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BOUNDARY INTEGRAL EQUATION AND THE EXISTENCE THEOREMS IN CONTACT PROBLEMS WITH FRICTION

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ABSTRACT

The mathematical statements for contact problems with unilateral restrictions and friction for elastic bodies with cracks are investigated. The unilateral restrictions and friction on the crack surfaces are formulated in the form of inequalities. An algorithm for the solution of this problem is considered as the superposition of the boundary integral operators and projection operators on the sets of the unilateral restrictions and friction. The theorems of existence and uniqueness are proved.

1. INTRODUCTION

Various mathematical aspects of problems with conditions in the form of inequalities with application in mechanics and physics were investigated by Duvaut and Lions (1972), Glowinski, Lions and Tremolieres (1981), Hlavacek, Haslinger, Necas and Lovisec (1988), Kikuchi and Oden (1987), Panagiotopoulos (1985), etc. Usually, for the mathematical investigation of such problems, variational inequalities methods are used. Duvaut and Lions (1972), Hlavacek, Haslinger, Necas and Lovisec (1988), and Panagiotopoulos (1985) have published mathematical investigations of existence and uniqueness in contact problems with friction. Different algorithms for the numerical solution of the problems with inequalities in mechanics and engineering were considered by Aliabadi and Brebbia (1993), Glowinski, Lions and Tremolieres (1981). Usually, such problems were solved using finite element (Kikuchi and Oden, 1987) or boundary element (Antes and Panagiotopoulos, 1992; Guz' and Zozulya, 1993) methods.

Unilateral contact problems with friction, in the theory of elasticity, have been investigated in numerous publications. In spite of this, some theoretical and applied problems have not yet been solved. For example, the existence and the uniqueness theorems for unilateral contact problems with friction. The information available on this field was presented in (Duvaut and Lions, 1972; Glowinski *et al.*, 1981; Hlavacek *et al.*, 1988; Kikuchi and Oden, 1987; Panagiotopoulos, 1985).

Elastodynamic contact problems with unilateral restrictions and friction for bodies with cracks have been investigated in (Guz' and Zozulya, 1993; 1995;

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Zozulya, 1990; 1992; Zozulya and Rivera, 1998). Especially, the mathematical statement and the solution algorithm for such problems have been presented by Zozulya (1990). The variational formulation and appropriate boundary functionals have also been investigated by Zozulya (1992). An approach with boundary integral equations has been used by Zozulya (1992). The results developed in these and other of our papers have been summarized and presented in the monograph Guz' and Zozulya (1993) in Russian and in the review Guz' and Zozulya (1995) in English.

This paper deals with problems related to the statement and the mathematical investigation of contact problems with unilateral restrictions and friction for bodies with cracks. In particular, it is shown that the boundary conditions in the form of inequalities can be considered as the projection operators on sets of unilateral restrictions and on sets where the conditions of friction take place. The problem under consideration is transformed into a system of boundary integral equations and the above mentioned projection operators. The algorithm for the solution is considered as the superposition of the projection and boundary integral operators that act in the appropriate functional spaces. Here, the results presented in publications mentioned before have been used.

II. STATEMENT OF THE PROBLEM

Let's consider an elastic body in the three-dimensional Euclidean space R^3 that occupies a volume V. The boundary of the body ∂V is a piecewise smooth and consists of two parts ∂V_u and ∂V_p, where displacements $\boldsymbol{u}(\boldsymbol{x})$ and surface traction $\boldsymbol{p}(\boldsymbol{x})$ vectors are assigned respectively. There are N arbitrarily oriented cracks in the body, which are described by their surfaces $\Omega_n^+ \cup \Omega_n^-$, where Ω_n^+ and Ω_n^- are the opposite edges. The body may be affected by the body forces $\boldsymbol{b}(\boldsymbol{x})$.

