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$$M(R,\theta;\rho,\phi)=\begin{cases} M^I=\sum_{n=1}^{\infty}\frac{\rho^{n-1}}{R^{n+1}}\,n\cos(n(\theta-\phi)),\ R>\rho \\ M^E=\sum_{n=1}^{\infty}\frac{R^{n-1}}{\rho^{n+1}}\,n\cos(n(\theta-\phi)),\ R<\rho \end{cases}$$

$$\int_0^{2\pi} M^I(R,\theta;\rho,\phi)\sin(m\theta)\,d\theta$$

$$=\int_0^{2\pi}\left[\sum_{n=1}^{\infty}\frac{\rho^{n-1}}{R^{n+1}}\,n\cos(n(\theta-\phi))\right][\sin(m\theta)]\,d\theta$$

$$=\int_0^{2\pi}\left[\sum_{n=1}^{\infty}\frac{\rho^{n-1}}{R^{n+1}}\,n(\cos(n\theta)\cos(n\phi)+\sin(n\theta)\sin(n\phi))\right][\sin(m\theta)]\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{\rho^{m-1}}{R^{m+1}}\,m\sin(m\phi)\sin^2(m\theta)\right]d\theta \qquad \text{(週期函數性質, n = m才會有值)}$$

$$=\int_0^{2\pi}\left[\frac{\rho^{m-1}}{R^{m+1}}\,m\sin(m\phi)\left(\frac{1-\cos(2m\theta)}{2}\right)\right]d\theta$$

$$=\frac{\rho^{m-1}}{R^{m+1}}\,m\sin(m\phi)\left(\frac{1}{2}\theta-\frac{1}{4m}\sin(2m\theta)\right)\Bigg|_0^{2\pi}$$

$$=\frac{\rho^{m-1}}{R^{m+1}}\,m\sin(m\phi)\,\pi$$

- $$\int_0^{2\pi} M^I(\rho,\phi;\bar{\rho},\bar{\phi})\int_0^{2\pi} M^I(R,\theta;\rho,\phi)\sin(m\theta)\,d\theta\,d\phi$$

$$=\int_0^{2\pi}\left[\sum_{i=1}^{\infty}\frac{\bar{\rho}^{i-1}}{\rho^{i+1}}\,i\cos\left(i\left(\phi-\bar{\phi}\right)\right)\right]\left[\frac{\rho^{m-1}}{R^{m+1}}\,m\sin(m\phi)\,\pi\right]d\phi$$

$$=\int_0^{2\pi}\left[\sum_{i=1}^{\infty}\frac{\bar{\rho}^{i-1}}{\rho^{i+1}}\,i\left(\cos(i\phi)\cos\left(i\bar{\phi}\right)+\sin(i\phi)\sin\left(i\bar{\phi}\right)\right)\right]\left[\frac{\rho^{m-1}}{R^{m+1}}\,m\sin(m\phi)\,\pi\right]d\phi$$

$$=\int_0^{2\pi}\left[\frac{\bar{\rho}^{m-1}}{\rho^2 R^{m+1}}\,m^2\pi\sin\left(m\bar{\phi}\right)\sin^2(m\phi)\right]d\phi \qquad \text{(週期函數性質, i = m才會有值)}$$

$$=\frac{\bar{\rho}^{m-1}}{\rho^2 R^{m+1}}\,m^2\pi\sin\left(m\bar{\phi}\right)\left(\frac{1}{2}\phi-\frac{1}{4m}\sin(2m\phi)\right)\Bigg|_0^{2\pi}$$

$$=\frac{\bar{\rho}^{m-1}}{\rho^2 R^{m+1}}\,m^2\pi^2\sin\left(m\bar{\phi}\right)$$

$$=m^2\pi^2\sin\left(m\bar{\phi}\right)=-\pi^2\frac{d^2}{d\bar{\phi}^2}\left(\sin\left(m\bar{\phi}\right)\right) \qquad \left(\text{when } \rho=\bar{\rho}=R=1\right)$$

