The rank-deficiency problem in BEM

(邊界元素法中秩降問題的探討)

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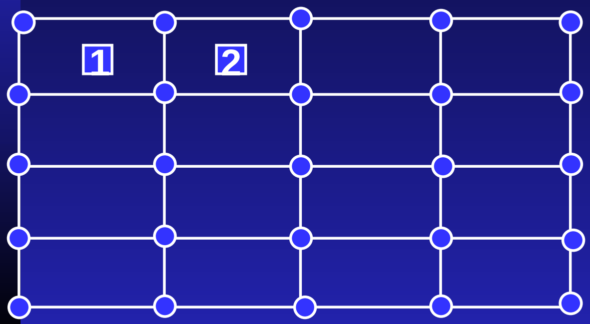
Outlines

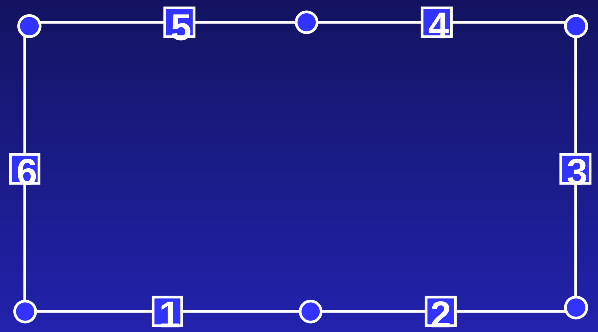
- Motivation
- FEM and BEM
- Rank-deficiency in BEM
- Treatment of rank-deficiency
- Singular value decomposition
- Fredholm alternative theorem
- **Conclusions**

Motivation

- What problems in BEM
- How to overcome the rank-deficiency problem
- Mathematical tools

FEM and BEM





● geometry node

N the Nth constant or linear element

American doctor !

Commercial software

NASTRAN

ANSYS

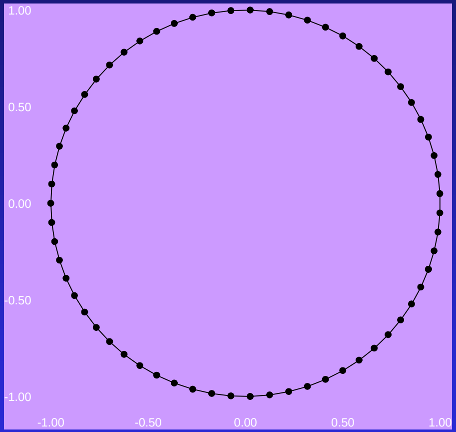
Chinese doctor !

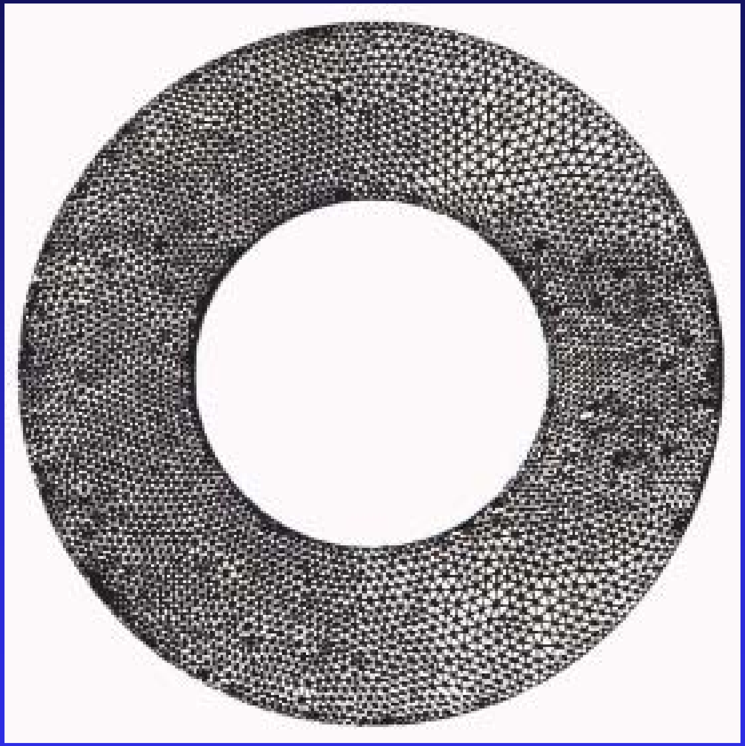
Commercial software

SYSNOISE

BEASY

The mesh of the BEM and FEM





The matrices of the FEM and BEM

FEM

BEM

$$[K]=\begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & b_3 & \ddots & \\ & & \ddots & \ddots & c \\ & & & a & b \end{bmatrix}_{m\times m}$$

$$[U]=\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}_{n\times n}$$

m > n

FEM : banded matrix

BEM : full matrix

What is the boundary element method

The wave equation

$$\nabla^2 u(x,t) = \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial^2 t}, \quad x \in D$$

Time domain

Fourier transformation

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D$$

Frequency domain

Helmholtz equation

$$\nabla^2 u(x) + k^2 u(x) = 0, \; x \in D$$

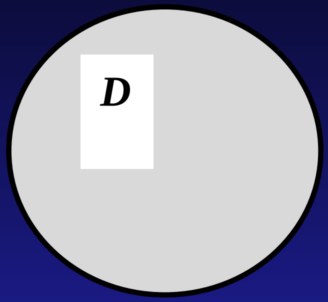
u : acoustic potential

k : wave number, $k = \omega / c$

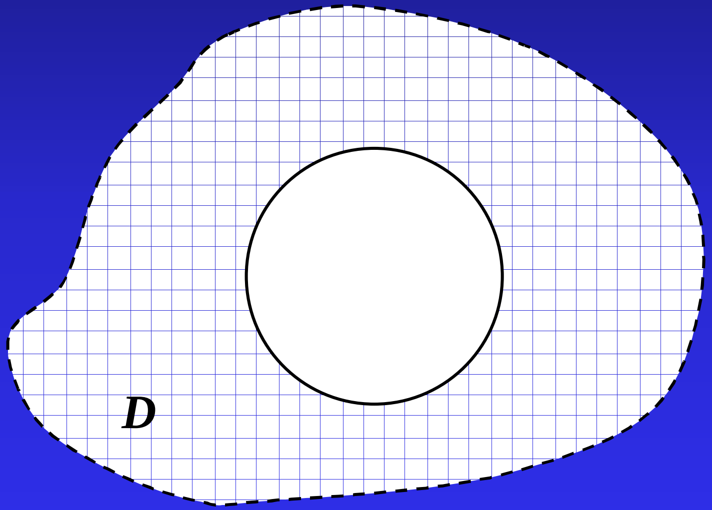
ω : angular frequency

c : sound speed

D : domain of interest



Membrane vibration



Radiation or scattering problems

Boundary integral equation

Original system

$$(\nabla^2 + k^2)u(x) = 0, \quad x \in D$$

Auxiliary system

$$(\nabla^2 + k^2)U(s,x) = \delta(s-x)$$

$$U(s,x) = \frac{-i\pi}{2} H_0^{(1)}(kr)$$

Kernel function

Reciprocal work theorem

$$\alpha\, u(x) = \int_B T(s,x)u(s)dB(s) - \int_B U(s,x)t(s)dB(s)$$

UT formulation

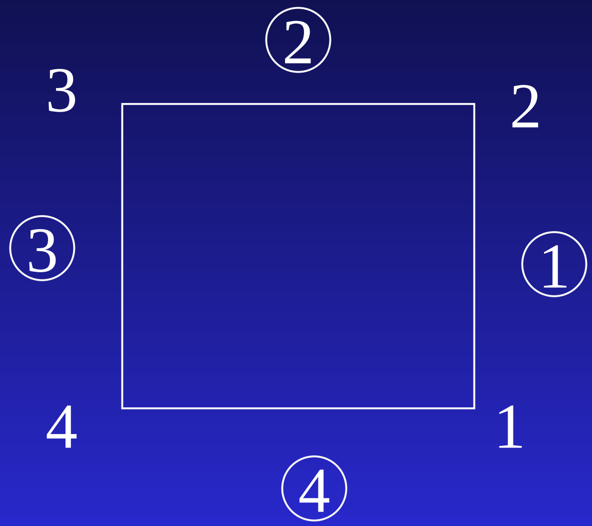
$$\alpha\, t(x) = \int_B M(s,x)u(s)dB(s) - \int_B L(s,x)t(s)dB(s)$$

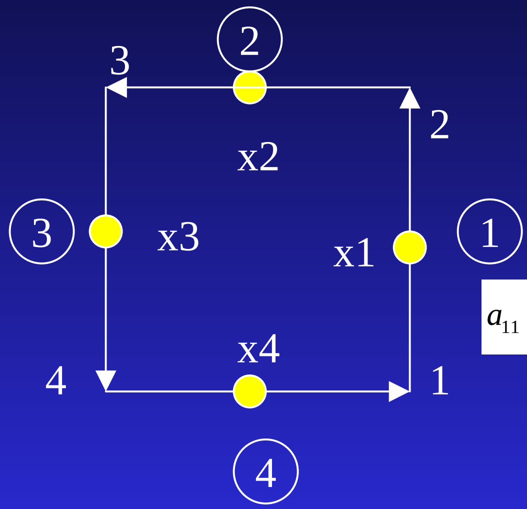
LM formulation

$$for\ 2D, \alpha = \begin{cases} 2\pi, & x \in D \\ \pi, & x \in B \\ 0, & x \in D^e \end{cases}$$