The displacement linear equations of elasticity with mixed boundary conditions may be presented in the form

$$A_{ij}u_j+b_i=0,\ \forall \boldsymbol{x}\in V,\ A_{ij}=\mu\delta_{ij}\partial_k\partial_k+(\lambda+\mu)\partial_i\partial_j.$$

$$p_i(\boldsymbol{x})=\sigma_{ij}(\boldsymbol{x})n_j(\boldsymbol{x})=\psi_i(\boldsymbol{x}),\ \forall \boldsymbol{x}\in \partial V_p,$$

$$u_i(\boldsymbol{x})=\varphi_i(\boldsymbol{x}),\ \forall \boldsymbol{x}\in \partial V_u \qquad (1)$$

In these formulas the following indications are introduced: $\partial_i=\partial/\partial x_i$ are derivatives with respect to coordinates, ρ is the density of the material, λ and μ are the Lame constants, δ_{ij} is the Kroneker's symbol, $\psi_i(\boldsymbol{x})$ and $\varphi_i(\boldsymbol{x})$ are functions that describe the boundary conditions on the parts ∂V_p and ∂V_u respectively. The summation convention applies to repeated indices.

Let us formulate the conditions which must be satisfied on the crack surfaces. The mutual displacements of the crack surfaces are characterized by the displacement discontinuity vector (Guz' and Zozulya, 1993).

$$\Delta\boldsymbol{u}(\boldsymbol{x})=\boldsymbol{u}^+(\boldsymbol{x})-\boldsymbol{u}^-(\boldsymbol{x})\quad \forall \boldsymbol{x}\in\Omega^+\cup\Omega^-,$$

$$\Omega^+=\bigcup_{n=1}^{N}\Omega_n^+,\quad \Omega^-=\bigcup_{n=1}^{N}\Omega_n^-$$

The contact interaction forces at the crack edges are connected with the components of the stress tensor by the relations

$$q_i(\boldsymbol{x})=-\sigma_{ij}(\boldsymbol{x})n_j(\boldsymbol{x}),\ q_i(\boldsymbol{x})=q_i^+(\boldsymbol{x})=-q_i^-(\boldsymbol{x}),$$

$$\forall \boldsymbol{x}\in\Omega^+\cup\Omega^-\cap\Omega_e$$

where $\Omega_e=\bigcup_{n=1}^{N}\Omega_n^e=\bigcup_{n=1}^{N}\Omega_n^+\cap\bigcup_{n=1}^{N}\Omega_n^-=\Omega^+\cap\Omega^-$ are the close contact areas of the cracks edges and $n_j^+(\boldsymbol{x})=-n_j^-(\boldsymbol{x})$ are the vectors normal to the surfaces Ω^+ and Ω^-.

The contact between the crack surfaces is supposed to be a unilateral, so, the normal component of the contact force vector cannot be a tensile one. It is usually assumed that friction occurs in accordance with Coulomb's law (Duvavt and Lions, 1972; Guz' and Zozulya, 1993; 1995; Hlavacek *et al.*, 1988; Kikuchi and Oden, 1987; Panagiotopoulos, 1985).

Taking into account all the considerations presented above for the contact forces of the crack surfaces interactions and the displacements discontinuity vectors, the one sided restrictions with friction in the form of inequalities on the crack surfaces take the form (Duravt and Lions, 1972; Guz' and Zozulya, 1993; Hlavacek *et al.*, 1988; Panagiotopoulos, 1985)

$$\Delta u_n\geq h_o,\ q_n\geq 0,\ (\Delta u_n-h_o)q_n=0,$$

$$|\boldsymbol{q}_\tau|\leq k_\tau q_n\rightarrow\Delta\boldsymbol{u}_\tau=0,\ |\boldsymbol{q}_\tau|=k_\tau q_n\rightarrow\Delta\boldsymbol{u}_\tau=-\lambda_t\boldsymbol{q},$$

$$\forall \boldsymbol{x}\in\Omega^+\cup\Omega^- \qquad (2)$$

where q_n, $\boldsymbol{q}_\tau$, Δu_n, $\Delta\boldsymbol{u}_\tau$ are the normal and tangential components of the contact forces and the displacement discontinuity vectors respectively, h_o is the initial opening of the cracks, k_τ and λ_τ are coefficients dependent on the contact surfaces properties.

In Zozulya and Rivera (1998), Guz' and Zozulya (1993) it was demonstrated that for the solution of contact problems with unilateral restrictions for elastic bodies with cracks a direct boundary integral equations method is very efficient.

