ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

Engineering Analysis with Boundary Elements 23 (1999) 611–617

Research note

A boundary formulation for calculating moments of an arbitrary closed planar region

W. Yeih[a], J.R. Chang[b], J.T. Chen[a,*]

[a]Department of Harbor and River Engineering, National Taiwan Ocean University, Keelung, Taiwan 202, R.O.C.
[b]Department of Naval Architecture, National Taiwan Ocean University, Keelung, Taiwan 202, R.O.C.

Received 30 September 1998; accepted 17 December 1998

Abstract

In this study, a boundary formulation for calculating moments of an arbitrary closed planar region is proposed instead of calculating moments using the domain integral. The Gauss' divergence theorem is used to transform moments of area into boundary integrals. Three examples are demonstrated to show the validity of the proposed method. Results obtained from the boundary formulation are compared with analytical solutions, and good accuracy is obtained. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Moments of area; Domain integral; Boundary integral

1. Introduction

Evaluation of the domain integral of an arbitrary closed planar region is a fundamental problem in various fields of engineering and applied science. The area of an arbitrary closed region is important in surveying [1]. The geometric center of an arbitrary closed region is of great importance for two-dimensional dynamics [2] as the planar motion of a rigid body can be decomposed into two parts: the linear motion of the mass center and the rotational motion about the mass center. The second moment of an arbitrary two-dimensional closed region is also meaningful in dynamics [2] and in the mechanics of materials [3]. The polar moment of the area is useful in 2-D planar rigid body dynamics when the angular acceleration is related to the resultant moment about the center [2]. The second moment of a cross-sectional area about its neutral axis appears in the formulation of the bending problem of an Euler–Bernoulli beam. The shear stress acting on the cross-section of a beam is also related to the moment of area [3]. The torsional rigidity is calculated using a domain integral in the Prandtl formulation for the Saint–Venant torsion problem, and the boundary formulation of the torsional rigidity has been proposed [4,5]. When the Poisson's equation is solved using the boundary element method, a domain integral exists in the formulation. Several kinds of domain integrals can be transformed into a boundary integral, e.g. when the body force distribution is a constant body force, centrifugal force or thermal loading [6]. However, general transformation from a domain integral with an arbitrary body force function into a boundary integral has not been proposed to the best of authors' knowledge.

To evaluate the domain integral of an arbitrary closed planar region, finite discretization of the domain is traditionally used. To avoid the complexity of domain discretization, it is quite promising to employ an easier discretization process if the domain integral can be transformed into a boundary integral formulation. This is quite similar to the original idea of BEM, which tries to solve the boundary value problem by discretizing boundary alone, developed by Kinoshita and Mura [7] in their pioneer works. Of course, not all the domain integrals of arbitrary functions can be transformed into boundary integrals. However, some functions (for example, the moments of area presented in this article) can be transformed. This concept, which transforms a domain integral into a boundary integral, has been employed in the multiple reciprocal method (MRM) [8–12] in boundary element analysis for acoustic problems and many other related problems.

Consider a closed planar region D bounded by the boundary Γ as shown in Fig. 1; the moments of area defined as follows:

$$M_x^n \equiv \int_D y^n \, \mathrm{d}D, \tag{1a}$$

$$M_y^n \equiv \int_D x^n \, \mathrm{d}D, \tag{1b}$$

* Corresponding author. Fax: +886-2-2462-2192-6140.
E-mail address: b0209@ntou66.ntou.edu.tw (J.T. Chen)