$$\boxed{M^I M^I(\sin(m\theta))=-\pi^2\frac{d^2}{d\theta^2}(\sin(m\theta))=m^2\pi^2\sin(m\theta)}$$

$$\int_0^{2\pi} M^I (R, \theta; \rho, \phi) \cos(m\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{\rho^{n-1}}{R^{n+1}}\, n \cos(n(\theta-\phi))\right][\cos(m\theta)]\, d\theta$$

$$= \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{\rho^{n-1}}{R^{n+1}}\, n (\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi))\right][\cos(m\theta)]\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\rho^{m-1}}{R^{m+1}}\, m \cos(m\phi)\cos^2(m\theta)\right] d\theta \qquad \text{(週期函數性質, } n = m \text{才會有值)}$$

$$= \int_0^{2\pi} \left[\frac{\rho^{m-1}}{R^{m+1}}\, m \cos(m\phi)\left(\frac{1+\cos(2m\theta)}{2}\right)\right] d\theta$$

$$= \frac{\rho^{m-1}}{R^{m+1}}\, m \cos(m\phi)\left(\frac{1}{2}\theta + \frac{1}{4m}\sin(2m\theta)\right)\Bigg|_0^{2\pi}$$

$$= \frac{\rho^{m-1}}{R^{m+1}}\, m \cos(m\phi)\,\pi$$

- $\int_0^{2\pi} M^I (\rho, \phi; \bar{\rho}, \bar{\phi}) \int_0^{2\pi} M^I (R, \theta; \rho, \phi) \cos(m\theta)\, d\theta\, d\phi$

$$= \int_0^{2\pi} \left[\sum_{i=1}^{\infty} \frac{\bar{\rho}^{i-1}}{\rho^{i+1}}\, i \cos\left(i\left(\phi-\bar{\phi}\right)\right)\right]\left[\frac{\rho^{m-1}}{R^{m+1}}\, m \cos(m\phi)\,\pi\right] d\phi$$

$$= \int_0^{2\pi} \left[\sum_{i=1}^{\infty} \frac{\bar{\rho}^{i-1}}{\rho^{i+1}}\, i \left(\cos(i\phi)\cos\left(i\bar{\phi}\right) + \sin(i\phi)\sin\left(i\bar{\phi}\right)\right)\right]\left[\frac{\rho^{m-1}}{R^{m+1}}\, m \cos(m\phi)\,\pi\right] d\phi$$

$$= \int_0^{2\pi} \left[\frac{\bar{\rho}^{m-1}}{\rho^2 R^{m+1}}\, m^2 \pi \cos\left(m\bar{\phi}\right)\cos^2(m\phi)\right] d\phi \quad \text{(週期函數性質, } i = m \text{才會有值)}$$

$$= \frac{\bar{\rho}^{m-1}}{\rho^2 R^{m+1}}\, m^2 \pi \cos\left(m\bar{\phi}\right)\left(\frac{1}{2}\phi + \frac{1}{4m}\sin(2m\phi)\right)\Bigg|_0^{2\pi}$$

$$= \frac{\bar{\rho}^{m-1}}{\rho^2 R^{m+1}}\, m^2 \pi^2 \cos\left(m\bar{\phi}\right)$$

$$= m^2 \pi^2 \cos\left(m\bar{\phi}\right) = -\pi^2 \frac{d^2}{d\bar{\phi}^2}\left(\cos\left(m\bar{\phi}\right)\right) \qquad \left(\text{when } \rho = \bar{\rho} = R = 1\right)$$

$$\boxed{M^I M^I (\cos(m\theta)) = -\pi^2 \frac{d^2}{d\theta^2}(\cos(m\theta)) = m^2 \pi^2 \cos(m\theta)}$$

$$U(R, \theta; \rho, \phi) = \begin{cases} U^I = \ln R - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\rho}{R}\right)^n \cos(n(\theta-\phi)), & R > \rho \\ U^E = \ln\rho - \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{R}{\rho}\right)^n \cos(n(\theta-\phi)), & R < \rho \end{cases}$$