In order to obtain an integral representation for the components of the surface displacement and traction vectors on the body ∂V and crack surfaces Ω, the Somiliano identity and the properties of the surface potentials on the boundary and crack surfaces are used. On the smooth parts of the boundary and the crack surfaces these integral representations have the following form

$$\frac{1}{2}u_i(x) = \int_{\partial V} [p_j(y)U_{ji}(y-x) - u_j(y)W_{ji}(y,x)]dS$$

$$-\int_\Omega \Delta u_j(y)W_{ji}(y,x)d\Omega + \int_V b_j(y)U_{ji}(y-x)dV,$$

$$\forall x \in \partial V$$

$$\frac{1}{2}p_i(x) = \int_{\partial V} [p_j(y)K_{ji}(y,x) - u_j(y)F_{ji}(y,x)]dS$$

$$-\int_\Omega \Delta u_j(y)K_{ji}(y,x)d\Omega + \int_V b_j(y)K_{ji}(y,x)dV,$$

$$\forall x \in \partial V$$

$$p_i(x) = \int_{\partial V} [p_j(y)K_{ji}(y,x) - u_j(y)F_{ji}(y,x)]dS$$

$$-\int_\Omega \Delta u_j(y)K_{ji}(y,x)d\Omega + \int_V b_j(y)K_{ji}(y,x)dV,$$

$$\forall x \in \Omega \qquad (3)$$

The kernels in these integral representations are the fundamental solutions of the elasticity theory. Any book regarding the boundary element methods contains the expressions for the kernels (see for example Guz' and Zozulya (1993)).

Different kinds of boundary integral equations may be constructed using the integral representations (3). In Guz' and Zozulya (1993) the following boundary integral equations were used for the solution of the contact problems for bodies with cracks

$$\int_{\partial V_u} p_j(y)K_{ji}(y,x)dS - \int_{\partial V_p} u_j(y)F_{ji}(y,x)dS$$

$$-\int_\Omega \Delta u_j(y)F_{ji}(y,x)d\Omega = \frac{1}{2}\psi_i(x) - B(x), \ \forall x \in \partial V_p$$

$$\frac{1}{2}p_i(x) - \int_{\partial V_u} p_j(y)K_{ji}(y,x)dS + \int_{\partial V_p} u_j(y)F_{ji}(y,x)dS$$

$$+\int_\Omega \Delta u_j(y)F_{ji}(y,x)d\Omega = B(x), \ \forall x \in \partial V_p$$

$$p_i(x) - \int_{\partial V_u} p_j(y)K_{ji}(y,x)dS + \int_{\partial V_p} u_j(y)F_{ji}(y,x)dS$$

$$+\int_\Omega \Delta u_j(y)F_{ji}(y,x)d\Omega = B(x), \ \forall x \in \Omega$$

where $B(x) = \int_V b_j(y)K_{ji}(y,x)dV + \int_{\partial V_p} \psi_j(y)K_{ji}(y,x)dS + \int_{\partial V_u} \varphi_j(y)F_{ji}(y,x)d\Omega$.

In the case of an unbounded body this system is transformed to the form

$$p_i(x) = -\int_\Omega \Delta u_j(y)F_{ji}(y,x)d\Omega, \ \forall x \in \Omega$$

III. VARIATIONAL FORMULATION AND ALGORITHM

Now, the variational formulation of the contact problems with unilateral restrictions for elastic bodies with cracks will be considered. Using the variational principles of the theory of elasticity and an approach developed by Zozulya (1992) and Guz' and Zozulya (1993; 1995), the following boundary variational inequality may be obtained

$$\int_{\partial V_p} (p_i - \psi_i)(v_i - u_i)dS - \int_{\partial V_u} (u_i - \varphi_i)P_{ik}(v_k - u_k)dS$$

$$+\Phi_i(\Delta v_i) - \Phi_i(\Delta u_i) \geq 0 \qquad (4)$$

$$\Phi_i(u_i) = \begin{cases} \int_\Omega k_\tau |p_n| \, |u_\tau| d\Omega, & \text{if } u_i \geq h_o, \ |p_\tau| = k_\tau |p_n| \\ \infty & \text{otherwise} \end{cases}$$

where $P_{ik}=\lambda n_i\partial_k+\mu(\delta_{ik}\partial_n+n_k\partial_i)$ is a differential operator, which transforms a displacement into the surface traction and u_i and $v_i \in K_u(u_i)$

$$K_u(\Delta u_i)=\{u_i \in \boldsymbol{H}^{1/2}(\partial V), \ |\boldsymbol{q}_\tau|<k_\tau|\boldsymbol{q}_n| \Rightarrow \Delta \boldsymbol{u}_\tau=0,$$

$$|\boldsymbol{q}_\tau|=k_\tau|\boldsymbol{q}_n| \Rightarrow \Delta \boldsymbol{u}_\tau=\lambda_\tau \boldsymbol{q}_\tau\}$$