PII: S0955-7997(99)00004-1

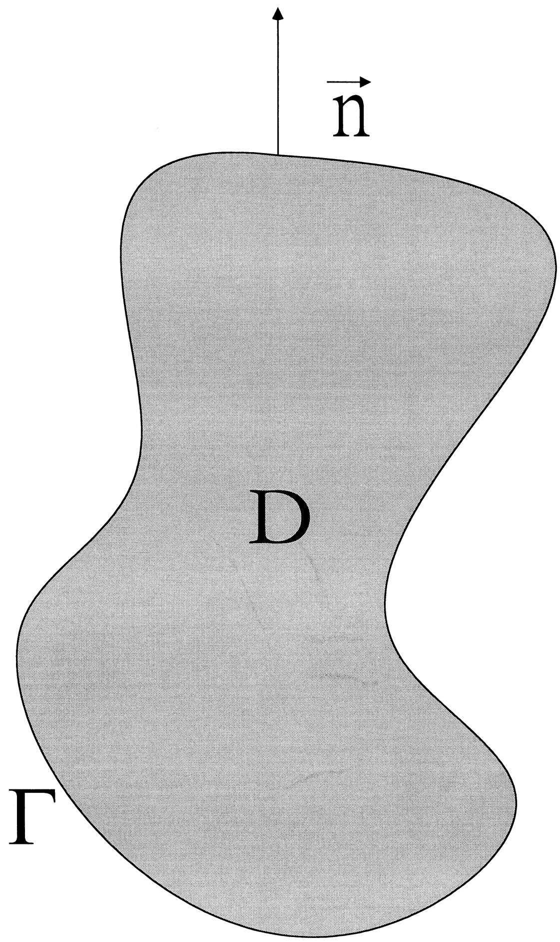

Fig. 1. A closed planar region.

$$M_{xy}^{nm} \equiv \int_D x^n y^m \, dD, \tag{1c}$$

$$M_P^n \equiv \int_D r^n \, dD, \tag{1d}$$

where n and m are non-negative integers, M_x^n is the n-th moment of area with respect to the x-axis, M_y^n is the n-th moment of area with respect to y-axis, M_{xy}^{nm} is the cross moment of area and M_P^n is the n-th polar moment of area with r defined as the distance from the origin.

Applications of moments of area have appeared in many engineering fields. Several commonly used applications are described below. The area of a closed planar region can be calculated from Eqs. (1a) or (1b) by letting n be zero. In surveying, the so-called area integrator can be used to calculate the area of a closed region by means of the coordinates of boundary points. However, the method employed is different from which we propose here, which will be elaborated on later. The coordinates of the geometric center, $(\bar{x}, \bar{y})$, for a closed planar region are known to be

$$\bar{x} = \frac{\int_D x \, dD}{\int_D dD}, \tag{2a}$$

$$\bar{y} = \frac{\int_D y \, dD}{\int_D dD}. \tag{2b}$$

It is clearly seen in Eqs. (2a) and (2b) that the coordinates of the geometric center are obtained by dividing the first moment of area by the zeroth moment of area.

In dynamics, the moments of inertia for a homogeneous planar region are defined as

$$I_{xx} \equiv \rho \int_D y^2 \, dD, \quad I_{xy} = I_{yx} = \rho \int_D xy \, dD \quad \text{and} \quad I_{yy} = \rho \int_D x^2 \, dD, \tag{3}$$

where ρ is the mass density function. It can be easily found that Eqs.(1a), (1b) and (1c) are required in order to calculate the terms in Eq. (3).

In mechanics of materials, the bending stress in an Euler–Bernoulli beam as shown in Fig. 2 is formulated as

$$\sigma_{xx} = \frac{My}{I_{zz}}, \tag{4}$$

where the neutral plane of the beam is the x–z plane (i.e. $y = 0$), σ_{xx} is the bending stress, M is the moment acting on the cross-section and I_{zz} is the second moment of area with respect to the z-axis.

To calculate the shear stress in an Euler–Bernoulli beam with a rectangular cross-section as shown in Fig. 2, the following is used:

$$\tau = \frac{VQ}{I_{zz} b}, \tag{5}$$

where V is the resultant shear force acting on the cross-section, Q is the first moment of the shaded area, A, in Fig. 2 (i.e. $Q = \int_A y \, dA$) with respect to the z-axis, τ is the shear stress and b is the width of rectangular cross-section.

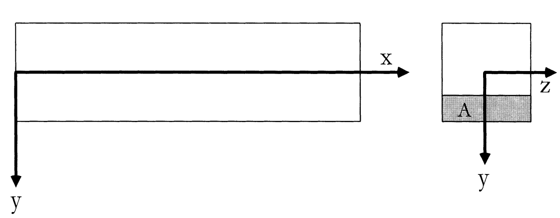

Fig. 2. A beam problem.

The applications mentioned earlier all require the calculation of moments of area. To transform a planar domain integral into a boundary integral, two theorems, the Gauss' divergence theorem and the Stokes' theorem [13], may be useful. In this study, the Gauss' divergence theorem is used to transform the moments of area from a domain integral into a boundary integral, and the discretization formulation, which uses the line element scheme is also derived. Finally, three examples are illustrated to show the validity of this proposed formulation.