$$\int_0^{2\pi} U^I (R, \theta; \rho, \phi) \sin(m\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[\ln R - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\rho}{R}\right)^n \cos(n(\theta - \phi))\right] [\sin(m\theta)]\, d\theta$$

$$= \int_0^{2\pi} [\ln R \sin(m\theta)] - \left[\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\rho}{R}\right)^n \cos(n(\theta - \phi)) \sin(m\theta)\right] d\theta$$

$$= -\frac{1}{m} \ln R \cos(m\theta) \Big|_0^{2\pi} - \frac{1}{m} \left(\frac{\rho}{R}\right)^m \sin(m\phi) \left(\frac{1}{2}\theta - \frac{1}{4m} \sin(2m\theta)\right) \Big|_0^{2\pi}$$

$$= -\frac{1}{m} \left(\frac{\rho}{R}\right)^m \sin(m\phi)\, \pi$$

- $$\int_0^{2\pi} U^I\left(\rho, \phi; \bar{\rho}, \bar{\phi}\right) \int_0^{2\pi} U^I (R, \theta; \rho, \phi) \sin(m\theta)\, d\theta\, d\phi$$

$$= \int_0^{2\pi} \left[\ln\rho - \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{\bar{\rho}}{\rho}\right)^i \cos\left(i\left(\phi - \bar{\phi}\right)\right)\right] \left[-\frac{1}{m} \left(\frac{\rho}{R}\right)^m \sin(m\phi)\, \pi\right] d\phi$$

$$= \int_0^{2\pi} \left[-\ln\rho \frac{1}{m} \left(\frac{\rho}{R}\right)^m \sin(m\phi)\, \pi\right] + \left[\frac{1}{m^2} \left(\frac{\bar{\rho}}{R}\right)^m \cos\left(m\left(\phi - \bar{\phi}\right)\right) \sin(m\phi)\, \pi\right] d\phi$$

$$= \ln\rho \frac{1}{m^2} \left(\frac{\rho}{R}\right)^m \cos(m\phi)\, \pi \Big|_0^{2\pi} + \frac{1}{m^2} \left(\frac{\bar{\rho}}{R}\right)^m \sin\left(m\bar{\phi}\right) \left(\frac{1}{2}\phi - \frac{1}{4m} \sin(2m\phi)\right) \Big|_0^{2\pi} \pi$$

$$= \frac{1}{m^2} \left(\frac{\bar{\rho}}{R}\right)^m \sin\left(m\bar{\phi}\right) \pi^2$$

$$= \frac{1}{m^2} \pi^2 \sin\left(m\bar{\phi}\right) = \pi^2 \int\int \sin\left(m\bar{\phi}\right) d\theta\, d\phi \qquad \left(\text{when } \rho = \bar{\rho} = R = 1\right)$$

$$\boxed{U^I U^I (\sin(m\theta)) = -\pi^2 \int\int \sin(m\theta)\, d\theta\, d\phi = \frac{1}{m^2} \pi^2 \sin(m\theta)}$$

$$T(R, \theta; \rho, \phi) = \begin{cases} T^I = \frac{1}{R} + \sum_{n=1}^{\infty} \frac{\rho^n}{R^{n+1}} \cos(n(\theta - \phi)),\ R > \rho \\ T^E = -\sum_{n=1}^{\infty} \frac{R^{n-1}}{\rho^n} \cos(n(\theta - \phi)),\ R < \rho \end{cases}$$

$$\int_0^{2\pi} T^E (R, \theta; \rho, \phi) \sin(m\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[-\sum_{n=1}^{\infty} \frac{R^{n-1}}{\rho^n} \cos(n(\theta - \phi))\right] [\sin(m\theta)]\, d\theta$$

$$= -\frac{R^{m-1}}{\rho^m} \sin(m\phi) \left(\frac{1}{2}\theta - \frac{1}{4m} \sin(2m\theta)\right) \Big|_0^{2\pi}$$

