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ON THE TRUE AND SPURIOUS EIGENSOLUTIONS USING CIRCULANTS FOR REAL-PART DUAL BEM

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Abstract It has been found recently that the multiple reciprocity method (MRM), the real-part BEM and the imaginary-part BEM result in spurious eigenvalues for eigenproblems. In this paper, a circular domain is considered as a demonstrative example. Based on the dual framework of real-part BEM, the true and spurious eigenvalues can be separated by using the singular value decomposition technique (SVD). To understand why the spurious eigenvalues occur, analytical derivation by discretizing the circular boundary into a discrete system is employed and results in a circulants. By using the SVD updating terms, the true eigensolutions can be extracted by merging the two influence matrices in dual BEM. The spurious eigensolutions can be filtered out by using the SVD updating documents where the other two influence matrices are combined.

Keywords: real-part BEM; spurious eigensolution;SVD updating technique

1. INTRODUCTION

It is well known that fictitious frequency occurs when the singular integral equation or hypersingular integral equation is used alone to solve the exterior acoustic problems (radiation or scattering cases). The reason has been clearly understood that the nonunique solution results from the zero in the denominator [1]. Although some researchers called this irregular value " spurious frequency " [2, 3], we will distinguish the differences between fictitious and spurious solutions in this paper.

For interior eigenproblems, complex-valued boundary integral formulation [4] has been used to determine the eigensolutions. To avoid the complex-valued computation, multiple reciprocity method [5], real-part [6], and imaginary-part [7] formulations have been tried to solve the same problem. However, spurious eigensolutions are embedded in the

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above three methods. To overcome this difficulty, the dual formulation in conjunction with the SVD technique [5, 6, 8] is a novel method to extract the true solutions. This technique has been successfully applied to rod [9], beam [10] and cavity problems using dual MRM [5] or real-part dual BEM [6]. A unified method to filter out the spurious solutions is not trivial.

In this paper, we employ the real-part dual BEM to solve the acoustic problem of a circular cavity. After assembling the dual equations, the singular value decomposition (SVD) technique is employed to extract the true and spurious eigenvalues for two-dimensional cavities. The spurious eigensolutions are analytically predicted in the discrete system of circulants and are compared with those obtained by using the real-part dual BEM program, DUALREAL. Finally, the true eigenvalues for a circular cavity are derived analytically by approaching the discrete system to continuous system using the analytical properties of circulants [11]. Two approaches, SVD updating terms and updating documents, are employed to extract the true and spurious solutions respectively. Three results, analytical solution, discrete system solution using circulants and numerical solution using BEM, are compared with each other.

2. DUAL INTEGRAL FORMULATION FOR A TWO-DIMENSIONAL INTERIOR ACOUSTIC PROBLEMS

The governing equation for an interior acoustic problem is the Helmholtz equation as follows:

$$(\nabla^2 + k^2)u(x_1, x_2) = 0, \quad (x_1, x_2) \in D,$$

where ∇^2 is the Laplacian operator, D is the domain of the cavity and k is the wave number, which is angular frequency over the speed of sound. The boundary conditions can be either the Neumann or Dirichlet type.

Based on the dual formulation [12], the dual equations for the boundary points are

$$\pi u(x) = C.P.V. \int_B T(s,x)u(s)dB(s) - R.P.V. \int_B U(s,x)t(s)dB(s), \quad x \in B \quad (1)$$

$$\pi t(x) = H.P.V. \int_B M(s,x)u(s)dB(s) - C.P.V. \int_B L(s,x)t(s)dB(s), \quad x \in B \quad (2)$$

where $R.P.V.$, $C.P.V.$ and $H.P.V.$ denote the Riemann principal value, the Cauchy principal value, and the Hadamard principal value, $t(s) = \frac{\partial u(s)}{\partial n_s}$, B denotes the boundary enclosing D and the explicit forms of the four kernels, U, T, L and M, can be found in [12].