Several methods from the duality theory are used (Glowinski *et al.*, 1981; Hlavacek *et al.*, 1988; Panagiotopoulos, 1985) for the solution of the variational inequality (4). Then the variational inequality (4) can be transformed into **infsup** problem, i.e. finding a saddle point of the functional

$$\Psi(u_i, q_i) = \inf\{\psi(u_i) + \sup[\langle q_n, \Phi_n(\Delta u_n)\rangle + \langle q_\tau, \Phi_\tau(\Delta u_\tau)\rangle]\}$$

$$u_i \in K_u(\Delta u_i) \quad q_i \in K_q(q_i)$$

$$K_q(q_i)=\{q_i : q_i \in \boldsymbol{H}^{-1/2}(\partial V), \ q_n \geq 0, \ |\boldsymbol{q}_\tau| \leq k_\tau q_n, \ \forall x \in \Omega$$

where

$$\sup\langle q_n, \Phi_n(\Delta u_n)\rangle = \begin{cases} 0, & \text{if } \Delta u_n - h_0 \geq 0, \\ \infty, & \text{if } \Delta u_n - h_0 < 0 \end{cases}$$

$$q_n \in K_q(q_i)$$

$$\sup\langle \boldsymbol{q}_\tau, \Phi_\tau(\Delta \boldsymbol{u}_\tau)\rangle = \begin{cases} \int_\Omega k_\tau q_n |\Delta \boldsymbol{u}_\tau| d\Omega, & \text{if } |\boldsymbol{q}_\tau| \leq k_\tau q_n, \\ \infty, & \text{if } |\boldsymbol{q}_\tau| > k_\tau q_n \end{cases}$$

$$\boldsymbol{q}_\tau \in K_q(q_i)$$

The saddle point problem for this functional is reduced to the successive **inf** problems for the functional $\Psi(u_i, p_i)$ and the projection on the sets of one-sided restrictions with frictions $K_u(\Delta u_i)$ and $K_p(p_i)$. For this purpose, an algorithm of the Udzavy type (Glowinski *et al.*, 1981; Hlavacek *et al.*, 1988) developed in (Zozulya, 1990; 1992) for the solution of the elastic contact problems with unilateral restrictions and friction for elastic bodies with cracks is used here.

The Algorithm Consists of The Following Steps

a) The initial distribution of the contact forces on the crack edges $q_i^0(\boldsymbol{x})$, $\forall \boldsymbol{x} \in \Omega$ is assigned.
b) The systems of the boundary integral Eq. (3) are solved and the unknowns functions on the boundary $p_i(\boldsymbol{x})$, $u_i(\boldsymbol{x})$ and crack surfaces $q_i(\boldsymbol{x})$, $\Delta u_i(\boldsymbol{x})$ are defined.
c) The normal and the tangential components of the contact forces vector are corrected to satisfy the restrictions (2)

$$q_n^1(\boldsymbol{x}) = P_n\{q_n^0(\boldsymbol{x}) - \rho_n[\Delta u_n^1(\boldsymbol{x}) - h_0(\boldsymbol{x})]$$
$$\boldsymbol{q}_\tau^1(\boldsymbol{x}) = \boldsymbol{P}_\tau\{\boldsymbol{q}_\tau^0(\boldsymbol{x}) - \rho_\tau \Delta \boldsymbol{u}_\tau^1(\boldsymbol{x})\} \quad (5)$$

where

$$P_n[q_n] = \begin{cases} 0, & \text{if } q_n \leq 0 \\ q_n, & \text{if } q_n > 0 \end{cases}$$

$$\boldsymbol{P}_\tau[\boldsymbol{q}_\tau] = \begin{cases} \boldsymbol{q}_\tau, & \text{if } |\boldsymbol{q}_\tau| \leq k_\tau q_n \\ k_\tau q_n \dfrac{\boldsymbol{q}_\tau}{|\boldsymbol{q}_\tau|}, & \text{if } |\boldsymbol{q}_\tau| > k_\tau q_n \end{cases} \quad (6)$$

are operators for orthogonal projection on the sets $q_n \geq 0$ and $|\boldsymbol{q}_\tau| \leq k_\tau q_n$. The coefficients ρ_n and ρ_τ are chosen based on the conditions that give the best convergence for the algorithm.
f) Proceed to the second step of the iteration.