Discrete the boundary integral equation

$$[T]\{u\}=[U]\{t\}$$





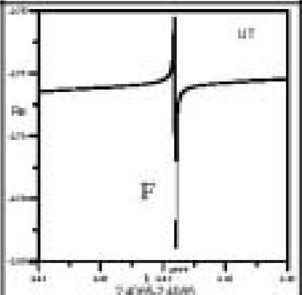
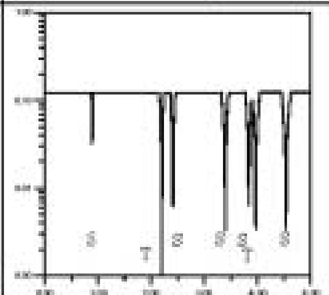
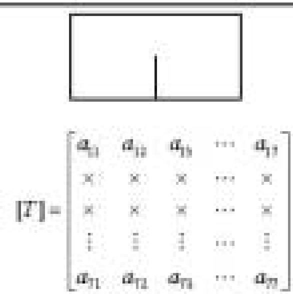
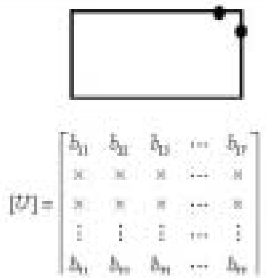
$$a_{11}=\int_{B_1}^{B_2} U(s,x)dB$$

$$[U]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Influence matrix

Four pitfalls in rank-deficiency problems using BEM

Table 1-1 The rank-deficiency problems using BEM.

Physical problems	Exterior acoustics	Interior acoustics (Eigen problem)	Degenerate boundary	Corner problem
Mathematical formulation & numerical method	Complex-valued BEM (UT or LM)	BEM (real-part BEM or imaginary-part BEM MRM)	Complex-valued BEM (UT or LM)	BEM (UT or LM)
Numerical trouble				
Treatment	1. Dual BEM 2. CHIEF method 3. Burton and Miller method	1. Dual BEM 2. CHEEF method 3. SVD updating technique	1. Dual BEM	1. Dual BEM

Methods in exterior problems using BEM

- Singular integral equation (*UT*)
- hypersingular integral equation (*LM*)
  (occurrence the fictitious frequency)
- Combined Helmholtz integral equation formulation (CHIEF) (1968) (take risk for CHIEF points)
- Burton & Miller method (1971) (*UT+i/k LM*)

Dimension for interior eigenproblems using BEM

- **Complex-value BEM (2NâN)**
- **Real-part BEM and Imaginary-part BEM, (N â N) (De Mey 1977)**
- **MRM in conjunction with SVD technique (2N â N) (Kamiya,1999)**
- **Real-part in conjunction with SVD technique, (2N â N) (Yeih *et al*. 1999)**
- **Combined Helmholtz exterior integral equation formulation (CHEEF) in conjunction with SVD technique, ((N+2) âN) (Chen *et al*. 2001)**

N : the number of boundary elements

The explicit forms of the fundamental solution

Direct BEM	Complex-valued BEM	Real-part BEM	Imag.-part BEM	MRM
U(s,x)	$\frac{-i\pi}{2}H_0^{(1)}(kr)$	$\frac{\pi}{2}Y_0(kr)$	$\frac{\pi}{2}J_0(kr)$	$\frac{\pi}{2}\bar{Y}_0(kr)$

Mathematical tools

1. Degenerate kernels
2. Circulants
3. Fredholm alternative theorem
4. SVD updating term and updating document

Degenerate kernels

$$U(s,x) = \frac{-i\pi H_0^{(1)}(kr)}{2} \quad \textbf{(closed-form)}$$

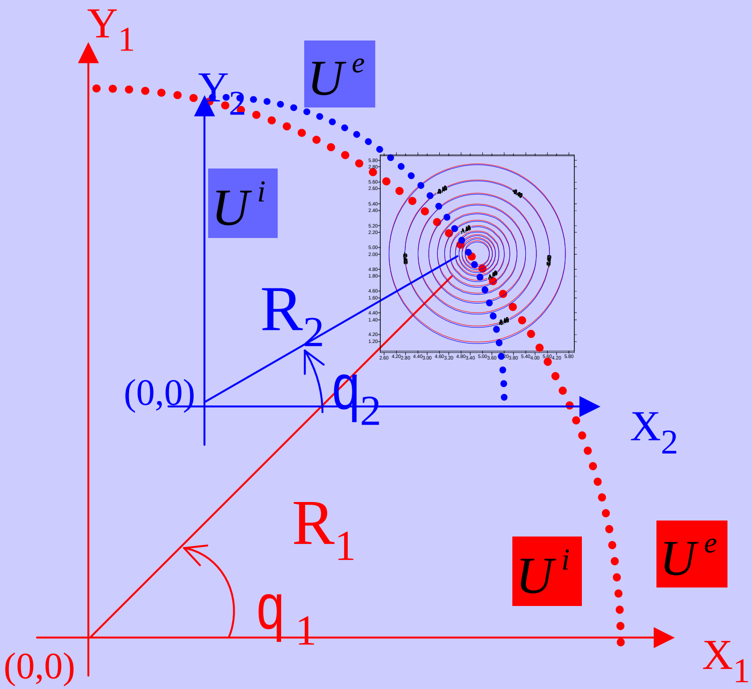
$$U(s,x) = \begin{cases} U^i(R,\theta;\rho,\phi) = \sum_{n=-\infty}^{\infty} \frac{\pi}{2}[-iJ_n(kR) + Y_n(kR)]J_n(k\rho)\cos(n(\theta-\phi)), \ R > \rho \\ U^e(R,\theta;\rho,\phi) = \sum_{n=-\infty}^{\infty} \frac{\pi}{2}[-iJ_n(k\rho) + Y_n(k\rho)]J_n(kR)\cos(n(\theta-\phi)), \ R < \rho \end{cases}$$

$H_0^{(1)}(kr)$:The first kind Hankel function with order *0*.