2. Boundary integral formulation of moments of area

Consider a planar closed region D on the x–y plane with boundary Γ, where the outnormal vector on the boundary is denoted as $\bar{n}$ (shown in Fig. 1); the domain integrals of the moments of area can be transformed into boundary integrals as follows.

First consider $n = 0$ in Eq. (1a) (or $n = 0$ in Eq. (1b)); the zeroth moment of area is simply the area. We have

$$\int_D y^0 \, \mathrm{d}D = \frac{1}{2}\int_D \nabla^2 y^2 \, \mathrm{d}D = \frac{1}{2}\int_D \vec{\nabla}\cdot(\vec{\nabla} y^2) \, \mathrm{d}D = \frac{1}{2}\int_\Gamma \vec{\nabla} y^2 \cdot \vec{n} \, \mathrm{d}\Gamma, \tag{6}$$

in which Gauss' divergence theorem is used.

Now consider that n is a non-zero positive integer in Eq. (1a) or (1b)); we have

$$\begin{aligned}\int_D y^n \, \mathrm{d}D &= \frac{1}{2n}\int_D \vec{\nabla} y^2 \cdot \vec{\nabla} y^n \, \mathrm{d}D \\ &= \frac{1}{2n}\left\{\int_\Gamma y^2 \vec{\nabla} y^n \cdot \vec{n} \, \mathrm{d}\Gamma - \int_D y^2 \nabla^2 y^n \, \mathrm{d}D\right\} \\ &= \frac{1}{2n}\left\{\int_\Gamma y^2 \vec{\nabla} y^n \cdot \vec{n} \, \mathrm{d}\Gamma - n(n-1)\int_D y^n \, \mathrm{d}D\right\}.\end{aligned} \tag{7}$$

Therefore, the following equation can be derived:

$$\int_D y^n \, \mathrm{d}D = \frac{1}{n(n+1)}\int_\Gamma y^2 \vec{\nabla} y^n \cdot \vec{n} \, \mathrm{d}\Gamma. \tag{8}$$

Similarly, we can derive

$$\int_D x^n \, \mathrm{d}D = \frac{1}{n(n+1)}\int_\Gamma x^2 \vec{\nabla} x^n \cdot \vec{n} \, \mathrm{d}\Gamma. \tag{9}$$

For the cross moment of area defined in Eq. (1c), the following equation can be obtained when $m \neq 0$:

$$\begin{aligned}\int_D x^n y^m \, \mathrm{d}D &= \frac{1}{2m}\int_D \vec{\nabla}(x^n y^2)\cdot\vec{\nabla}(y^m) \, \mathrm{d}D \\ &= \frac{1}{2m}\left\{\int_\Gamma (x^n y^2)\vec{\nabla}(y^m)\cdot\vec{n} \, \mathrm{d}\Gamma - \int_D (x^n y^2)\nabla^2(y^m) \, \mathrm{d}D\right\} \\ &= \frac{1}{2m}\left\{\int_\Gamma (x^n y^2)\vec{\nabla}(y^m)\cdot\vec{n} \, \mathrm{d}\Gamma - m(m-1) \times \int_D (x^n y^2) y^{m-2} \, \mathrm{d}D\right\} \\ &= \frac{1}{2m}\left\{\int_\Gamma (x^n y^2)\vec{\nabla}(y^m)\cdot\vec{n} \, \mathrm{d}\Gamma - m(m-1)\int_D x^n y^m \, \mathrm{d}D\right\}.\end{aligned} \tag{10}$$

Therefore, we have

$$\int_D x^n y^m \, \mathrm{d}D = \frac{1}{m(m+1)}\int_\Gamma x^n y^2 \vec{\nabla}(y^m)\cdot\vec{n} \, \mathrm{d}\Gamma. \tag{11}$$

If $n \neq 0$, the cross moment of area can be also represented as

$$\int_D x^n y^m \, \mathrm{d}D = \frac{1}{n(n+1)}\int_\Gamma x^2 y^m \vec{\nabla}(x^n)\cdot\vec{n} \, \mathrm{d}\Gamma. \tag{12}$$