This algorithm has been applied for the numerical solution of various static and dynamic contact problems with unilateral restriction and friction for elastic bodies with cracks by Zozulya (1990; 1992), Guz' and Zozulya (1993; 1995) and Zozulya and Rivera (1998). The mathematical investigation of the algorithm convergence and the theorems of existence and uniqueness in contact problems with unilateral restriction and friction for elastic bodies with cracks have been studied by Zozulya (1990). Other algorithms and their mathematical investigations have been studied by Aliabadi and Brebbia (1993), Antes and Panagiotopoulos (1992), Duvaut and Lions (1972), Glowinski, Lions and Tremolieres (1981), Hlavacek, Haslinger, Necas and Lovisec (1988), Kikuchi and Oden (1987), Panagiotopoulos (1985).

IV. EXISTENCE THEOREMS

The iterative algorithm (5) and (6) may be used not only for numerical solution, but also, to prove its convergence and the existence and uniqueness theorems for various contact problems with friction. The Banach fixed-point theorem and its modification will be used here for the mathematical investigation of the problem.

Theorem 1. (Zeidler, 1993)

Let the operator A: $\boldsymbol{H} \rightarrow \boldsymbol{H}$ to be k-contractive on the Banach space $\boldsymbol{H}$, i.e. there is a $k \in [0,1]$ such that

$$\| A[f] - A[g] \| \leq k \| f - g \|, \text{ for } f \in \mathbf{H} \text{ and } g \in \mathbf{H}$$

Then, there exists a unique function $f \in \mathbf{H}$ (point in the functional space $\boldsymbol{H}$) such that $f = A[f]$ (the so-called fixed point of the operator A) and this point may be found by the iterations $f_n = A[f_{n-1}]$, where $f_0 \in \boldsymbol{H}$ and $f_n \rightarrow f$. Moreover, the error estimate gives us the inequality

$$\| f - f_n \| \leq k^n (1-k)^{-1} \| A[f_0] - f_0 \|$$

The algorithm (5) and (6) has been transformed to a form which is convenient for the application of the Banach fixed-point theorem. For this purpose, the system of boundary integral Eqs. (3) has been considered. The vector of the displacement discontinuity of the crack surfaces is the solution of these integral equations and it may be written in the form

$$\Delta u_i(\boldsymbol{x}) = B_{ij}[q_j(\boldsymbol{x})] + F_i(\boldsymbol{x})$$

where B_{ij} are operators which result from the solution of the boundary integral Eq. (3) and F_i are functionals that depend on the boundary conditions, the body forces and other solutions of the boundary integral equations.

The normal and tangential components of the displacement discontinuity vector are defined in the form

$$\Delta u_n(\boldsymbol{x})=B_n[q_n(\boldsymbol{x})]+F_n(\boldsymbol{x}) \tag{7}$$

$$\Delta \boldsymbol{u}_\tau(\boldsymbol{x})=\boldsymbol{B}_\tau[\boldsymbol{q}_\tau(\boldsymbol{x})]+\boldsymbol{F}_\tau(\boldsymbol{x}) \tag{8}$$

The functional F_n depends on q_τ and the functional $\boldsymbol{F}_\tau$ depends also on q_n.

Substituting Δu_n and $\Delta \boldsymbol{u}_\tau$ from these equations into (5) gives us the operator equations in the form

$$q_n(\boldsymbol{x})=P_n[q_n(\boldsymbol{x})-\rho_n(B_n[q_n(\boldsymbol{x})]+F_n(\boldsymbol{x})-h_o)] \tag{9}$$

$$\boldsymbol{q}_\tau(\boldsymbol{x})=\boldsymbol{P}_\tau[\boldsymbol{q}_\tau(\boldsymbol{x})-\rho_\tau(\boldsymbol{B}_\tau[\boldsymbol{q}_\tau(\boldsymbol{x})]+\boldsymbol{F}_\tau(\boldsymbol{x}))] \tag{10}$$

These equations may be rewritten in a suitable form for the application of *Theorem 1*:

$$q=A[q]=P[q-\rho(B[q]+F)]$$

The letters A, P, B, F, q and ρ, in this equation must be supplied with the index n or τ to correspond to the Eqs. (9) and (10) respectively.