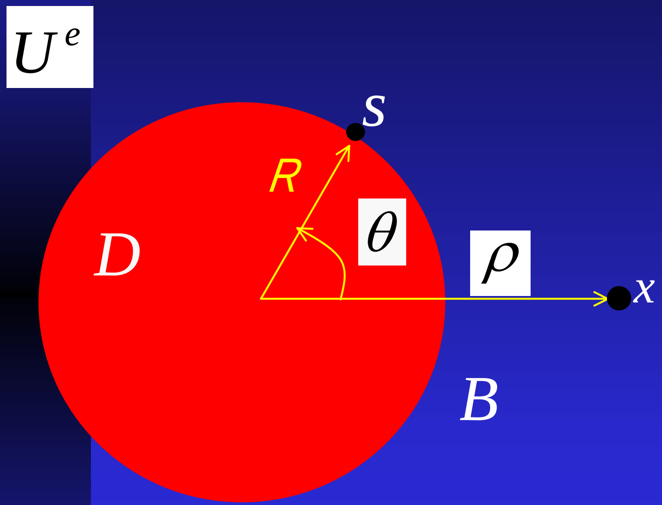
$r = |s - x|$

$s = (R,\theta)$: source point

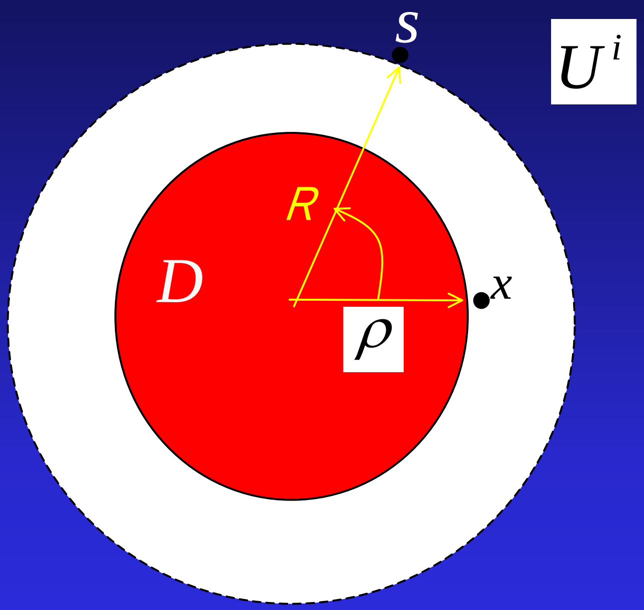
$x = (\rho,\phi)$: field point



Choice of U^i and U^e for interior problems

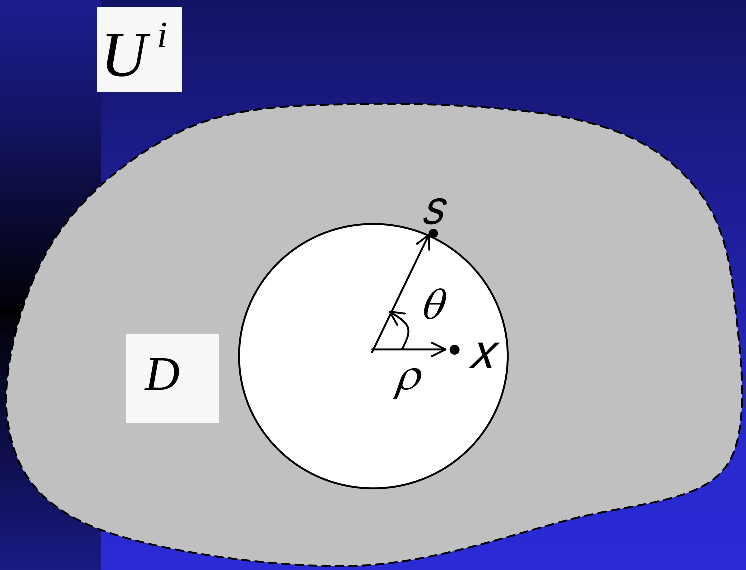


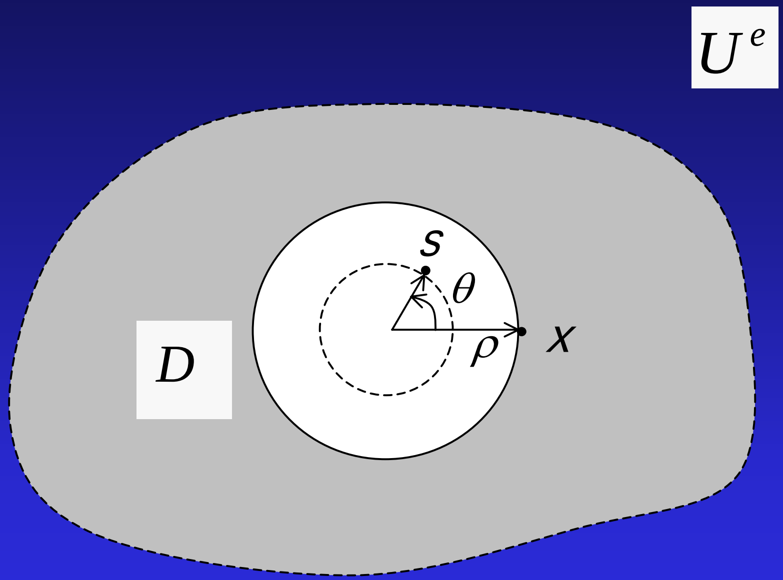
Direct method



Indirect method

Choice of U^i or U^e for exterior problems

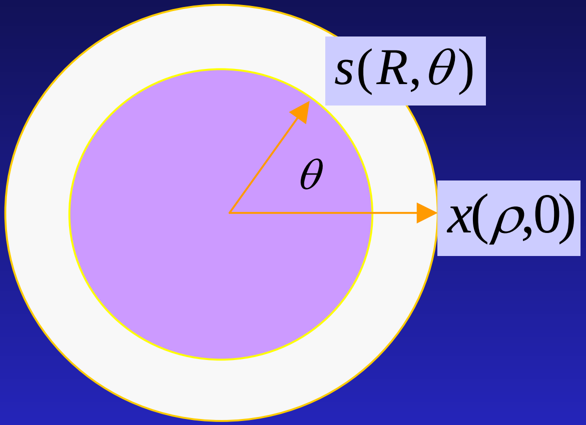




Direct method **Indirect method**

Circulants and properties of circulant

Discretizing 2N constant elements for a circular boundary



$$[G]=\begin{bmatrix} a_0 & a_1 & \cdots & a_{2N-1} \\ a_{2N-1} & a_0 & \cdots & a_{2N-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix}$$

$$[G]=a_0 I + a_1 C_{2N}^{1} + a_2 C_{2N}^{2} + \cdots + a_{2N-1} C_{2N}^{2N-1}$$

$$C_{2N}^{1}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2N\times 2N}$$

Eigenvalue and eigenvector of circulants

The eigenvalue for the circulants is the root for $\alpha^{2N} = 1$

$$\alpha_n = e^{i\frac{2\pi n}{2N}}, \quad n = 0, \pm 1, \pm 2, \cdots, \pm (N-1), N$$

The eigenvector is $\{\phi_n\} = \begin{Bmatrix} \alpha_n^0 \\ \alpha_n^1 \\ \alpha_n^2 \\ \vdots \\ \alpha_n^{2N-1} \end{Bmatrix}$

Determinants of the influence matrices

$$\det[U^i] = \lambda_0\lambda_N(\lambda_1\cdots\lambda_{N-1})(\lambda_{-1}\cdots\lambda_{-(N-1)})$$

$$\det[U^e] = \lambda_0\lambda_N(\lambda_1\cdots\lambda_{N-1})(\lambda_{-1}\cdots\lambda_{-(N-1)})$$

$$\det[T^e] = \mu_0\mu_N(\lambda\mu\cdots\mu_{N-1})(\mu_{-1}\cdots\mu_{-(N-1)})$$

$$\det[L^i] = \mu_0\mu_N(\lambda\mu\cdots\mu_{N-1})(\mu_{-1}\cdots\mu_{-(N-1)})$$

$$\det[T^i] = \upsilon_0\upsilon_N(\upsilon_1\cdots\upsilon_{N-1})(\upsilon_{-1}\cdots\upsilon_{-(N-1)})$$

$$\det[L^e] = \upsilon_0\upsilon_N(\upsilon_1\cdots\upsilon_{N-1})(\upsilon_{-1}\cdots\upsilon_{-(N-1)})$$

$$\det[M^i] = k_0k_N(k_1\cdots k_{N-1})(k_{-1}\cdots k_{-(N-1)})$$

$$\det[M^e] = k_0k_N(k_1\cdots k_{N-1})(k_{-1}\cdots k_{-(N-1)})$$

$$l = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N$$

Eigenvalue of the influence matrices

$$1.\ \lambda_l = \pi^2 \rho(-iJ_l(k\rho) + Y_l(k\rho))J_l(k\rho), \quad [U^i]\ and\ [U^e]$$

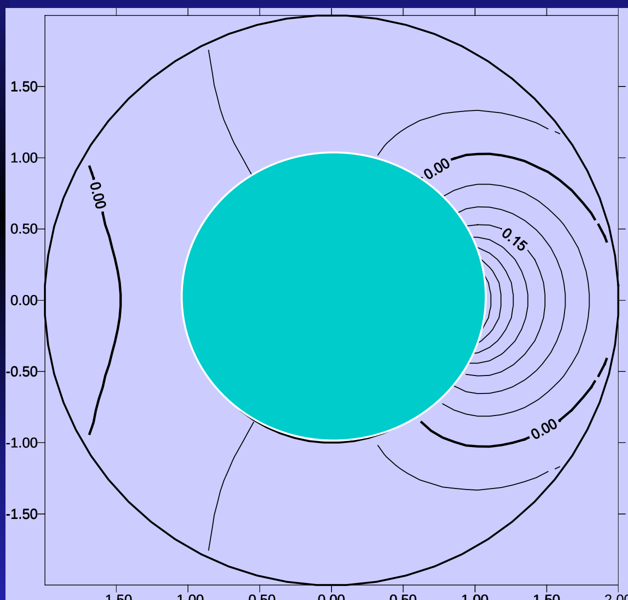
$$2.\ \mu_l = \pi^2 k\rho(-iJ_l(k\rho) + Y_l(k\rho))J_l'(k\rho), \quad [T^e]\ and\ [L^i]$$

$$3.\ \upsilon_l = \pi^2 k\rho(-iJ_l'(k\rho) + Y_l'(k\rho))J_l(k\rho), \quad [T^i]\ and\ [L^e]$$

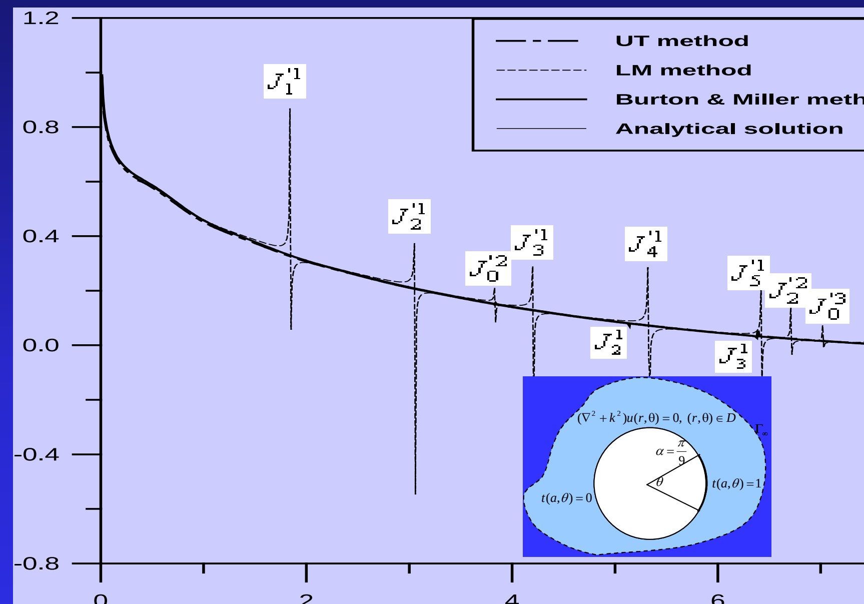
$$4.\ k_l = \pi^2 k^2 \rho(-iJ_l'(k\rho) + Y_l'(k\rho))J_l'(k\rho). \quad [M^i]\ and\ [M^e]$$

$$l = 0, \pm 1, \pm 2, \cdots \pm (N-1), N$$

Fictitious frequency (exterior problem)



u(a,0)



