Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Mechanics Research Communications 32 (2005) 75–92
MECHANICS
RESEARCH COMMUNICATIONS
www.elsevier.com/locate/mechrescom

Fictitious frequency for the exterior Helmholtz equation subject to the mixed-type boundary condition using BEMs

Jeng-Tzong Chen [a,*], Tzong-Wey Lin [a], I-Lin Chen [b], Yang-Jye Lee [c]

[a] Department of Harbor and River Engineering, National Taiwan Ocean University, P.O. Box 7-59, Keelung, 20224, Taiwan
[b] Department of Naval Architecture, National Kaohsiung Marine University, Kaohsiung, 811, Taiwan
[c] Department of Civil Engineering, National Ilan University, Ilan, 260, Taiwan

Available online 20 May 2004

Abstract

Boundary integral equations and boundary element methods were employed analytically, semi-analytically and numerically to study the occurrence of fictitious frequency for the exterior Helmholtz equations subject to the mixed-type boundary conditions. A semi-infinite rod and a circular radiator of problems were addressed. Degenerate kernel of the fundamental solution and Fourier series for boundary density were utilized in the null-field integral equation to examine the occurrence of fictitious frequency semi-analytically. The BEM was utilized to solve the solution numerically. The CHIEF technique and Burton and Miller method were adopted to suppress the occurrence of the fictitious frequency. It is emphasized that the occurrence of fictitious frequency depend on the adopted method (singular or hypersingular formulation) no matter what the given type of boundary condition for the problem is. The illustrative examples were tested to verify this finding successfully.

Keywords: Mixed-type boundary condition; Boundary element method; Semi-analytical approach; Fictitious frequency; Singular formulation; Hypersingular formulation

1. Introduction

Boundary integral equation methods and boundary element methods have been used to solve exterior acoustic problems (radiation and scattering) for many years. The importance of the integral equation in the solution, both theoretical and practical, for certain types of boundary value problems is universally recognized. One of the problems frequently addressed in BEM is the problem of irregular frequency (fictitious frequency) in boundary integral formulations for exterior acoustics and water wave problems. Fictitious frequency does not represent any kind of physical resonance but are due to the numerical method, which

* Corresponding author. Tel.: +886-2-24622192-6140; fax: +886-2-24632375.
E-mail address: jtchen@mail.ntou.edu.tw (J.-T. Chen).

0093-6413/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mechrescom.2004.03.009

has no unique solution at some certain frequencies for a corresponding interior problem (Dokumaci, 1990; Lee and Sclavounos, 1989; Lee et al., 1996; Malenica and Chen, 1998; Ohmatsu, 1983; Ursell, 1981). The non-uniqueness problem is numerically manifested in rank-deficiency of the BEM coefficient matrix. In order to obtain the unique solution that is known to exist analytically, several modified integral equation formulations that provide additional constraints to the original system of equations have been proposed. Burton and Miller (1971) proposed an integral equation that is valid for all wave numbers by forming a linear combination of the singular integral equation and its normal derivative with an imaginary number. However, the calculation for the hypersingular integration is required. To avoid this computation, an alternative one, CHIEF method, was proposed by Schenck (1968) and Benthien and Schenck (1997). Many researchers (Juhl, 1994; Poulin, 1997; Seybert and Rengarajan, 1989) applied the CHIEF method to deal with the problem of fictitious frequencies. Schenck proposed the CHIEF method, which employed the null-field integral equations by collocating the interior point as an auxiliary condition to make up deficient constraint condition. The fictitious frequency for the problems subject to either the Dirichlet or the Neumann boundary condition have been discussed by analytical and numerical methods (Chen et al., 2000, 2003; Chen and Kuo, 2000; Chen, 1998) but only a few papers focused on the mixed-type boundary conditions. The existence theorem and uniqueness theorems for the Helmholtz equation with mixed-type boundary condition problem had been proved by Kress and Roach in the three-dimensional problem by using a modified layer approach. Nevertheless, no numerical results were provided (Kress and Roach, 1977). It is found that the occurrence of the fictitious frequency only depends on the formulation no matter what the boundary condition in Dirichlet or Neumann type (Chen et al., 2000, 2003; Chen and Kuo, 2000; Chen, 1998). It is interesting to extend the conclusion to the problem subject to the mixed-type boundary conditions. The fictitious frequency for the two-dimensional radiation problem will be derived by using the degenerate kernels and the Fourier series and numerical examples will be tested. Two examples will be considered by using BEMs. One is the semi-infinite rod, and the other one is the circular radiator. Both the Burton and Miller approach and the CHIEF method will be adopted to overcome the problem of fictitious frequencies in the radiation problem.

