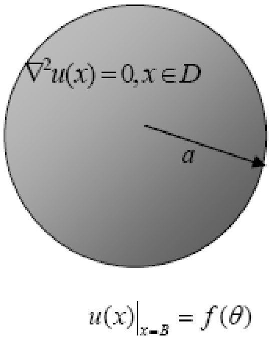
BEM H.W .010 M93520010 陳柏源 [***HW010.nb***]

Please extend to interior case

$$u(\rho, \phi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - \rho^2}{a^2 + \rho^2 - 2a\rho\cos(\phi - \theta)} f[\theta]\, d\theta, \quad 0 < \rho < a, \quad 0 < \phi < 2\pi$$



ANS.

$$2\pi u(x) = \int_B T_G(s, x)\, u(s)\, dB(s) - \int_B U_G(s, x)\, t(s)\, dB(s)$$

when $\nabla^2 U_G(s, x) = \delta(s, x)$ and $U_G(s, x)\,|_{s=B} = 0$

$\underset{\sim}{x} = (\rho, \phi) = (\rho\cos\phi, \rho\sin\phi)$

$\underset{\sim}{s} = (R, \theta) = (R\cos\theta, R\sin\theta)$

$\underset{\sim}{s}' = (R', \theta) = (R'\cos\theta, R'\sin\theta) = \left(\frac{a^2}{R}\cos\theta, \frac{a^2}{R}\sin\theta\right)$

$U_G(s, x)\,|_{x=B} = 0 \qquad (R = a)$

$\Rightarrow U_G(s, x) = \ln|x - s| - \ln|x - s'| - c = 0$

$\Rightarrow c = \ln|x - s| - \ln|x - s'|\ ;$

$$c = \ln\sqrt{(\rho\cos\phi - R\cos\theta)^2 + (\rho\sin\phi - R\sin\theta)^2} - \ln\sqrt{(\rho\cos\phi - R'\cos\theta)^2 + (\rho\sin\phi - R'\sin\theta)^2}$$

$$= \ln\sqrt{(\rho\cos\phi - R\cos\theta)^2 + (\rho\sin\phi - R\sin\theta)^2} - \ln\sqrt{\left(\rho\cos\phi - \frac{a^2}{R}\cos\theta\right)^2 + \left(\rho\sin\phi - \frac{a^2}{R}\sin\theta\right)^2}$$

$$= \frac{1}{2}\ln\left|a^2 + R^2 - 2\rho R(\cos\phi\cos\theta + \sin\phi\sin\theta)\right|$$
$$- \frac{1}{2}\ln\left|\rho^2 + \frac{a^4}{R^2} - \frac{2\rho a^2}{R}(\cos\phi\cos\theta + \sin\phi\sin\theta)\right|$$

$$= \frac{1}{2}\ln\left|a^2 + R^2 - 2\rho R\cos(\phi - \theta)\right| - \frac{1}{2}\ln\left|\frac{\rho^2 R^2 + a^4 - 2\rho a^2 R\cos(\phi - \theta)}{R^2}\right|$$

$U_G(s, x)\,|_{x=B} = 0, \ (\rho = a)$

$$c = \frac{1}{2}\ln\left|a^2 + R^2 - 2aR\cos(\phi - \theta)\right| - \frac{1}{2}\ln\left|\frac{a^2(R^2 + a^2 - 2aR\cos(\phi - \theta))}{R^2}\right|$$

$$= \frac{1}{2}\ln\left|a^2 + R^2 - 2aR\cos(\phi - \theta)\right|$$
$$- \left(\frac{1}{2}\ln\left|a^2\right| + \frac{1}{2}\ln\left|a^2 + R^2 - 2aR\cos(\phi - \theta)\right| - \frac{1}{2}\ln\left|R^2\right|\right)$$