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Radiation problem with Neumann B. C. $t=\bar{t}$

1.Singular equation (*UT*)

$$[T^i]\{u\}=[U^i]\{\bar{t}\}=\{p_1\}$$

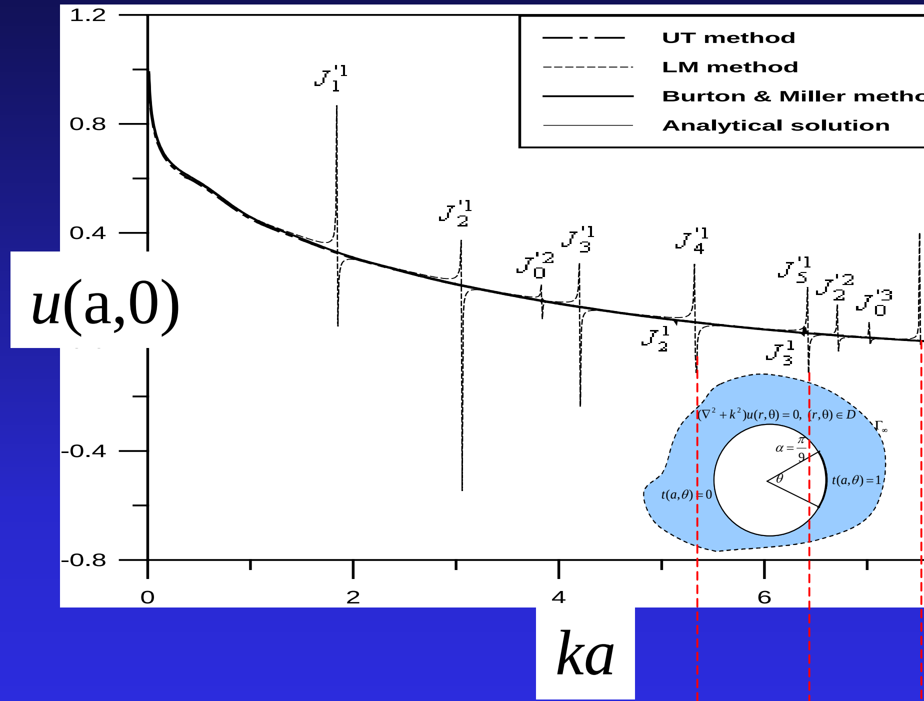
$$\{u\}=[T^i]^{-1}\{p_1\}$$

$$\upsilon_l=\pi^2 k\rho(-iJ_l'(k\rho)+Y_l'(k\rho))J_l(k\rho)$$

$$(-iJ_l'(k\rho)+Y_l'(k\rho))J_l(k\rho)=0$$

$$\because -iJ_l'(k\rho)+Y_l'(k\rho)\neq 0$$

$$J_l(k\rho)=0$$



Radiation problem with Neumann B. C.

2.Hypersingular equation (*LM*)

$$[M^i]\{u\} = [L^i]\{\bar{t}\} = \{p_2\},$$

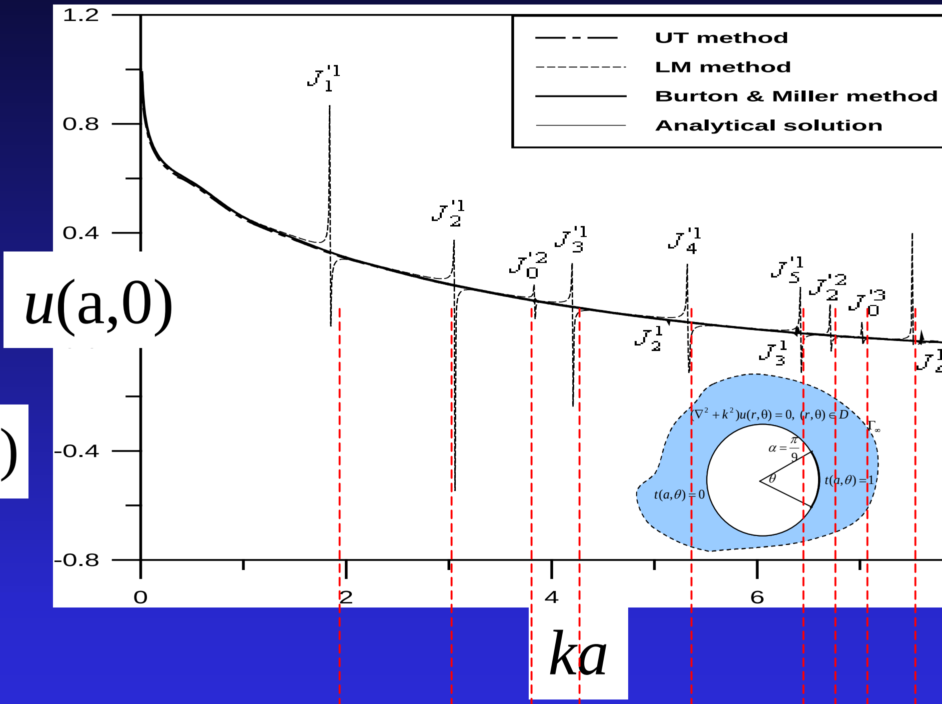
$$\{u\} = [M^i]^{-1}\{p_2\},$$

$$k_l = \pi^2 k^2 \rho(-iJ_l'(k\rho) + Y_l'(k\rho))J_l'(k\rho)$$

$$(-iJ_l'(k\rho) + Y_l'(k\rho))J_l'(k\rho) = 0$$

$$\because -iJ_l'(k\rho) + Y_l'(k\rho) \neq 0$$

$$J_l'(k\rho) = 0$$



Radiation problem with Dirichlet B. C. $u(x)=\overline{u}$

1.Singular equation (*UT*)

$$[U^i]\{t\}=[T^i]\{\overline{u}\}=\{q_1\},$$

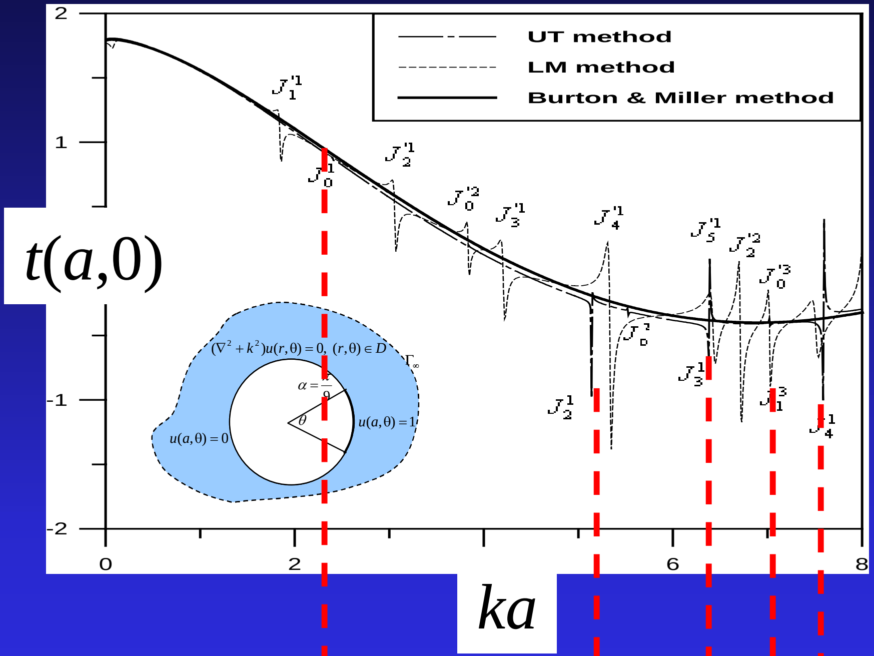
$$\{t\}=[U^i]^{-1}\{q_1\},$$

$$\lambda_l=\pi^2\rho(-iJ_l(k\rho)+Y_l(k\rho))J_l(k\rho)$$

$$(-iJ_l(k\rho)+Y_l(k\rho))J_l(k\rho)=0$$

$$\because -iJ_l(k\rho)+Y_l(k\rho)\neq 0$$

$$\therefore J_l(k\rho)=0$$



Radiation problem with Dirichlet B. C., $u(x) = \bar{u}$

2.Hypersingular equation (*LM*)

$$[L^i]\{t\} = [M^i]\{\bar{u}\} = \{q_2\},$$

$$\{t\} = [L^i]^{-1}\{q_2\},$$

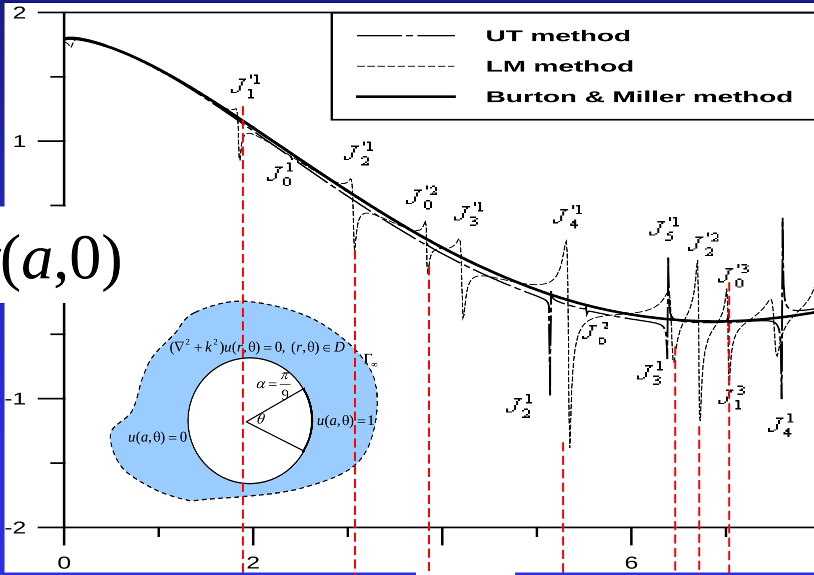
$$\mu_l = \pi^2 k\rho(-iJ_l(k\rho) + Y_l(k\rho))J_l'(k\rho)$$

