BEM H.W .007 M93520010 陳柏源

$\nabla^2 U(x, s) = \delta(x - s)$

ANS.

$\nabla^2 U(x, s) = \delta(x - s)$

右式 $\rightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} (U(r, \theta, \phi))$

$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \mathrm{Sin}[\theta]} \frac{\partial}{\partial \theta} \left(\mathrm{Sin}[\theta] \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \mathrm{Sin}^2[\theta]} \left(\frac{\partial^2 U}{\partial \phi^2} \right)$

θ, ϕ不影響

$= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right)$

$= \frac{1}{r^2} \left(2 r \frac{\partial U}{\partial r} + r^2 \frac{\partial^2 U}{\partial r^2} \right)$

令等於零 (求補解)

$\frac{2}{r} \frac{\partial U[r]}{\partial r} + \frac{\partial^2 U[r]}{\partial r^2} = 0. \ldots\ldots (1)$

令$r = e^t$, $t = \mathrm{Log}[r]$, $dt = \frac{1}{r} dr$

$U[r] = U[e^t] = u[t]$

$U' = \frac{du}{dt} \frac{dt}{dr} = u' \frac{1}{r}$

$U'' = \frac{1}{r^2} (u'' - u')$

(1) 式

$\rightarrow \frac{1}{r^2} (u'' - u') + \frac{2}{r} u' \frac{1}{r} = 0$

$\rightarrow u'' - u' + 2 u' = 0$

$\rightarrow u'' + u' = 0$

$e^{\lambda t}$ 代入

$\lambda = 0, -1$

$U(e^t) = u(t) = C_1 e^{\lambda t} + C_2 e^{\lambda t}$

$\rightarrow U(r) = C_1 e^{\lambda \mathrm{Log}[r]} + C_2 e^{\lambda \mathrm{Log}[r]}$

$\rightarrow U(r) = C_1 e^{0 * \mathrm{Log}[r]} + C_2 e^{-1 * \mathrm{Log}[r]}$

$= C_1 + C_2 * \frac{1}{r} \ldots\ldots$ (C_1 : 剛體運動項)

$= \frac{C_2}{r} \ldots\ldots\ldots$ (補解)

$U''[r] + \frac{2}{r} U[r] = \delta(x - s) \ldots\ldots$ (特解)

$\int\int\int \nabla^2 U(x, s)\, dD_v = \int\int\int \delta(x - s)\, dD_v$

右式 $\rightarrow \int\int\int \nabla^2 U(x, s)\, dD_v = \int\int \nabla(\nabla . n)\, dD_a = \int\int \frac{\partial U}{\partial n} dD_a$

$\int_0^{2\pi} \int_0^{\pi} \frac{\partial U}{\partial r} d\theta\, d\phi = \int_0^{2\pi} \int_0^{\pi} \frac{-C_2}{r^2} (r \,\mathrm{Sin}[\theta])\, r\, d\theta\, d\phi$

$= -C_2 \int_0^{2\pi} \int_0^{\pi} \mathrm{Sin}[\theta]\, d\theta\, d\phi$

$= -C_2 \int_0^{2\pi} -\mathrm{Cos}[\theta] \Big|_0^{\pi} d\phi$

$$= -C_2 \left(2\phi \Big|_{0}^{2\pi} \right)$$

$$= -4\pi * C_2$$

左式 → $\iiint \delta\,(x - s)\; d\underset{v}{D} = 1$

$-4\pi * C_2 = 1$

$C_2 = -\frac{1}{4\pi}$

$U = -\frac{1}{4\pi}\;\frac{1}{r}$ ☺