3. CIRCULANT MATRICES FOR INTERIOR PROBLEMS USING THE REAL-PART DUAL BEM

By using the two sets of bases, $J_m(kx)$ and $Y_m(kx)$, we can decompose the two-dimensional real-part kernel functions into

$$U(s,x) = \begin{cases} U^i(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi}{2} Y_m(kR)J_m(k\rho)cos(k\theta), & R > \rho \\ U^e(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi}{2} Y_m(k\rho)J_m(kR)cos(k\theta), & R < \rho \end{cases} \quad (3)$$

$$T(s,x) = \begin{cases} T^i(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi k}{2} Y'_m(kR)J_m(k\rho)cos(k\theta), & R > \rho \\ T^e(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi k}{2} Y_m(k\rho)J'_m(kR)cos(k\theta), & R < \rho \end{cases} \quad (4)$$

$$L(s,x) = \begin{cases} L^i(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi k}{2} Y_m(kR)J'_m(k\rho)cos(k\theta), & R > \rho \\ L^e(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi k}{2} Y'_m(k\rho)J_m(kR)cos(k\theta), & R < \rho \end{cases} \quad (5)$$

$$M(s,x) = \begin{cases} M^i(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi k^2}{2} Y'_m(kR)J'_m(k\rho)cos(k\theta), R > \rho \\ M^e(\theta,0) = \sum_{m=-\infty}^{\infty} \frac{\pi k^2}{2} Y'_m(k\rho)J'_m(kR)cos(k\theta), R < \rho \end{cases} \quad (6)$$

where the superscripts "i" and "e" denotes interior and exterior domain, J_m and Y_m are the m-th order Bessel functions of the first and second kinds, respectively, $x = (\rho, 0)$ and $s = (R, \theta)$ in polar coordinate. By discretizing $2N$ constant elements on a circular boundary, we have the influence matrix in a form of the circulants as shown belows.

$$[K] = \begin{bmatrix} k_0 & k_1 & k_2 & \cdots & k_{2N-2} & k_{2N-1} \\ k_{2N-1} & k_0 & k_1 & \cdots & k_{2N-3} & k_{2N-2} \\ k_{2N-2} & k_{2N-1} & k_0 & \cdots & k_{2N-4} & k_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k_1 & k_2 & k_3 & \cdots & k_{2N-1} & k_0 \end{bmatrix} \quad (7)$$

where

$$k_m = \int_{(m-\frac{1}{2})\Delta\theta}^{(m+\frac{1}{2})\Delta\theta} F^e(\theta,0)\rho\, d\theta \approx F^e(\theta_m,0)\rho\, \Delta\theta, \quad m = 0,1,2,\cdots,2N-1 \quad (8)$$

in which $[K]$ can be the influence matrices of U, T, L or M , F^e can be U^e, T^e, L^e or M^e kernels as shown in Eqs.(3)-(6), $\theta_m = m\Delta\theta$ and $\Delta\theta = \frac{2\pi}{2N}$. It is interesting to find that all the matrices in Eq.(7) are symmetric circulants. By using the properties of circulants for the matrices, we have the determinants,

$$det[U] = \lambda_0(\lambda_1\lambda_2\cdots\lambda_{N-1})^2\lambda_N \tag{9}$$

$$det[L] = \mu_0(\mu_1\mu_2\cdots\mu_{N-1})^2\mu_N \tag{10}$$

$$det[T] = \nu_0(\nu_1\nu_2\cdots\nu_{N-1})^2\nu_N \tag{11}$$

$$det[M] = \kappa_0(\kappa_1\kappa_2\cdots\kappa_{N-1})^2\kappa_N \tag{12}$$

where

$$\lambda_\ell = \pi^2\rho Y_\ell(k\rho)J_\ell(k\rho),\ \ \ell = 0, \pm1, \cdots, \pm(N-1), N. \tag{13}$$

$$\mu_\ell = \pi^2 k\rho Y_\ell'(k\rho)J_\ell(k\rho),\ \ \ell = 0, \pm1, \cdots, \pm(N-1), N. \tag{14}$$

$$\nu_\ell = \pi^2 k\rho Y_\ell(k\rho)J_\ell'(k\rho),\ \ \ell = 0, \pm1, \cdots, \pm(N-1), N. \tag{15}$$

$$\kappa_\ell = \pi^2 k^2\rho Y_\ell'(k\rho)J_\ell'(k\rho),\ \ \ell = 0, \pm1, \cdots, \pm(N-1), N. \tag{16}$$