$$(-iJ_l'(k\rho) + Y_l'(k\rho))J_l'(k\rho) = 0$$

$$-iJ_l'(k\rho) + Y_l'(k\rho) \neq 0$$

$$J_l'(k\rho) = 0$$

$t(a,0)$



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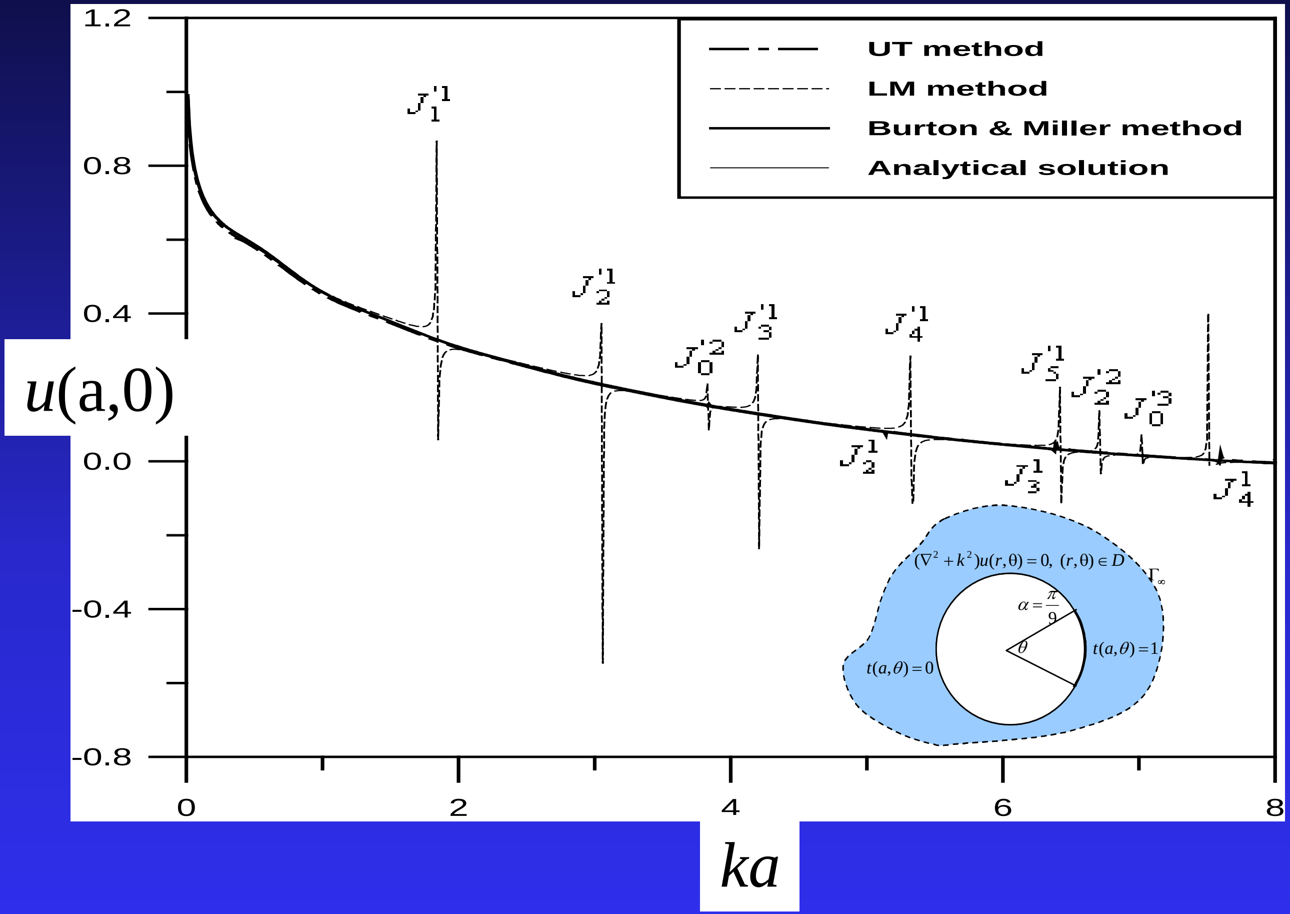
Occurrence of fictitious frequency

	Direct method		Indirect method	
	UT	*LM*	*UL*	*TM*
Dirichlet B.C.	$J_n(k\rho)$	$J'_n(k\rho)$	$J_n(k\rho)$	$J'_n(k\rho)$
Neumann B.C.	$J_n(k\rho)$	$J'_n(k\rho)$	$J_n(k\rho)$	$J'_n(k\rho)$

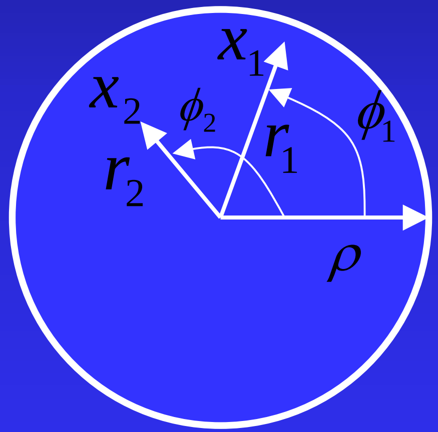
Suppression of the fictitious frequency

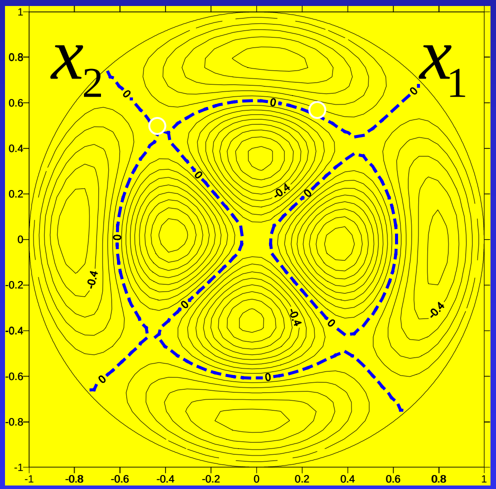


1. The CHIEF method

Combine the boundary and null field integral equation

$$\begin{bmatrix} U^{B}_{2N\times 2N} \\ U^{i}_{a\times 2N} \end{bmatrix}\{t\} = \begin{bmatrix} T^{B}_{2N\times 2N} \\ T^{i}_{a\times 2N} \end{bmatrix}\{u\}$$