2. Analytical derivation for one-dimensional exterior problems subject to the mixed-type boundary condition using the dual BEM (DBEM)

For a semi-infinite rod, the governing equation for the exterior problem is the Helmholtz equation,

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2}+k^2u(x)=0,\quad x_0<x<\infty, \tag{1}$$

where k and $u(x)$ denote the wave number and potential, respectively.

By utilizing an auxiliary system, the fundamental solution $U(s,x)$ satisfies

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2}+k^2U(x,s)=\delta(x-s), \tag{2}$$

where $\delta(x-s)$ is the Dirac delta function and explicit form of the kernel $U(s,x)$ is shown below:

$$U(s,x)=\frac{\mathrm{e}^{ik|x-s|}}{2ik}. \tag{3}$$

By employing the reciprocal theorem, we have the singular formulation

$$u(x)=T(s,x)u(s)|_{s_0}^{s_1}-U(s,x)t(s)|_{s_0}^{s_1}, \tag{4}$$

where $T(s,x)=\partial U(s,x)/\partial s$ and $t(s)=\mathrm{d}u(s)/\mathrm{d}s$.

After taking the spatial derivative with respect to Eq. (4), the second equation (hypersingular formulation) of the dual boundary integral equations can be derived

$$t(x) = M(s,x)u(s)\big|_{s_0}^{s_1} - L(s,x)t(s)\big|_{s_0}^{s_1}, \tag{5}$$

where $L(s,x) = \partial U(s,x)/\partial x$ and $M(s,x) = \partial^2 U(s,x)/\partial s \partial x$.

Although the auxiliary system $U(s,x)$ in Eq. (3) is a free-space Green function (fundamental solution), two other choices (symmetric and anti-symmetric systems) can be adopted as follows:

For symmetric case of free end at the origin ($T(0,s) = 0$), we have

$$U(s,x) = \begin{cases} U^l(s,x) = \frac{i}{k}\mathrm{e}^{-iks}\cos kx, & s > x \\ U^r(s,x) = \frac{i}{k}\mathrm{e}^{-ikx}\cos ks, & x > s > 0, \end{cases} \tag{6}$$

$$T(s,x) = \begin{cases} T^l(s,x) = \mathrm{e}^{-iks}\cos kx, & s > x \\ T^r(s,x) = -i\mathrm{e}^{-ikx}\sin ks, & x > s > 0, \end{cases} \tag{7}$$

$$L(s,x) = \begin{cases} L^l(s,x) = -i\mathrm{e}^{-iks}\sin kx, & s > x \\ L^r(s,x) = \mathrm{e}^{-ikx}\cos ks, & x > s > 0, \end{cases} \tag{8}$$

$$M(s,x) = \begin{cases} M^l(s,x) = -k\mathrm{e}^{-iks}\sin kx, & s > x \\ M^r(s,x) = -k\mathrm{e}^{-ikx}\sin ks, & x > s > 0. \end{cases} \tag{9}$$

For anti-symmetric case of fixed end at the origin ($U(0,s) = 0$), we have

$$U(s,x) = \begin{cases} U^l(s,x) = \frac{-1}{k}\mathrm{e}^{-iks}\sin kx, & s > x \\ U^r(s,x) = \frac{-1}{k}\mathrm{e}^{-ikx}\sin ks, & x > s > 0, \end{cases} \tag{10}$$

$$T(s,x) = \begin{cases} T^l(s,x) = i\mathrm{e}^{-iks}\sin kx, & s > x \\ T^r(s,x) = -\mathrm{e}^{-ikx}\cos ks, & x > s > 0, \end{cases} \tag{11}$$

$$L(s,x) = \begin{cases} L^l(s,x) = -\mathrm{e}^{-iks}\cos kx, & s > x \\ L^r(s,x) = i\mathrm{e}^{-ikx}\sin ks, & x > s > 0, \end{cases} \tag{12}$$

$$M(s,x) = \begin{cases} M^l(s,x) = ik\mathrm{e}^{-iks}\cos kx, & s > x \\ M^r(s,x) = ik\mathrm{e}^{-ikx}\cos ks, & x > s > 0. \end{cases} \tag{13}$$

