x^2+\epsilon^2)^2}\,dx$$

$$2\left(-\frac{x}{2(x^2+\epsilon^2)}+\frac{\mathrm{ArcTan}[\frac{x}{\epsilon}]}{2\epsilon}\right)$$

$$\mathrm{Limit}\left[\frac{-b}{b^2+\epsilon^2}+\frac{\frac{\pi}{2}-\mathrm{ArcTan}[\frac{\epsilon}{b}]}{\epsilon}+\frac{a}{a^2+\epsilon^2}-\frac{\frac{\pi}{2}-\mathrm{ArcTan}[\frac{\epsilon}{a}]}{\epsilon},\ \epsilon\to 0\right]$$

$$\frac{2}{a}-\frac{2}{b}$$

手算

$$\int_a^b\frac{2x^2}{(x^2+\epsilon^2)^2}\,dx=\frac{-b}{b^2+\epsilon^2}+\frac{\frac{\pi}{2}-\mathrm{ArcTan}[\frac{\epsilon}{b}]}{\epsilon}+\frac{a}{a^2+\epsilon^2}-\frac{\frac{\pi}{2}-\mathrm{ArcTan}[\frac{\epsilon}{a}]}{\epsilon}$$

$$\mathrm{Limit}_{\epsilon\to 0}\left[\frac{-b}{b^2+\epsilon^2}+\frac{\frac{\pi}{2}-\mathrm{ArcTan}[\frac{\epsilon}{b}]}{\epsilon}+\frac{a}{a^2+\epsilon^2}-\frac{\frac{\pi}{2}-\mathrm{ArcTan}[\frac{\epsilon}{a}]}{\epsilon}\right]=\frac{2}{a}-\frac{2}{b}$$

$a=-1$ & $b=1\to -4$