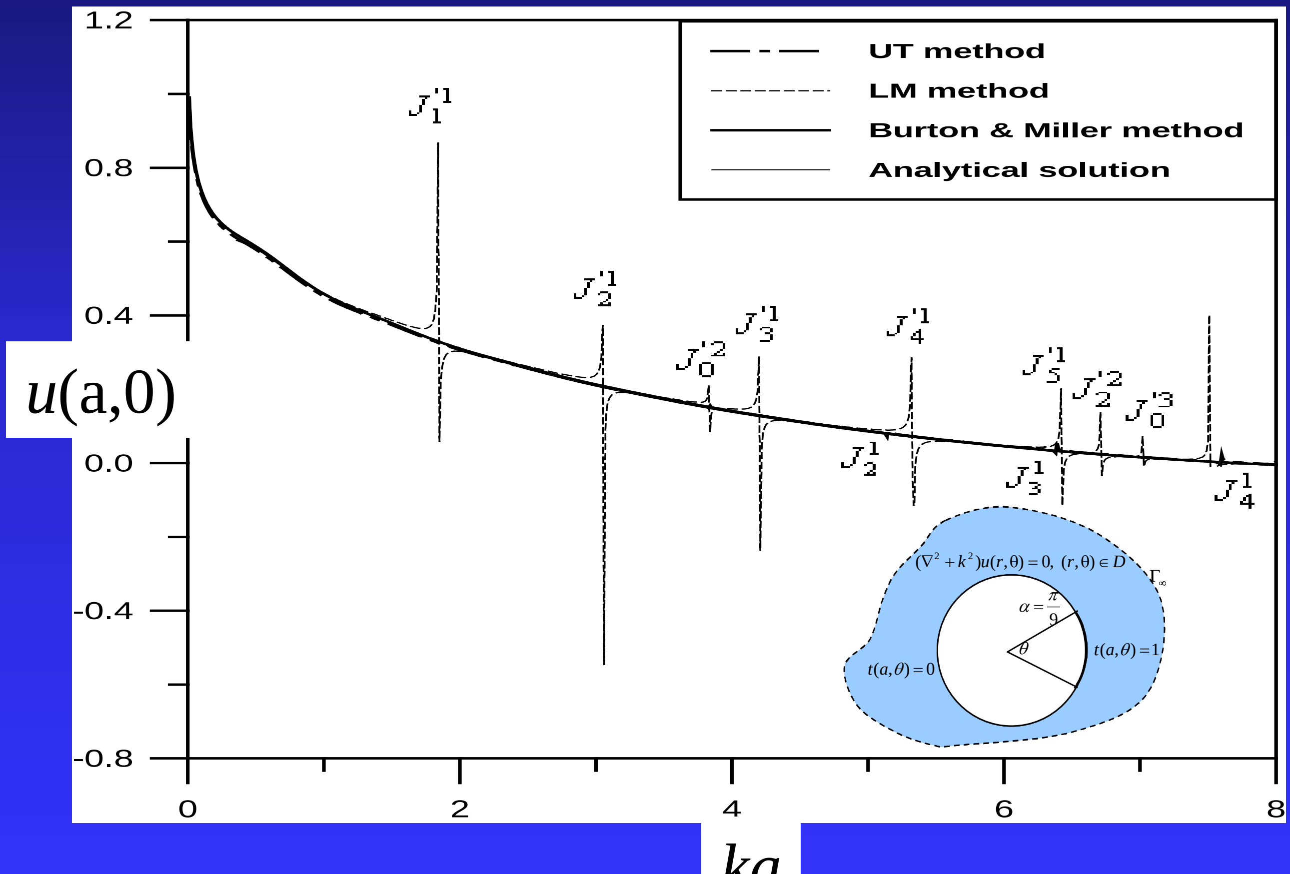




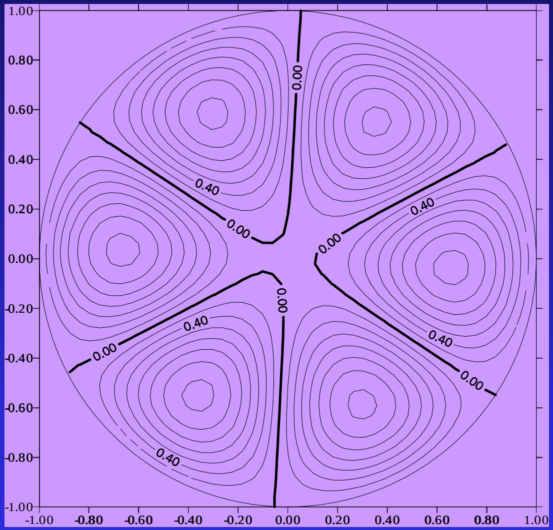
2. The Burton and Miller method

Combine the singular equation (*UT*) and hypersingular equation (*LM*) through an imaginary constant

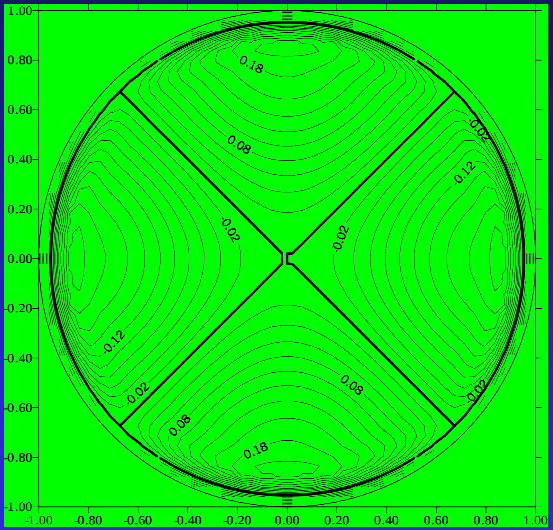
$$[U+\frac{i}{k}L]\{t\}=[T+\frac{i}{k}M]\{u\}$$



Spurious eigenvalue (interior problem)