$$= \frac{1}{2}\ln\left|R^2\right| - \frac{1}{2}\ln\left|a^2\right|$$

$$U_G(s,x) = \ln|x-s| - \ln|x-s'| - c$$
$$= \ln|r| - \ln|r'| - \ln|R| + \ln|a|$$

$$T_G(s,x) = \frac{\partial}{\partial R} U_G(s,x)$$

$$= \frac{\partial}{\partial R}(\ln|r| - \ln|r'| - \ln|R| + \ln|a|)$$
$$= \frac{\partial}{\partial R}\left(\ln\sqrt{(\rho\cos\phi - R\cos\theta)^2 + (\rho\sin\phi - R\sin\theta)^2} - \ln\sqrt{(\rho\cos\phi - R'\cos\theta)^2 + (\rho\sin\phi - R'\sin\theta)^2} - \ln\left|R\right| + \ln\left|a\right|\right)$$
$$= \frac{\partial}{\partial R}\left(\frac{1}{2}\ln\left|a^2 + R^2 - 2\rho R(\cos\phi\cos\theta + \sin\phi\sin\theta)\right| - \frac{1}{2}\ln\left|\rho^2 + \frac{a^4}{R^2} - \frac{2\rho a^2}{R}(\cos\phi\cos\theta + \sin\phi\sin\theta)\right| - \ln\left|R\right| + \ln\left|a\right|\right)$$
$$= \frac{\partial}{\partial R}\left(\frac{1}{2}\ln\left|a^2 + R^2 - 2\rho R\cos(\phi-\theta)\right| - \frac{1}{2}\ln\left|\rho^2 + \frac{a^4}{R^2} - \frac{2\rho a^2}{R}\cos(\phi-\theta)\right| - \ln\left|R\right| + \ln\left|a\right|\right)$$
$$= \frac{1}{2}\frac{2R - 2\rho\cos(\phi-\theta)}{a^2 + R^2 - 2\rho R\cos(\phi-\theta)} - \frac{1}{2}\frac{R^2}{\rho^2 R^2 + a^4 - 2\rho a^2 R\cos(\phi-\theta)}\left(-\frac{2a^2(a^2 + \mathrm{Cos}\,R\rho(\theta-\phi))}{R^3}\right) - \frac{1}{R}$$
$$= \frac{1}{2}\frac{2R - 2aR\cos(\phi-\theta)}{a^2 + R^2 - 2aR\cos(\phi-\theta)} + \frac{a^2(a^2 + \mathrm{Cos}\,R\rho(\theta-\phi))}{R(\rho^2 R^2 + a^4 - 2\rho a^2 R\cos(\phi-\theta))} - \frac{1}{R}$$

$$2\pi u(x) = \int_B T_G(s,x)\,u(s)\,dB(s) - \int_B U_G(s,x)\,t(s)\,dB(s)$$
$$= \int_{s=(a,0)}^{s=(a,2\pi)} T_G(s,x)\,u(s)\,dB(s) - \int_B 0 * t(s)\,dB(s)$$
$\Rightarrow R = a$代入
$$= \int_0^{2\pi}\left(\frac{a - \rho\cos(\phi-\theta)}{\rho^2 + a^2 - 2\rho a\cos(\phi-\theta)} - \frac{-a^3 + \rho a^2\cos(\phi-\theta)}{\rho^2 a^2 + a^4 - 2\rho a^3\cos(\phi-\theta)} - \frac{1}{a}\right) f(\theta)\,a\,d\theta$$
$$= \int_0^{2\pi}((a^3 - a^2\rho\cos(\phi-\theta) + a^3 - \rho a^2\cos(\phi-\theta) - \rho^2 a - a^3 + 2\rho a^2\cos(\phi-\theta)) / (\rho^2 a^2 + a^4 - 2\rho a^3\cos(\phi-\theta)))\,f(\theta)\,a\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{a^3 - \rho^2 a}{\rho^2 a^2 + a^4 - 2\rho a^3\cos(\phi-\theta)}\right) f(\theta)\,a\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{a(a^2 - \rho^2)}{a^2(\rho^2 + a^2 - 2\rho a\cos(\phi-\theta))}\right) f(\theta)\,a\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{(a^2 - \rho^2)}{a(\rho^2 + a^2 - 2\rho a\cos(\phi-\theta))}\right) f(\theta)\,a\,d\theta$$
$$= \int_B \frac{a^2 - \rho^2}{\rho^2 + a^2 - 2\rho a\cos(\phi-\theta)}\,f(\theta)\,d\theta$$