For the n-th polar moment, we have

$$\begin{aligned}\int_D r^n \, \mathrm{d}D &= \frac{1}{n+1}\int_D \vec{\nabla} r \cdot \vec{\nabla} r^{n+1} \, \mathrm{d}D \\ &= \frac{1}{n+1}\left\{\int_\Gamma r \vec{\nabla} r^{n+1}\cdot\vec{n} \, \mathrm{d}\Gamma - \int_D r \nabla^2 r^{n+1} \, \mathrm{d}D\right\} \\ &= \frac{1}{n+1}\left\{\int_\Gamma r \vec{\nabla} r^{n+1}\cdot\vec{n} \, \mathrm{d}\Gamma - n\int_D r^n \, \mathrm{d}D\right\}.\end{aligned} \tag{13}$$

Finally, we have

$$\int_D r^n \, \mathrm{d}D = \frac{1}{2n+1}\int_\Gamma r \vec{\nabla} r^{n+1}\cdot\vec{n} \, \mathrm{d}\Gamma. \tag{14}$$

3. Discretization formula using the line element scheme

The moment of area can be expressed in terms of the boundary integration as shown in the preceding section. In the numerical implementation, the boundary integral, which can be viewed as $\int_\Gamma f(x,y) \, \mathrm{d}\Gamma$, is obtained by an approximated summation after the discretization process. In general, the so-called shape function is used to approximate the unknown kernel function $f(x,y)$ in which method the value of the unknown kernel function at any point (x,y) inside the element is approximated by the summation of the products of shape function values and nodal values of the unknown kernel function. In the case we study here, the kernel itself is always a known function; therefore, it is not necessary to approximate the kernel. However, the boundary we encountered may be approximated by the shape function. Consider the m-th order element ($m \geq 1$) to approximate the boundary, in order to describe the m-th order polynomial 2-D curve in the x–y plane, it requires $m(m+3)/2$ nodes in one element. The shape functions can be set up as $m(m+3)/2 - 1$ order polynomials of the local coordinate s by requiring that the component of shape function corresponding to the node k of the element should become 1 when the local coordinate s matches the very value of node k and become zero when the local coordinate s matches values of other nodes except node k. This means that the coordinate of points on the element can be expressed as:

$$x(s) = \sum_K N_K(s) x_K; \tag{15}$$

$$y(s) = \sum_K N_K(s) y_K, \tag{16}$$

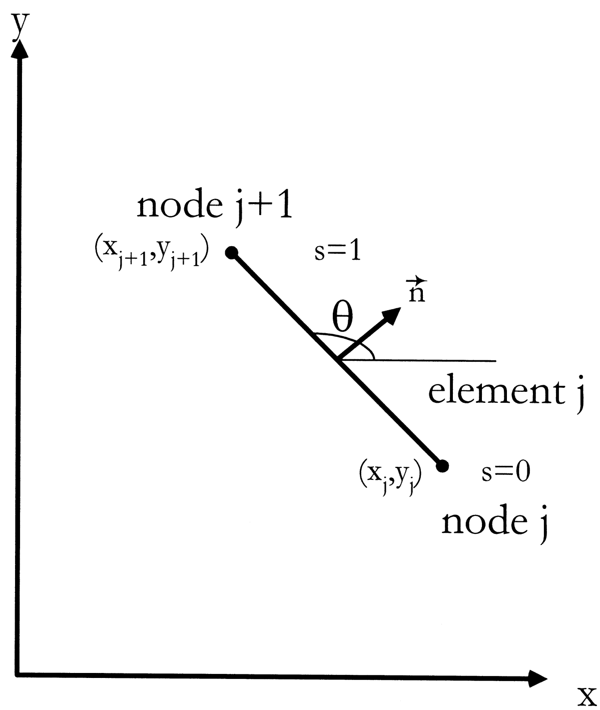

Fig. 3. Notations related to the integration of an element.

where the summation over K means to have the summation for all the nodes inside this element, $N_K(s)$ is the component of the shape function corresponding to the K-th node and (x_K, y_K) is the coordinate of the K-th node. According to this algorithm, the integral then can be written as

$$\sum_{\alpha} \int f(x(s_\alpha), y(s_\alpha)) J_\alpha \, \mathrm{d}s_\alpha, \tag{17}$$

where α is the element label and J_α is the Jacobian value of the mapping from the global coordinate (x, y) to the local coordinate value s.