In Guz' and Zozulya (1993) it has been shown that the operators A_n and $\boldsymbol{A}_\tau$ transform the functional spaces $\boldsymbol{H}^{-1/2}(\Omega)$ and $(\boldsymbol{H}^{-1/2}(\Omega))^2$ into themselves respectively. Such kinds of operators were considered by Zeidler (1990) and additional conditions which are necessary for the application of the Banach fixed-point theorem were established. These conditions are contained in the following theorem.

Theorem 2. (Zeidler, 1990)

Let the operator B: $\boldsymbol{H}_1 \rightarrow \boldsymbol{H}_2$ which acts in the Banach spaces $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ to be strictly monotone

$$\left\langle B[q_1]-B[q_2], q_1-q_2\right\rangle \geq 0 \quad \forall q_1, q_2 \in \boldsymbol{H}_1 \tag{11}$$

if $\left\langle B[q_1]-B[q_2], q_1-q_2\right\rangle=0 \Rightarrow q_1=q_2$

and Lipschitz continuous

$$\left\| B[q_1]-B[q_2]\right\|_{H_2} \leq C_2\left\| q_1-q_2\right\|_{H_1} \tag{12}$$

Then, the operator A is a contraction operator for which

$$\left\| A[q_1]-A[q_2]\right\|_{H_1} \leq k\left\| q_1-q_2\right\|_{H_1}$$

where $k=(1-2\rho C_1+\rho^2 C_2^2)$ and $\rho\in(0, 2C_1/C_2)$.

In theorems 1 and 2 $\|\ \|$ denotes the norm and $\langle\ \rangle$ denotes the duality operation in the Banach spaces.

V. APPLICATION OF THE EXISTENCE THEOREMS FOR CONTACT PROBLEM WITH FRICTION

Now, the possibility to apply theorem 1 and theorem 2 for the investigation of contact problems with unilateral restrictions and friction for elastic bodies with cracks will be considered. The conditions formulated in *Theorem 2* may be checked more simply than the ones formulated in *Theorem 1*. This is because *Theorem 1* deals with the complicated nonlinear operators A_n and $\boldsymbol{A}_\tau$ which act in the functional spaces $\boldsymbol{H}^{-1/2}(\Omega)$ and $(\boldsymbol{H}^{-1/2}(\Omega))^2$, but *Theorem 2* deals with the linear operators B_n which map $\boldsymbol{H}^{-1/2}(\Omega)$ in $\boldsymbol{H}^{1/2}(\Omega)$ and $\boldsymbol{B}_\tau$ which maps $(\boldsymbol{H}^{-1/2}(\Omega))^2$ in $(\boldsymbol{H}^{1/2}(\Omega))^2$. For our specific problem, in a previous section it was shown that for application of the Banach fixed-point theorem (*Theorem 1*), it is sufficient to check the fulfillment of the conditions formulated in *Theorem 2*.

According to *Theorem 2*, the operators B_n and $\boldsymbol{B}_\tau$ must be strictly monotone (11) and Lipschitz continuous (12). The property needed for the integral operators (3) to be Lipschitz continuous has been studied by Kupradze *et al.* (1979) and Zeidler (1993). They have shown that the integral Eq. (3) have unique solutions and the corresponding direct and inverse operators are Lipschitz continuous.

The property to be strictly monotone (11) for linear operators has the form

$$\left\langle B[q], q\right\rangle \geq 0$$

Since the operators B_n and $\boldsymbol{B}_\tau$ in (7) and (8) are linear, this condition for the normal component of the contact force vector in the cracks surfaces may be presented in the form

$$\left\langle B_n[q_n], q_n\right\rangle \geq 0$$

and for the tangential component may be presented in the form

$$\left\langle \boldsymbol{B}_\tau[\boldsymbol{q}_\tau], \boldsymbol{q}_\tau\right\rangle \geq 0$$

Taking into account the definitions (7) and (8) of the operators B_n and $\boldsymbol{B}_\tau$ and the definition of the duality operation between the Banach spaces