4. METHODS TO EXTRACT THE TRUE EIGENSOLUTIONS — SVD UPDATING TERMS

Since real-part BEM loses imaginary information, we can reconstruct the independent equation by differentiation. This results in dual formulation in order to extract the true eigensolutions. To obtain an overdetermined system, we can combine $[U]$ and $[L]$ matrices by using updating terms,

$$[C] = \begin{bmatrix} U \\ L \end{bmatrix}_{4N\times 2N} \tag{17}$$

for the Dirichlet problem. Since the eigensolution is nontrival, the rank of $[C]$ must be smaller than $2N$. Therefore, the $2N$ singular values for $[C]$ matrix must have at least one zero value. Based on the equivalence between the SVD technique and the least-squares method in mathematical essence, we have

$$\begin{bmatrix} U^T & L^T \end{bmatrix} \begin{bmatrix} U \\ L \end{bmatrix} = [U^2 + L^2] \tag{18}$$

since $[U]$ and $[L]$ are symmetric. For a circular cavity, we have

$$[U] = [\Phi] \begin{bmatrix} \ddots & & \\ & \lambda_\ell & \\ & & \ddots \end{bmatrix} [\Phi^T] \tag{19}$$

$$[L] = [\Phi] \begin{bmatrix} \ddots & & \\ & \mu_\ell & \\ & & \ddots \end{bmatrix} [\Phi^T] \tag{20}$$

where $[\Phi]$ is a modal matrix. By substituting Eq.(19) and (20) into Eq.(18), we can obtain

$$[C^T]\,[C] = [\Phi] \begin{bmatrix} \ddots & & \\ & (\lambda_\ell^2 + \mu_\ell^2) & \\ & & \ddots \end{bmatrix} [\Phi^T] \tag{21}$$

Therefore, we have the singular values of $\sqrt{\lambda_\ell^2 + \mu_\ell^2}$, $\ell = 0, \pm 1, \pm 2,$ $\pm(N-1), N$. By plotting the minimum singular value for C versus k, only true eigenvalues have dips. The results will be elaborated on later.

5. METHODS TO FILTER OUT THE SPURIOUS EIGENSOLUTIONS — SVD UPDATING DOCUMENTS

Based on the dual formulation, the $[U]$ and $[T]$ matrices have the same spurious eigenvalues. This results in spurious eigensolutions. In order to extract the spurious eigenvalues, we can combine the $[U^T]$ and $[T^T]$ matrices by using updating documents,

$$[D] = \begin{bmatrix} U^T \\ T^T \end{bmatrix}_{4N \times 2N} \tag{22}$$

Similarly, we have

$$\begin{bmatrix} U & T \end{bmatrix} \begin{bmatrix} U^T \\ T^T \end{bmatrix} = [U^2 + T^2] \tag{23}$$

since $[U]$ and $[T]$ are symmetric. According to Eqs.(13)-(16), the spurious eigenvalues are embedded in the transposes of $[U]$ and $[T]$ matrices. The singular values for $[D]$ must have at least one zero value. To determine the singular values for $[D]$, we have

$$[D^T]\,[D] = [\Phi] \begin{bmatrix} \ddots & & \\ & (\lambda_\ell^2 + \nu_\ell^2) & \\ & & \ddots \end{bmatrix} [\Phi^T] \tag{24}$$

By plotting the minimum singular values of $\sqrt{\lambda_\ell^2 + \nu_\ell^2}, \ell = 0, \pm 1, \pm 2, \pm (N-1), N$ versus k , we can extract the spurious eigenvalues where dips occur.

6. NUMERICAL RESULTS AND DISCUSSIONS

A circular cavity with a radius ($\rho = 1\,m$) subjected to the Dirichlet boundary condition ($u = 0, \rho = 1$) is considered. In this case, an analytical solution is available as follows:
eigenequation: $J_m(k_{mn}) = 0,\ m, n = 0, 1, 2, 3 \cdots$;
eigenmode : $u(a, \theta) = J_m(k_{mn}a)e^{im\theta}, 0 < a < \rho, 0 < \theta < 2\pi$.

Twenty constant elements with N=10 are adopted in the discrete system. Since two alternatives, the UT or LM equation, can be used to collocate on the circular boundary, two results from the UT and LM methods can be obtained. Fig.1 shows the minimum singular value versus k using the UT method. The true eigenvalues contaminated by the spurious eigenvalues can be obtained as shown in Fig.1 by considering the near zero minimum singular values if only the UT equation is chosen. The true eigenvalues occur at the positions of zeros for $J_m(k_{mn}\rho)$ while the spurious eigenvalues occur at the positions of zeros for $Y_m(k_{mn}\rho)$. Fig.2 shows the minimum singular value versus k only using the LM equation. In a similar way, the true eigenvalues contaminated by spurious eigenvalues can be obtained as shown in Fig.2 by considering the near zero minimum singular values if the LM equation is chosen alone. The true eigenvalues occur at the positions of zeros for $J_m(k_{mn}\rho)$ while the spurious eigenvalues occur at the positions of zeros for $Y'_m(k_{mn}\rho)$. It is interesting to find that no spurious eigenvalues occur as shown in Fig.3 when the U and L matrices are combined as shown in Eq.(18). To extract the spurious eigensolutions, we combine the U^T and T^T matrices as shown in Eq.(23). The minimum singular value versus k is shown in Fig.4, it is found that dips occur only at the positions of spurious eigenvalues.