For simplicity, only the line element (first order element) is given here. Now consider the element shown in Fig. 3; the angle between the positive x-axis and the circulation vector, which is the position vector from the j-th node to the $(j+1)$-th node, is θ_j. If we set the local coordinate $s = 0$ at j-th node and $s = 1$ at $(j+1)$-th node, the shape function then can be written as

$$\mathbf{N} = \begin{bmatrix} N_j(s) \\ N_{j+1}(s) \end{bmatrix} = \begin{bmatrix} 1 - s \\ s \end{bmatrix}. \tag{18}$$

In addition, the value of $J_\alpha = L_j$ where L_j is the element length for j-th element.

The outnormal vector on this element, $\vec{n}_j$, can be denoted as

$$\vec{n}_j = (\sin\theta_j, -\cos\theta_j), \tag{19}$$

where θ_j can be calculated from

$$\theta_j = \cos^{-1}\frac{x_{j+1} - x_j}{L_j}, \tag{20}$$

in which x_j is the x component for the j-th node.

From the definition of shape function, we have

$$x = \{(1 - s)x_j + s x_{j+1}\} \tag{21}$$

and

$$y = \{(1 - s)y_j + s y_{j+1}\}. \tag{22}$$

Therefore, $r = \sqrt{x^2 + y^2}$ can also be expressed as a function of the local coordinate s.

After discretization on the boundary with N elements, Eq. (6) can be written as

$$\int_D y^0 \, \mathrm{d}D = \sum_{j=1}^{N} \frac{\cos\theta_j}{2}(y_{j+1} + y_j)L_j. \tag{23}$$

Similarly, we have

$$\int_D y^n \, \mathrm{d}D = \frac{1}{(n+1)} \sum_{j=1}^{N} \frac{-\cos\theta_j}{L_j^{n+1}} \frac{[L_j y_{j+1}]^{n+2} - [L_j y_j]^{n+2}}{(y_{j+1} - y_j)(n+2)} \tag{24}$$

and

$$\int_D x^n \, \mathrm{d}D = \frac{1}{(n+1)} \sum_{j=1}^{N} \frac{\sin\theta_j}{L_j^{n+1}} \frac{[L_j x_{j+1}]^{n+2} - [L_j x_j]^{n+2}}{(x_{j+1} - x_j)(n+2)} \tag{25}$$

from Eqs. (8) and (9).

Further, the cross moment of area can be found as

$$\int_D x^n y^m \, \mathrm{d}D = \frac{-1}{m+1} \sum_{j=1}^{N} \left\{ \frac{\cos\theta_j}{L_j^{n+m+1}} \sum_{k=0}^{n} \sum_{l=0}^{m+1} \binom{n}{k} \times \binom{m+1}{l} a_j^k c_j^l b_j^{n-k} d_j^{m+1-l} \frac{(0.5L_j)^{k+l+1} - (-0.5L_j)^{k+l+1}}{k+l+1} \right\}, \tag{26}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad a_j = (x_{j+1} - x_j),$$

$$b_j = 0.5L_j(x_{j+1} + x_j), \qquad c_j = (y_{j+1} - y_j)$$

and

$$d_j = 0.5L_j(y_{j+1} + y_j).$$

Similarly, Eq. (12) can be rewritten as

$$\int_D x^n y^m \, \mathrm{d}D = \frac{1}{n+1} \sum_{j=1}^{N} \left\{ \frac{\sin\theta_j}{L_j^{n+m+1}} \sum_{k=0}^{n+1} \sum_{l=0}^{m} \binom{n+1}{k} \times \binom{m}{l} a_j^k c_j^l b_j^{n+1-k} d_j^{m-l} \frac{(0.5L_j)^{k+l+1} - (-0.5L_j)^{k+l+1}}{k+l+1} \right\}. \tag{27}$$

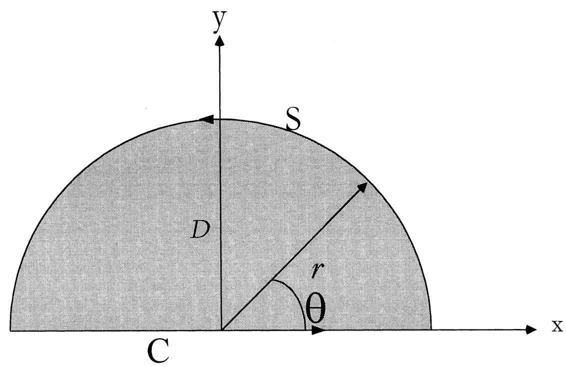

Fig. 4. A semicircular region for Example 1.