$\boldsymbol{H}^{1/2}(\Omega)$ and $\boldsymbol{H}^{-1/2}(\Omega)$ and between the Banach spaces $(\boldsymbol{H}^{1/2}(\Omega))^2$ and $(\boldsymbol{H}^{-1/2}(\Omega))^2$, these inequalities may be presented in the form

$$\int_\Omega q_n(\boldsymbol{x})\Delta u_n(\boldsymbol{x})d\Omega \geq 0 \tag{13}$$

and

$$\int_\Omega q_\tau(\boldsymbol{x}) \bullet \Delta u_\tau(\boldsymbol{x})d\Omega \geq 0 \tag{14}$$

These inequalities are not convenient for the investigation of the property of the operators B_n and $\boldsymbol{B}_\tau$ to be strictly monotone. Therefore they will be transformed to a form more convenient for that investigation. For this purpose the equation of motion (1) is multiplied by a displacement vector $\boldsymbol{u}$ and is integrated over the volume V

$$\int_V u_i(A_{ij}u_j + b_i)dV = 0$$

This expression, using the Gauss-Osstrogradskii theorem can be transformed to

$$\int_V u_i A_{ij} u_j dV = \int_{\partial V} p_i u_i dS - \int_V 2e(\boldsymbol{u})dV$$

The following equation will be used to estimate the inequalities (13) and (14)

$$E(\boldsymbol{u}) - A(\boldsymbol{b},\boldsymbol{p}) = \int_\Omega q_i \Delta u_i d\Omega \tag{15}$$

Here

$$E(\boldsymbol{u}) = \int_V e(\boldsymbol{u})dV, \ e(\boldsymbol{u}) = \sigma_{ij}\varepsilon_{ij}/2 = \lambda\varepsilon_{ii}^2/2 + \mu\varepsilon_{ij}\varepsilon_{ij}$$

is the total strain energy and

$$A(\boldsymbol{b},\boldsymbol{p}) = \int_V b_i u_i dV + \int_{\partial V} p_i u_i dS$$

is the work of the body force and the surface traction.

Because here it is considered the linear elastic model, the total strain energy may be presented in the form

$$E(\boldsymbol{u}) = E^p(\boldsymbol{u}^p) + E^q(\boldsymbol{u}^q)$$

where $E^p(\boldsymbol{u}^p)$ and $E^q(\boldsymbol{u}^q)$ are the total strain energy produced by the body forces and the surface traction and the total strain energy produced by the contact forces acting in the crack surfaces respectively.

From a mathematical point of view, such separation means that the boundary-value problem (1) with nonhomogeneous boundary conditions and cracks may be presented as two separated problems. The first one is the boundary-value problem (1) with nonhomogeneous boundary conditions and without cracks and the second one is the boundary-value problem (1) with homogeneous boundary conditions and with cracks. In this second problem the fictitious load is applied on the cracks surfaces (Guz' and Zozulya, 1993). It has been found as a solution of the first problem

Since

$$E^p(\boldsymbol{u}) - A(\boldsymbol{b},\boldsymbol{p}) = 0$$

the Eq. (15) may be transformed to the following one

$$E^q(\Delta\boldsymbol{u}) = \int_\Omega q_i \Delta u_i d\Omega \tag{16}$$

In the inequalities (13) and (14) q_n, Δu_n are presented as normal and $\boldsymbol{q}_\tau$, $\Delta\boldsymbol{u}_\tau$ as tangential components of the vectors of contact forces and displacement discontinuity. Therefore it is better to transform Eq. (16) in the following way

$$E^q(\Delta\boldsymbol{u}) = E^n(\Delta u_n) + E^\tau(\Delta\boldsymbol{u}_\tau),$$

$$\int_\Omega q_i \Delta u_i d\Omega = \int_\Omega q_n \Delta u_n d\Omega + \int_\Omega \boldsymbol{q}_\tau \bullet \Delta\boldsymbol{u}_\tau d\Omega$$

and

$$E^n(\Delta u_n) = \int_\Omega q_n \Delta u_n d\Omega, \ E^\tau(\Delta\boldsymbol{u}_\tau) = \int_\Omega \boldsymbol{q}_\tau \bullet \Delta\boldsymbol{u}_\tau d\Omega$$

Now the inequalities (13) and (14) can be presented in the form and

$$E_n(\Delta u_n) \geq 0 \text{ and } E^\tau(D\boldsymbol{u}_\tau) \geq 0 \tag{17}$$

The total strain energy is always positively determined, as was shown in (Duvavt and Lions, 1972; Kupradze *et al.*, 1979). It means that the inequalities (13) and (14) take place and the operators B_n and $\boldsymbol{B}_\tau$ are strictly monotone ones. Consequently all the conditions in theorem 2 have been fulfilled and the problem under consideration has a unique solution.