7. CONCLUSIONS

The real-part dual BEM in conjunction with the SVD technique using circulants has been applied to determine the true and spurious eigensolutions of a circular cavity. The spurious eigenvalues have been successfully extracted for discrete system. The true eigenvalues obtained by the real-part dual BEM also match very well the exact solutions. Two approaches, SVD updating terms and updating documents, were proposed to extract the true eigensolutions and to filter out the spurious eigensolutions by using the dual formulation. Three solutions, analytical solution, discrete system solution using circulants and numerical solution using real-part BEM program, are found to be in good agreement.

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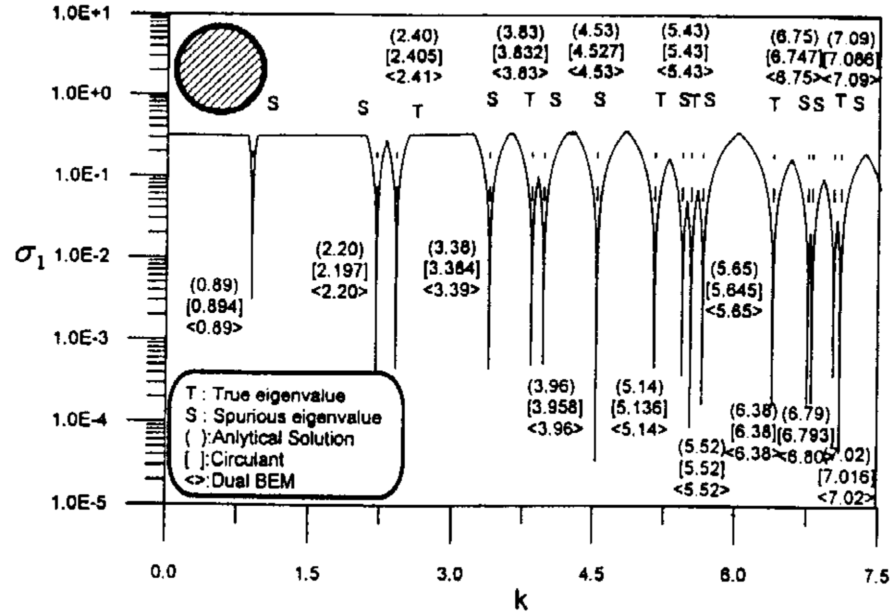

Fig 1. The first minimum singular value for different wave numbers using the [U] circulant of real-part dual BEM for the Dirichlet problem (u=0).

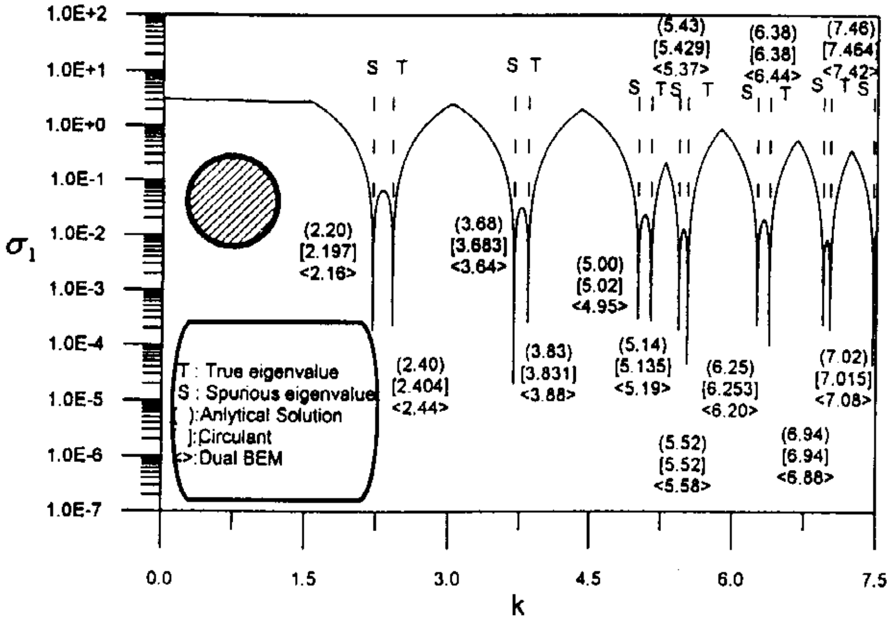

Fig 2. The first minimum singular value for different wave numbers using the [L] circulant of real-part dual BEM for the Dirichlet problem (u=0).