To evaluate the polar moment of area, we define

$$e_j \equiv L_j^2,$$

$$f_j \equiv -2(x_j^2 + y_j^2 - x_j x_{j+1} - y_j y_{j+1}),$$

$$g_j \equiv x_j^2 + y_j^2,$$

$$\omega_j \equiv 2e_j s + f_j,$$

$$\gamma_j \equiv 4e_j g_j - f_j^2$$

and

$$\lambda_j = \frac{\gamma_j}{8e_j}.$$

Then, the following integral forms can be used in the calculation:

$$\int \sqrt{(e_j s^2 + f_j s + g_j)^n}\, ds = \frac{\omega_j \sqrt{(e_j s^2 + f_j s + g_j)^n}}{2(n+1)e_j} + \frac{2n\lambda_j}{n+1} \int \sqrt{(e_j s^2 + f_j s + g_j)^{n-2}}\, ds;$$

$$\int \sqrt{(e_j s^2 + f_j s + g_j)}\, ds = \frac{\omega_j}{4e_j} \sqrt{(e_j s^2 + f_j s + g_j)} + \frac{\lambda_j}{\sqrt{e_j}} \ln\left|\omega_j + 2\sqrt{e_j}\sqrt{(e_j s^2 + f_j s + g_j)}\right| \text{ for } e_j > 0;$$

$$\int s\sqrt{(e_j s^2 + f_j s + g_j)^n}\, ds = \frac{1}{(n+2)e_j}\left[\sqrt{(e_j s^2 + f_j s + g_j)^{n+2}} - \frac{(2+n)f_j}{2} \int \sqrt{(e_j s^2 + f_j s + g_j)^n}\, ds\right].$$

For the boundary formulation of the polar moment of area, the integral in each element involves

$$K_j = \frac{n+1}{2n+1} \int_0^1 \sqrt{(e_j s^2 + f_j s + g_j)^n} \times \{[(1-s)x_j + s x_{j+1}](y_{j+1} - y_j) - [(1-s)y_j + s y_{j+1}](x_{j+1} - x_j)\}\, ds, \tag{28}$$

in which the integral can be performed using the formulae listed above. Finally, we have

$$\int_D r^n\, dD = \sum_{j=1}^{N} K_j. \tag{29}$$

Table 1
I_{xx} for a semicircular region with radius $r = 1$

Elements on the semicircular arc	10	20	50	100	500
$\Delta\theta$ (degrees)	18	9	3.6	1.8	0.36
$I_{xx} = \pi/8(0.393)$: analytical	0.380	0.389	0.392	0.393	0.393

4. Demonstrated examples

Three examples will be given to show the validity of the proposed method.

Example 1. A semicircular region with radius r as shown in Fig. 4 is considered. To calculate $I_{xx} = \int_D y^2\, dD$ we can use Eq. (8) to get

$$I_{xx} = \frac{1}{6}\left[\int_C y^2 \cdot 2y \cdot (-1)\, dx + \int_S r y^2 2y \sin\theta\, d\theta\right]$$

$$= \frac{1}{3}\left[\int_{-r}^{r} 0\, dx + r^4 \int_0^{\pi} \sin^4\theta\, d\theta\right]$$

$$= \frac{r^4}{3}\left[\frac{-1}{4}\cos\theta \sin^3\theta - \frac{3}{8}\cos\theta \sin\theta + \frac{3}{8}\theta\right]_0^{\pi}$$

$$= \frac{\pi r^4}{8},$$

where C is the integration path along the bottom edge of the semicircular region and S is the integration path along the circular arc of the semicircular region. The value obtained using the proposed method is exactly the same as the analytical solution. Numerical results for different element numbers on S are tabulated in Table 1. It is not surprising that the numerical values converge to the exact solution as the number of element increases.