$$= -\frac{R^{m-1}}{\rho^m} \sin(m\phi)\, \pi$$

- $\int_0^{2\pi} \mathbf{T}^{\mathbf{I}}\left(\rho,\ \phi;\ \bar{\rho},\ \bar{\phi}\right) \int_0^{2\pi} \mathbf{T}^{\mathbf{E}}\ (\mathbf{R},\ \theta;\ \rho,\ \phi)\ \sin(m\theta)\ d\theta\, d\phi$

$$= \int_0^{2\pi} \left[\frac{1}{\rho} + \sum_{i=1}^{\infty} \frac{\bar{\rho}^{i}}{\rho^{i+1}} \cos\left(i\left(\phi - \bar{\phi}\right)\right)\right]\left[-\frac{R^{m-1}}{\rho^m} \sin(m\phi)\ \pi\right] d\phi$$

$$= \int_0^{2\pi} \left[-\frac{1}{\rho}\ \frac{R^{m-1}}{\rho^m} \sin(m\phi)\ \pi\right] - \left[\frac{\bar{\rho}^{m} R^{m-1}}{\rho^{2m+1}} \cos\left(m\left(\phi - \bar{\phi}\right)\right) \sin(m\phi)\ \pi\right] d\phi$$

$$= \frac{1}{\rho}\ \frac{1}{m}\ \frac{R^{m-1}}{\rho^m} \cos(m\phi)\ \pi \Big|_0^{2\pi} - \frac{\bar{\rho}^{m} R^{m-1}}{\rho^{2m+1}} \sin\left(m\bar{\phi}\right) \left(\frac{1}{2}\phi - \frac{1}{4m}\sin(2m\phi)\right)\Big|_0^{2\pi} \pi$$

$$= -\frac{\bar{\rho}^{m} R^{m-1}}{\rho^{2m+1}} \sin\left(m\bar{\phi}\right) \pi^2$$

$$= \boxed{-\pi^2 \sin\left(m\bar{\phi}\right)} \qquad \left(\text{when } \rho = \bar{\rho} = R = 1\right)$$

$$L\ (R,\ \theta;\ \rho,\ \phi) = \begin{cases} L^{I} = -\sum_{n=1}^{\infty} \frac{\rho^{n-1}}{R^n} \cos(n(\theta - \phi)),\ R > \rho \\ L^{E} = \frac{1}{\rho} + \sum_{n=1}^{\infty} \frac{R^n}{\rho^{n+1}} \cos(n(\theta - \phi)),\ R < \rho \end{cases}$$

$$\int_0^{2\pi} L^{E}\ (R,\ \theta;\ \rho,\ \phi)\ \sin(m\theta)\ d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{\rho} + \sum_{n=1}^{\infty} \frac{R^n}{\rho^{n+1}} \cos(n(\theta - \phi))\right] [\sin(m\theta)]\ d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{\rho} \sin(m\theta)\right] + \left[\frac{R^m}{\rho^{m+1}} \cos(m(\theta - \phi)) \sin(m\theta)\right] d\theta$$

$$= -\frac{1}{m}\ \frac{1}{\rho} \cos(m\theta) \Big|_0^{2\pi} + \frac{R^m}{\rho^{m+1}} \sin(m\phi) \left(\frac{1}{2}\theta - \frac{1}{4m}\sin(2m\theta)\right)\Big|_0^{2\pi}$$

$$= \frac{R^m}{\rho^{m+1}} \sin(m\phi)\ \pi$$

- $\int_0^{2\pi} \mathbf{L}^{\mathbf{I}}\left(\rho,\ \phi;\ \bar{\rho},\ \bar{\phi}\right) \int_0^{2\pi} \mathbf{L}^{\mathbf{E}}\ (\mathbf{R},\ \theta;\ \rho,\ \phi)\ \sin(m\theta)\ d\theta\, d\phi$