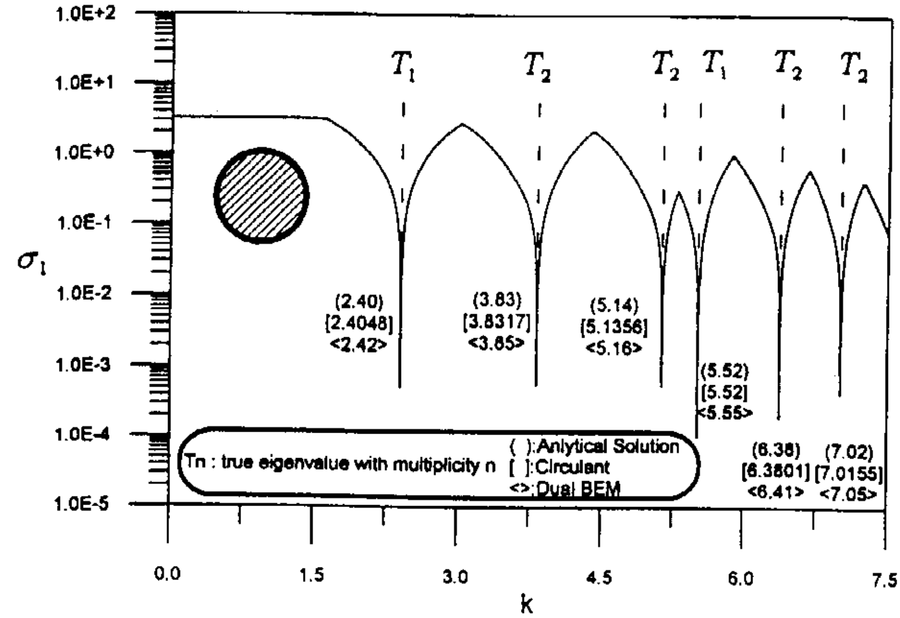

Fig 3. The first minimum singular value for different wave numbers using the $\begin{bmatrix} U \\ L \end{bmatrix}$ circulant of real-part dual BEM for the Dirichlet problem (u=0).

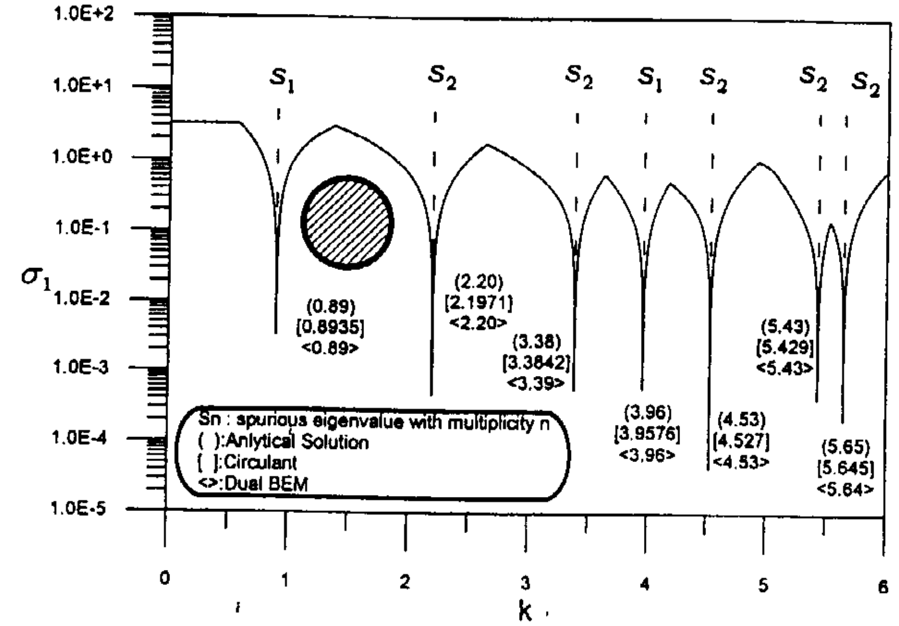

Fig 4. The first minimum singular value for different wave numbers using the $\begin{bmatrix} U^T \\ T^T \end{bmatrix}$ circulant of real-part dual BEM.