Example 2. A parabolic spandrel is considered as shown in Fig. 5. In order to calculate the geometric center, we first calculate the area of this region as

$$A = \int_D y^0\, dD$$

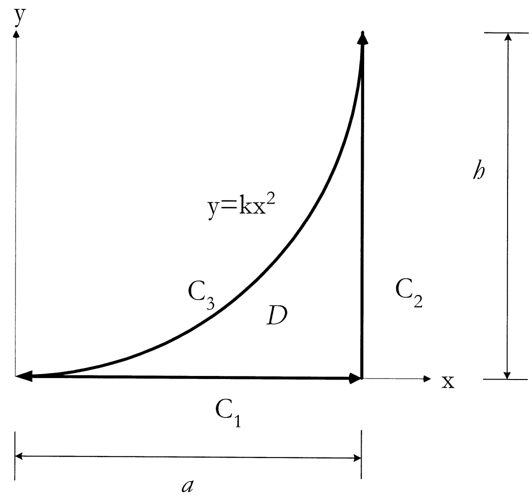

Fig. 5. A parabolic spandrel for Example 2.

$$= \int_{C_1} y(-1)\,dx + \int_{C_2} y\cdot 0\,dy + \int_{C_3} y\cdot n_y\,dC_3$$

$$= \int_a^0 -kx^2\,dx$$

$$= \frac{ka^3}{3}$$

$$= \frac{ah}{3}.$$

Then the first moment with respect to the x-axis and y-axis can be calculated as

$$\int_D y\,dD = \frac{1}{2}\left[\int_{C_1} y^2(-1)\,dx + \int_{C_2} y^2\cdot 0\,dy + \int_{C_3} y^2 n_y\,dC_3\right]$$

$$= \frac{1}{2}\int_a^0 -k^2x^4\,dx$$

$$= \frac{k^2a^5}{10}$$

$$= \frac{ah^2}{10}$$

and

$$\int_D x\,dD = \frac{1}{2}\left[\int_{C_1} x^2\cdot 0\,dx + \int_{C_2} x^2\cdot 1\,dy + \int_{C_3} x^2 n_x\,dC_3\right]$$

Table 2
Coordinates of the geometric center for a parabolic spandrel with $a = 1$, $k = 1$ and $h = 1$

Elements on the parabolic arc	10	20	40	100	500
Δx	0.1	0.05	0.025	0.01	0.002
$\bar{x} = 0.75$	0.744	0.748	0.750	0.750	0.750
$\bar{y} = 0.3$	0.300	0.300	0.300	0.300	0.300

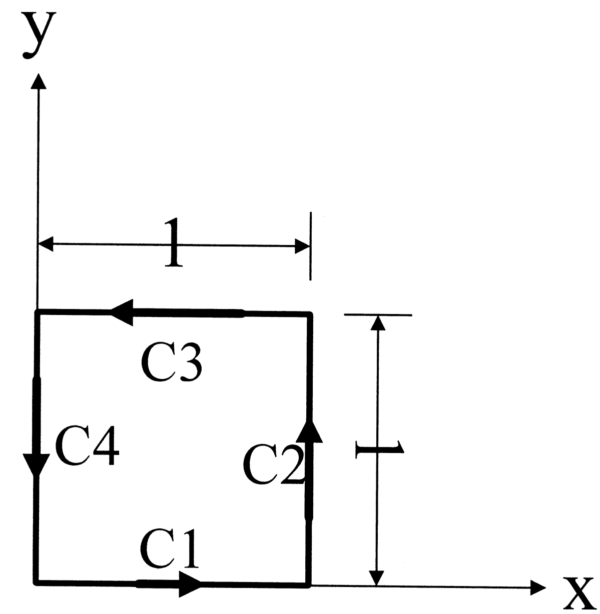

Fig. 6. A square region for Example 3.

$$= \frac{a^2h}{2} + \frac{1}{2}\int_h^0 \frac{y}{k}\,dy$$

$$= \frac{a^2h}{2} - \frac{a^2h}{4}$$

$$= \frac{a^2h}{4}.$$

Therefore, the coordinate of the geometric center can be found as

$$\bar{x} = \frac{\int_D x\,dD}{A} = \frac{3a}{4} \qquad \text{and} \qquad \bar{y} = \frac{\int_D y\,dD}{A} = \frac{3h}{10}.$$

The numerical results for different element numbers on the parabolic arc are tabulated in Table 2. Good convergence is obtained as the number of element increases.