$$= \int_0^{2\pi} \left[-\sum_{i=1}^{\infty} \frac{\bar{\rho}^{i-1}}{\rho^{i}} \cos\left(i\left(\phi - \bar{\phi}\right)\right)\right]\left[\frac{R^{m}}{\rho^{m+1}} \sin(m\phi)\ \pi\right] d\phi$$

$$= \int_0^{2\pi} \left[-\frac{\bar{\rho}^{m-1} R^{m}}{\rho^{2m+1}} \cos\left(m\left(\phi - \bar{\phi}\right)\right) \sin(m\phi)\ \pi\right] d\phi$$

$$= -\frac{\bar{\rho}^{m-1} R^{m}}{\rho^{2m+1}} \sin\left(m\bar{\phi}\right) \pi \left(\frac{1}{2}\phi - \frac{1}{4m}\sin(2m\phi)\right)\Big|_0^{2\pi}$$

$$= -\frac{\bar{\rho}^{m-1} R^{m}}{\rho^{2m+1}} \sin\left(m\bar{\phi}\right) \pi^2$$

$$= \boxed{-\pi^2 \sin\left(m\bar{\phi}\right)} \qquad \left(\text{when } \rho = \bar{\rho} = R = 1\right)$$

$$\int_0^{2\pi} M^I (R, \theta; \rho, \phi) \sin(m\theta)\, d\theta$$

$$= \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{\rho^{n-1}}{R^{n+1}} n \cos(n(\theta - \phi))\right] [\sin(m\theta)]\, d\theta$$

$$= \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \frac{\rho^{n-1}}{R^{n+1}} n (\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi))\right] [\sin(m\theta)]\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{\rho^{m-1}}{R^{m+1}} m \sin(m\phi) \sin^2(m\theta)\right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{\rho^{m-1}}{R^{m+1}} m \sin(m\phi) \left(\frac{1 - \cos(2m\theta)}{2}\right)\right] d\theta$$

$$= \frac{\rho^{m-1}}{R^{m+1}} m \sin(m\phi) \left(\frac{1}{2}\theta - \frac{1}{4m}\sin(2m\theta)\right)\Big|_0^{2\pi}$$

$$= \frac{\rho^{m-1}}{R^{m+1}} m \sin(m\phi)\, \pi$$

- $$\int_0^{2\pi} U^I (\rho, \phi; \bar{\rho}, \bar{\phi}) \int_0^{2\pi} M^I (R, \theta; \rho, \phi) \sin(m\theta)\, d\theta\, d\phi$$

$$= \int_0^{2\pi} \left[\ln\rho - \sum_{i=1}^{\infty} \frac{1}{i} \left(\frac{\bar{\rho}}{\rho}\right)^i \cos\left(i\left(\phi - \bar{\phi}\right)\right)\right] \left[\frac{\rho^{m-1}}{R^{m+1}} m \sin(m\phi)\, \pi\right] d\phi$$

$$= \int_0^{2\pi} \left[\ln\rho \frac{\rho^{m-1}}{R^{m+1}} m \sin(m\phi)\, \pi\right] - \left[\bar{\rho}^m \rho^{m-2} R^{m+1} \cos\left(m\left(\phi - \bar{\phi}\right)\right) \sin(m\phi)\, \pi\right] d\phi$$

$$= -\ln\rho \frac{\rho^{m-1}}{R^{m+1}} \cos(m\phi)\, \pi\Big|_0^{2\pi} - \bar{\rho}^m \rho^{m-2} R^{m+1} \sin\left(m\bar{\phi}\right) \left(\frac{1}{2}\phi - \frac{1}{4m}\sin(2m\phi)\right)\Big|_0^{2\pi} \pi$$

$$= -\bar{\rho}^m \rho^{m-2} R^{m+1} \sin\left(m\bar{\phi}\right) \pi^2$$

$$= \boxed{-\pi^2 \sin\left(m\bar{\phi}\right)} \qquad \left(\text{when } \rho = \bar{\rho} = R = 1\right)$$

Calderon projector : $T^I T^E = U M$
$L^I L^E = M U$