VI. CONCLUSIONS

Contact problems with unilateral restrictions and friction for elastic bodies with cracks have been transformed into superposition of the boundary integral equations and operators of projection on the set of one-sided restrictions with friction.

Some theorems from the functional analysis have been used for the investigation of the solution existence and uniqueness. The main condition for existence and uniqueness in those theorems was that the properties of the operators B_n and $\boldsymbol{B}_\tau$ are a strictly monotone and they have been transformed into the property of the total strain energy to be a positively determined one. But, for linear elastic models the total strain energy is always positively determined.

The existence and uniqueness of contact problem with sunilateral restrictions and friction for elastic bodies with cracks has been proved.

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NOMENCLATURE

V	volume of elastic body
∂V	boundary of the body
∂V_u and ∂V_p	parts of the boundary, where displacements and surface traction are assigned
$\boldsymbol{u}(\boldsymbol{x})$	displacements
$\boldsymbol{p}(\boldsymbol{x})$	surface traction
$\boldsymbol{b}(\boldsymbol{x})$	body forces
N	number of cracks
Ω_n^+ and Ω_n^-	opposite edges of crack
A_{ij}	differential operator of elasticity
$\partial_i=\partial/\partial x_i$	derivatives with respect to coordinates
ρ	density of the material
λ and μ	Lame constants
δ_{ij}	Kroneker's symbol
$\psi_i(\boldsymbol{x})$ and $\varphi_i(\boldsymbol{x})$	functions that describe the boundary conditions on the parts ∂V_p and ∂V_u
$\Delta \boldsymbol{u}$	displacements of the crack surfaces
$\boldsymbol{q}$	forces of the crack edges contact interaction
q_n and $\boldsymbol{q}_\tau$	normal and tangential components of the contact force vector
Δu_n and $\Delta \boldsymbol{u}_\tau$	normal and tangential components of the displacement discontinuity vector
h_0	initial opening of the cracks
k_τ and λ_τ	coefficients dependent on the contact surfaces properties
Ω_e	close contact area
P_{ik}	differential operator that transforms a displacement into the surface traction
$\Phi_i(u_i)$	subdifferential functional
$K_u(u_i)$ and $K_q(u_i)$	sets of one-sided restrictions with friction
$\Psi(u_i, q_i)$	boundary functional
$P_n[q_n]$ and $\boldsymbol{P}_\tau[\boldsymbol{q}_\tau]$	operators for orthogonal projection on the sets of one-sided restrictions with friction
ρ_n and ρ_τ	coefficients which are used for best convergence for the algorithm
$\boldsymbol{H}$, $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$	Banach functional spaces
B_{ij}, B_n and $\boldsymbol{B}_\tau$	operators which result from the solution of the boundary integral equations
F_i, F_n and $\boldsymbol{F}_\tau$	functionals that depend on the boundary integral equations
$\boldsymbol{H}^{1/2}(\Omega)$, $(\boldsymbol{H}^{1/2}(\Omega))^2$, $\boldsymbol{H}^{-1/2}(\Omega)$ and $(\boldsymbol{H}^{-1/2}(\Omega))^2$	Sobolev functional spaces
$\Vert\ \Vert$ and $\langle\ \rangle$	norm and duality operation in the Banach spaces
$e(\boldsymbol{u})$	local strain energy
$E(\boldsymbol{u})$	total strain energy
$E^p(\boldsymbol{u}^p)$	total strain energy produced by the body forces and the surface traction
$E^q(\boldsymbol{u}^q)$	total strain energy produced by the contact forces acting in the cracks surfaces
$A(\boldsymbol{b}, \boldsymbol{p})$	work of the body force and the surface traction

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摩擦接觸問題之邊界積分方程式及其存在定理探討

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摘 要

本文探討含裂縫彈性體受到單邊限制及含摩擦效應之接觸問題數學描述，在裂縫表面的單邊限制及摩擦效應可以不等式來推導。解決這個問題的方法可以疊加使用邊界積分運算元與投影運算元來描 述單邊限制及摩擦問題，存在與唯一定理在本文也予以證明。

關鍵字：邊界積分方程式，接觸，摩擦力，裂縫，不動點。