Example 3. A square region is considered as shown in Fig. 6. To calculate $I_{xy} = \int_D xy\,dD$, we can evaluate it using Eq. (11) and get

$$\int_D xy\,dD = \frac{1}{2}\int_{C_1+C_2+C_3+C_4} xy^2\,\vec{\nabla}(y)\cdot\vec{n}\,d\Gamma$$

$$= -\frac{1}{2}\int_{C_1} x\cdot 0\,dx + \frac{1}{2}\int_{C_3} x\cdot 1^2(-dx)$$

Table 3
I_{xy} for a square region

Elements	4	8	16	20	32
Element length	1.0	0.5	0.25	0.2	0.125
$I_{xy} = 0.25$	0.25	0.25	0.25	0.25	0.25

$$= \frac{1}{2}\int_1^0 x(-\mathrm{d}x)$$

$$= \frac{1}{4}.$$

Numerical results obtained using the equal-length element mesh are tabulated in Table 3. It is found that the results match the analytical value very well even when only four elements are used. As the formula listed in Eq. (26) and other discretized formulae are based on the line element, the result will be exact when a polygon is considered.

5. Integration of an analytical function in a planar region

It is well known that an analytic function, $h(x,y)$, can be expanded into a Taylor's series with the expansion center at the origin, which has the following form:

$$h(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}x^m y^n. \tag{30}$$

Then the area integral of $h(x,y)$ can be evaluated numerically using

$$\int_D h(x,y)\,\mathrm{d}D \cong \sum_{m=0}^{N}\sum_{n=0}^{N} h_{mn}\int_D x^m y^n\,\mathrm{d}D, \tag{31}$$

provided that the higher order terms could be neglected. Depending on the values of n and m, the domain integrals in Eq. (31) can be transformed into boundary integrals easily by using Eqs.(6), (8), (9), (11) and (12). Therefore, the proposed method is useful in evaluating the area integral for an analytical function and especially for an irregular shape domain. Further, the proposed method is valid for 1-D, 2-D and 3-D cases. In 1-D cases, the Gauss divergence theorem simply reduces to integration by parts.

6. Conclusions

In this study, the general boundary integral formulae for the moments of area have been derived analytically using Gauss' divergence theorem. Formulae for the finite discretization using the line elements have also been derived. Three examples involving analytical derivation and numerical implementation have been given to show the validity of the proposed method. The area integral for an analytical function can be evaluated using the boundary integral formulation while employing the Taylor's series expansion of the analytical function.

References

[1] Wolf PR, Brinker RC. Elementary surveying. 9th ed.. New York: Harper Collins College Publishers, 1994.
[2] Beer FP, Johnston Jr. ER. Vector mechanics for engineers: dynamics. 5th ed.. New York: McGraw-Hill, 1988.
[3] Gere JM, Timoshenko SP. Mechanics of materials. 4th ed. Boston: PWS Publishing Company, 1990.
[4] Chen JT, Chen KH, Yeih W, Shieh NC. Dual boundary element analysis for cracked bars under torsion. Engineering Computations 1998;15:732–749.
[5] Petrov EP. Analysis of torsion and shear characteristics of beam cross-sections by the boundary element method. The International Journal of BEM Communications 1997;8:239–245.
[6] Banerjee PK. The boundary element methods in engineering. 2. New York: McGraw-Hill, 1994.
[7] Kinoshita N, Mura T. On boundary value problem of elasticity. Research Reports of the Faculty of Engineering, Meiji University, Vol. 8. 1956:56–82.
[8] Nowak AJ, Brebbia CA. Solving Helmholtz equation by boundary elements using multiple reciprocity method. In: Calomagno GM, Brebbia CA, editors. Computers and Experiments Fluid Flow, Berlin: Comp. Mech. Pub./Springer, 1989. pp. 265–270.
[9] Kamiya N, Andoh E. Eigenvalue analysis by boundary element method. Journal of Sound and Vibration 1993;160:279–287.
[10] Kamiya N, Andoh E, Nogae K. Eigenvalue analysis by boundary element method: new developments. Engineering Analysis with Boundary. Elements 1993;16:203–207.
[11] Chen JT, Wong FC. Analytical derivation for one-dimensional eigenproblems using dual BEM and MRM. Engineering Analysis with Boundary. Elements 1997;20:25–33.
[12] Yeih W, Chen JT, Wong FC, Chen KH. A study on the multiple reciprocity method and complex-valued formulation for the Helmhotz equation. Advances in Engineering Software 1998;29(1):1–6.
[13] Kreyszig E. Advanced engineering mathematics. 7th ed.. New York: Wiley, 1993.