True mode



Spurious mode

Interior problem with Dirichlet B.C. $u=0$

a. Complex-valued BEM *UT* equation

$$[U^e]\{t\}=[T^e]\{u\}=0,$$

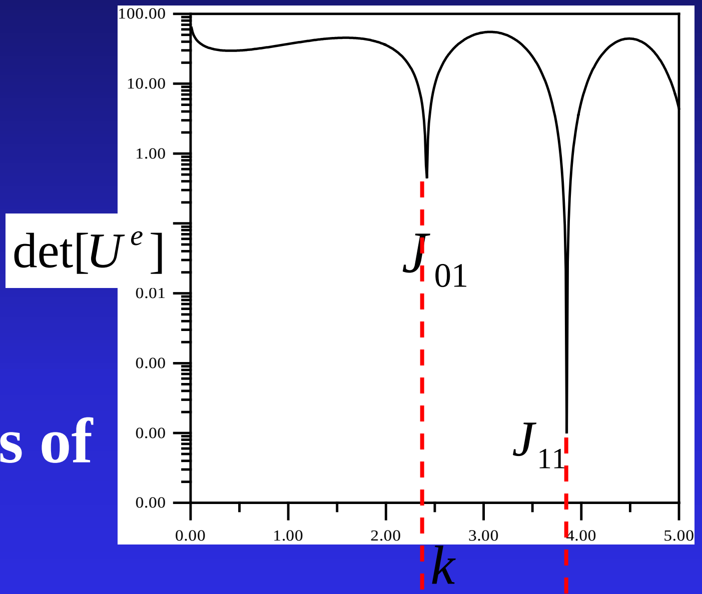
The eigenequation is derived

$$\therefore(-iJ_l(k\rho)+Y_l(k\rho))J_l(k\rho)=0$$

$$-iJ_l(k\rho)+Y_l(k\rho)\neq 0$$

The true eigenvalues are the roots of

$$J_l(k\rho)=0$$



Interior problem with Dirichlet B.C. $u = 0$

b. Complex-valued BEM *LM* equation

$$[L^e]\{t\} = [M^e]\{u\} = 0,$$

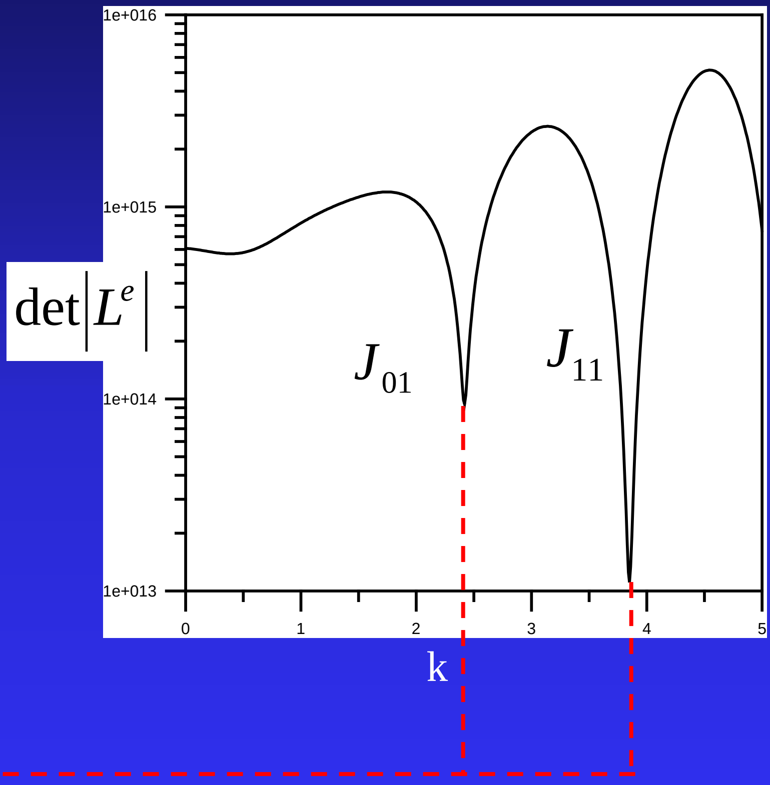
The eigenequation is derived

$$\therefore (-iJ'_l(k\rho) + Y'_l(k\rho))J_l(k\rho) = 0$$

$$-iJ'_l(k\rho) + Y'_l(k\rho) \neq 0$$

The true eigenvalues

are the roots of $J_l(k\rho) = 0$



Interior problem with Dirichlet B.C. $u = 0$

Real-part BEM

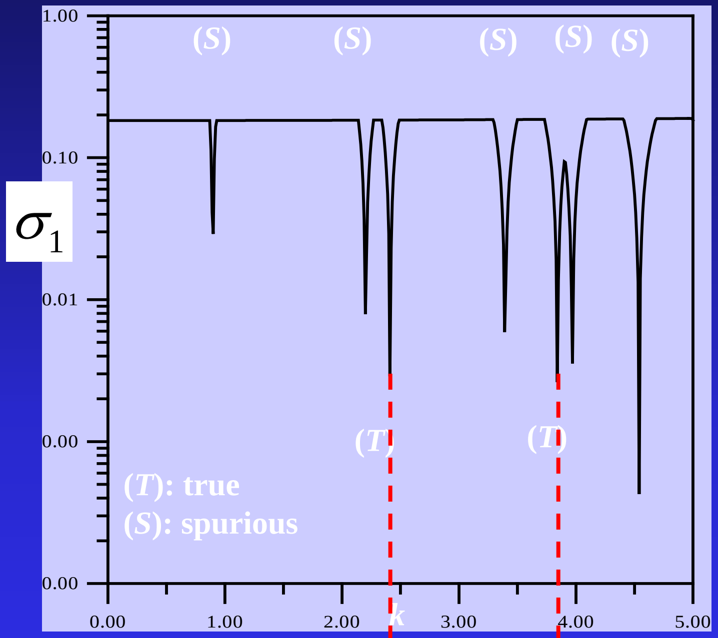
a. The *UT* equation

$$[U_R^e]\{t\} = [T_R^e]\{u\} = 0$$

***R* denotes the real part.**

The true and spurious eigenvalues are the roots of

$$Y_l(k\rho)J_l(k\rho) = 0$$



$J_l(k\rho) = 0$ **, true eigenvalue**

$Y_l(k\rho) = 0$ **, spurious eigenvalue**

Interior problem with Dirichlet B.C. $u = 0$

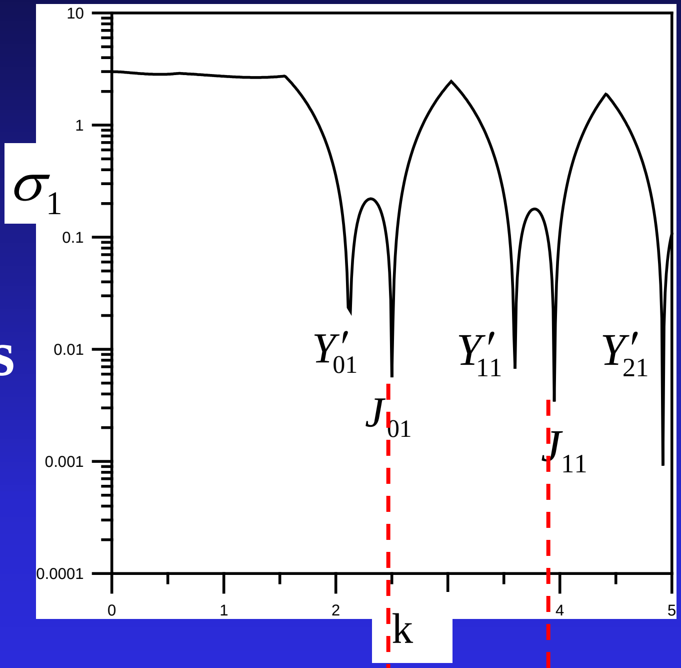
Real-part BEM

b. The *LM* equation

$$[L_R^e]\{t\} = [M_R^e]\{u\} = 0$$

The true and spurious eigenvalues are the roots of

$$Y'_l(k\rho)J_l(k\rho) = 0$$



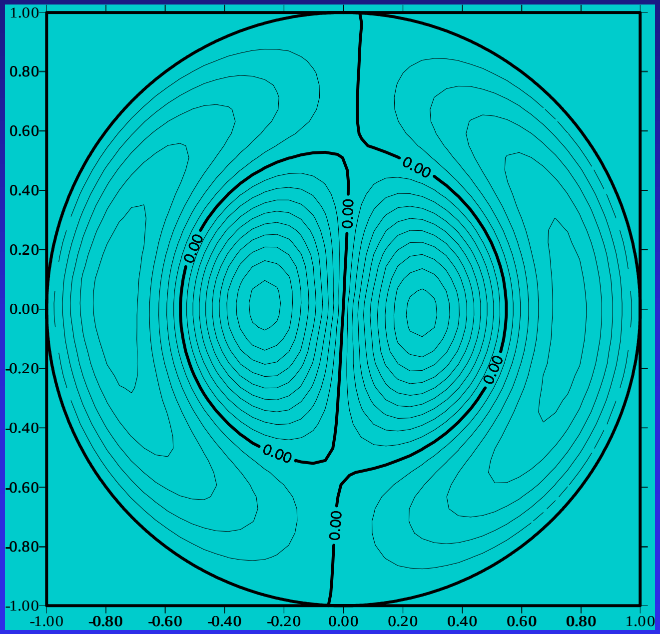
$J_l(k\rho) = 0$, **true eigenvalue**

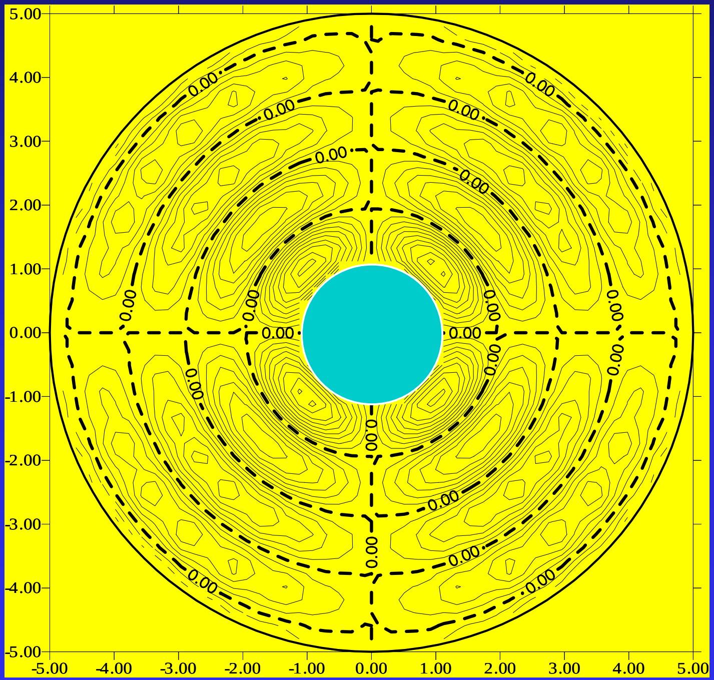
$Y'_l(k\rho) = 0$, **spurious eigenvalue**

		Direct method					
		UT formulation			*LM* formulation		
		Comp. valued BEM	Real-part BEM	Imag.-part BEM	Comp. valued BEM	Real-part BEM	Imag.-part BEM
Dirichlet B.C.	True	J_n	J_n	J_n	J_n	J_n	J_n
	Spurious		Y_n	J_n *		Y'_n	J'_n
Neumann B.C.	True	J'_n	J'_n	J'_n	J'_n	J'_n	J'_n
	Spurious		Y_n	J_n		Y'_n	J'_n *

* Denote the spurious multiplicity

Suppression of the spurious eigenvalue (CHEEF)

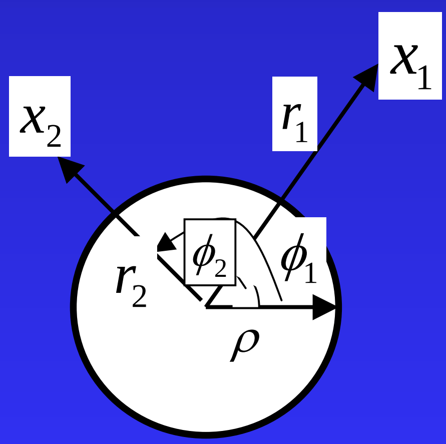


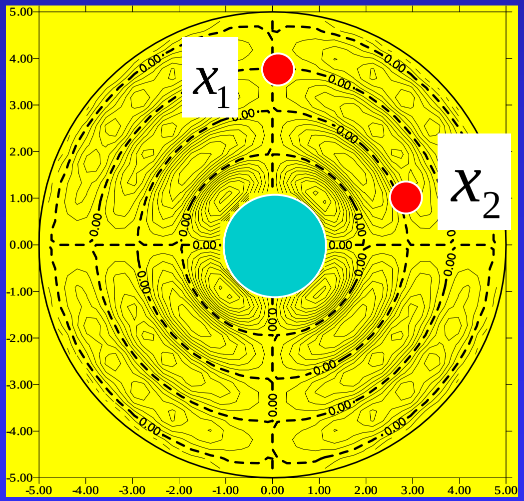


The CHEEF method

Combined Helmholtz exterior integral equation formulation (CHEEF) in conjunction with SVD technique

$$\begin{bmatrix} U^{B}_{2N\times 2N} \\ U^{e}_{a\times 2N} \end{bmatrix} \{t\} = \begin{bmatrix} T^{B}_{2N\times 2N} \\ T^{e}_{a\times 2N} \end{bmatrix} \{u\}$$





Fredholm alternative theorem

For solving an algebraic system:

$[A]\{x\} = \{b\} \neq \{0\}$ → $\{x\}$ **has a unique solution,**

If and only if

$[A]\{x\} = \{0\}$ → $\{x\} = \{0\}$

Alternatively

$\{x\}$ **has at least one solution,**

if

$[A]^{+}\{\phi\} = \{0\}$ → $\{\phi\}$ **: Nontrivial solutio**

$\{b\}^{+}\{\phi\} = 0$

"+" denotes transpose conjugate

Filter out the spurious eigenvalue

UT **formulation :** $[U_R^e]\{t\}=[T_R^e]\{u\}$

k_s **: spurious eigenvalue**

Neumann problem $(t=\bar{t})$

$\Rightarrow [T_R^e]\{u\}=\{b\}$

Dirichlet problem $(u=\bar{u})$

$\Rightarrow [U_R^e]\{t\}=\{q\}$

Fredholm Alternative therrem

$\{b^T\}\{\phi_s\}=0$

$[T^T]\{\phi_s\}=\{0\}$

$\{q^T\}\{\phi_s\}=0$

$[U^T]\{\phi_s\}=\{0\}$

$b^T\}\{\phi_s\}=\{\bar{t}^T\}[U_R^e]^T\{\phi_s\}=0$

$\{q^T\}\{\phi_s\}=\{\bar{u}^T\}\{T^T\}\{\phi_s\}=0$

SVD updating term

$\Rightarrow \begin{bmatrix} U^T \\ T^T \end{bmatrix}\{\phi_s\}=\{0\}$

$\Rightarrow \begin{bmatrix} U^T \\ T^T \end{bmatrix}\{\phi_s\}=\{0\}$

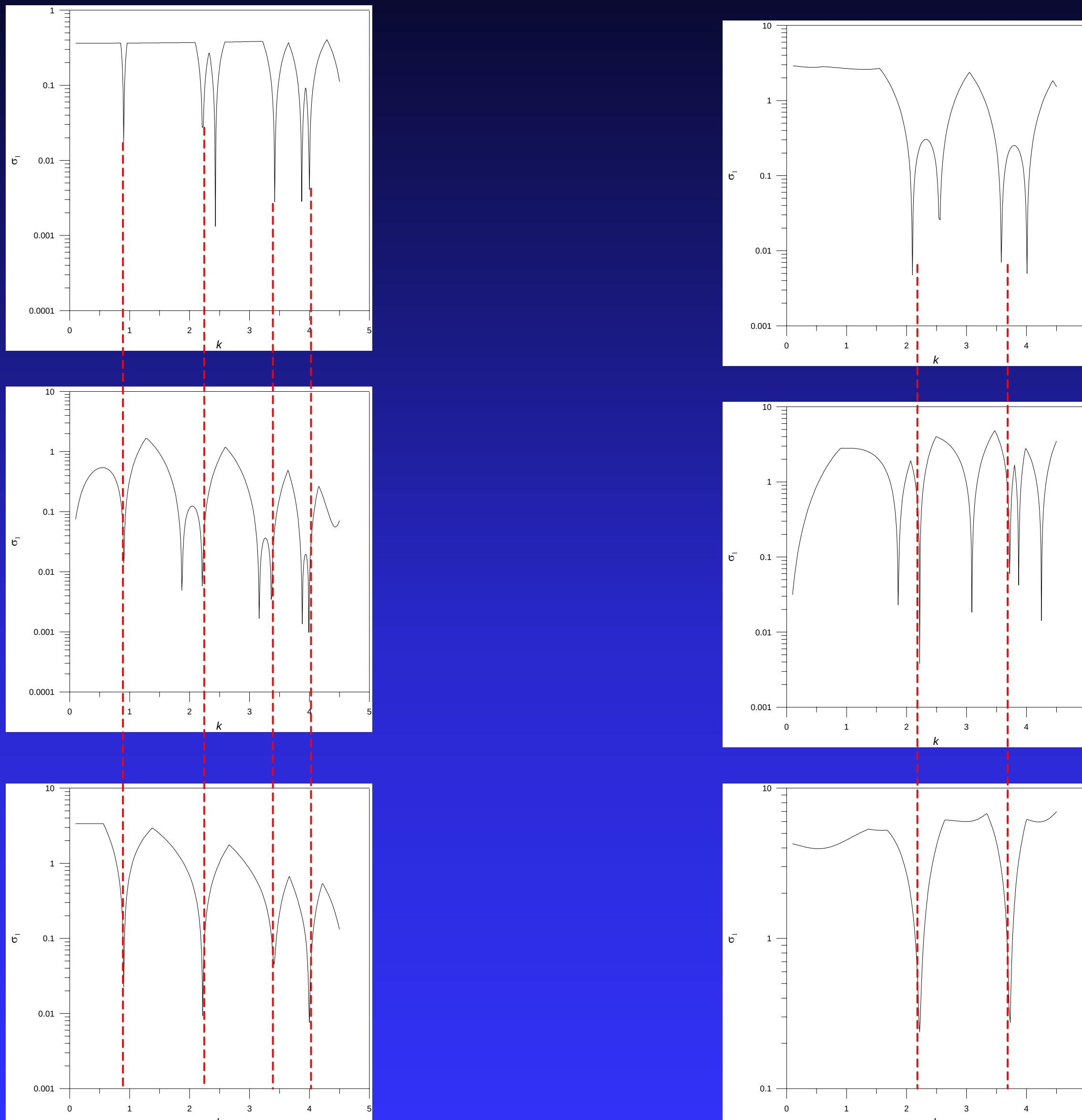
$\{\phi_s\}$**: spurious mode**

SVD updating terms for the spurious eigensolutions

[U]

[T]

$$\begin{bmatrix} U^T \\ T^T \end{bmatrix}$$

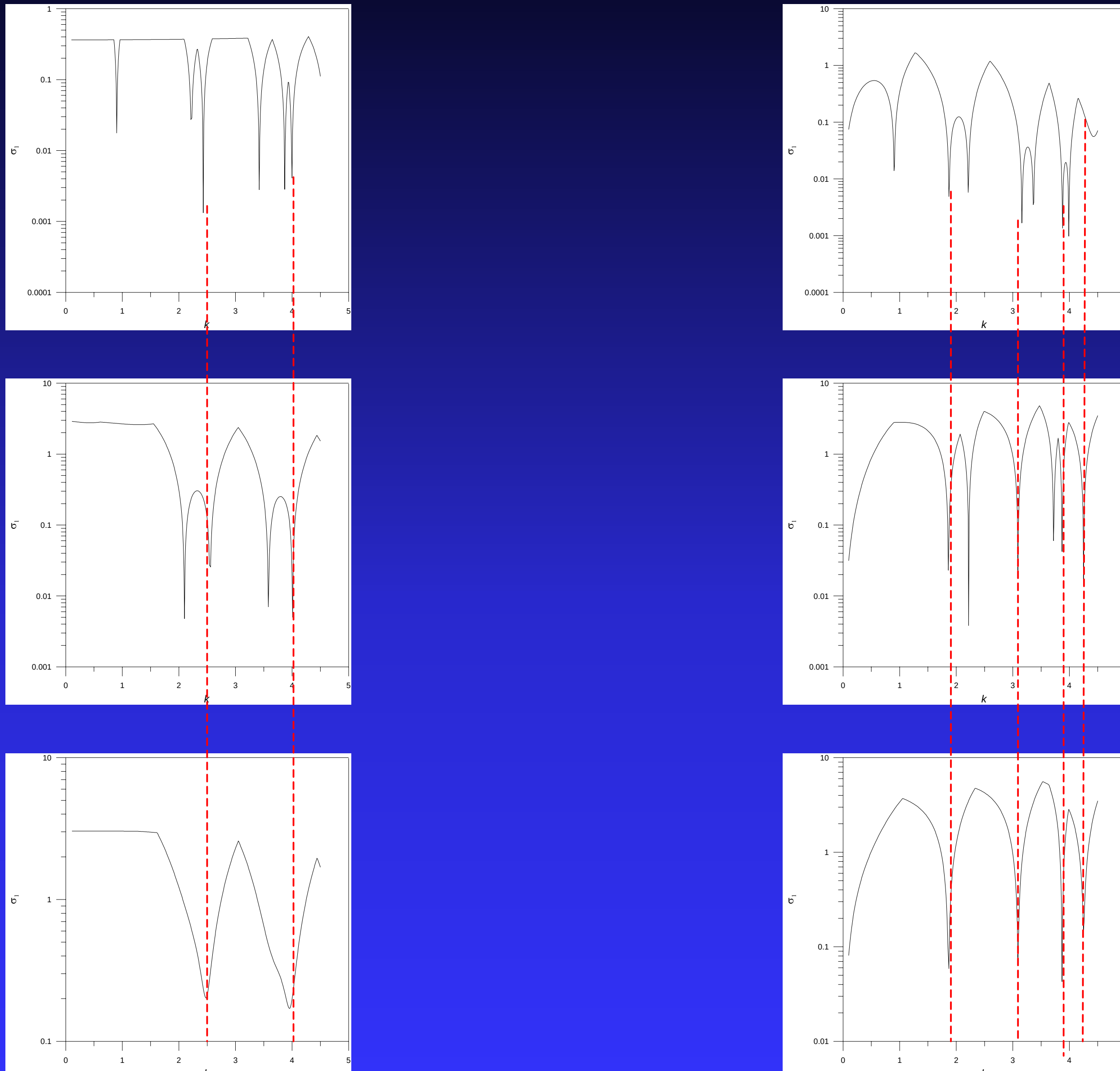


[L]

[M]

$$\begin{bmatrix} L^T \\ M^T \end{bmatrix}$$

SVD updating terms for the true eigensolutions



[U]

[T]

[L]

[M]

$$\begin{bmatrix} U \\ L \end{bmatrix}$$

$$\begin{bmatrix} T \\ M \end{bmatrix}$$

Conclusions

1. By using the degenerate kernels and circulants properties, the rank-deficiency mechanism were studied for interior and exterior problems.

2. By using the Fredholm alternative theorem in conjunction with the SVD updating technique, the spurious and true boundary modes can be extracted out from the left and right unitary vectors, respectively.

3. The CHIEF method adopted to overcome the fictitious frequency and the criterion for choosing the better CHIEF points was suggested.

4. To overcome the spurious eigenvalue, the CHEEF method was proposed. The optimum numbers and proper positions for the CHEEF points were analytically studied