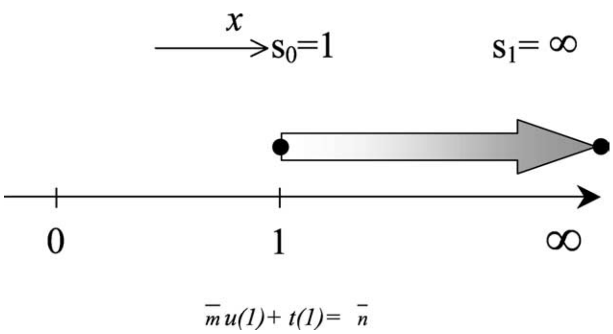

Fig. 1. The 1-D radiation problem subject to the mixed-boundary condition.

By moving the field point x to be close to the end of rod as shown in Fig. 1, we have

UT equation (singular formulation)

$$0 = u(1)\mathrm{e}^{-ik}\cos k + t(1)\frac{i}{k}\mathrm{e}^{-ik}\cos k \quad \text{(auxiliary system : symmetric Green function)}, \tag{14}$$

$$0 = u(1)i\mathrm{e}^{-ik}\sin k + t(1)\frac{1}{k}\mathrm{e}^{-ik}\sin k \quad \text{(auxiliary system : anti-symmetric Green function)}. \tag{15}$$

LM equation (hypersingular formulation)

$$0 = -u(1)k\mathrm{e}^{-ik}\sin k - t(1)i\mathrm{e}^{-ik}\sin k \quad \text{(auxiliary system : symmetric Green function)}, \tag{16}$$

$$0 = u(1)ik\mathrm{e}^{-ik}\cos k - t(1)\mathrm{e}^{-ik}\cos k \quad \text{(auxiliary system : anti-symmetric Green function)}. \tag{17}$$

By utilizing the mixed-type (Robin) boundary conditions,

$$\bar{m}u(1) + t(1) = \bar{n}, \tag{18}$$

where $\bar{m}$ and $\bar{n}$ are constants, the boundary integral equations can be derived as follows:
By choosing the auxiliary system of symmetric Green function,

UT equation yields

$$0 = u(1)\mathrm{e}^{-ik}\cos k + (\bar{n} - \bar{m}u(1))\frac{i}{k}\mathrm{e}^{-ik}\cos k,$$

$$u(1) = \frac{\bar{n}}{\bar{m} + ik}\frac{\cos k}{\cos k},$$

the possible irregular values occur at the positions where k satisfies $\cos k = 0$;

LM equation yields

$$0 = -u(1)k\mathrm{e}^{-ik}\sin k - (\bar{n} - \bar{m}u(1))i\mathrm{e}^{-ik}\sin k,$$

$$u(1) = \frac{\bar{n}}{\bar{m} + ik}\frac{\sin k}{\sin k},$$

the possible irregular values occur at the positions where k satisfies $\sin k = 0$.
By choosing the auxiliary system of anti-symmetric Green function,

UT equation yields

$$0 = u(1)i\mathrm{e}^{-ik}\sin k + (\bar{n} - \bar{m}u(1))\frac{1}{k}\mathrm{e}^{-ik}\sin k,$$

$$u(1) = \frac{\bar{n}}{\bar{m} + ik}\frac{\sin k}{\sin k},$$

the possible irregular values occur at the positions where k satisfies $\sin k = 0$;

LM equation yields

$$0 = u(1)ik\mathrm{e}^{-ik}\cos k - (\bar{n} - \bar{m}u(1))\mathrm{e}^{-ik}\cos k,$$

Table 1
The fictitious eigenvalues of the one-dimensional exterior problem subject to the Dirichlet, Neumann and mixed-type boundary conditions using the dual formulations

Exterior problem		Fictitious eigenequations		
		Dirichlet $u(1)=u$	Neumann $t(1)=t$	Mixed-type $mu(1)+t(1)=n$
Symmetric auxiliary system	*UT*	$\cos k$	$\cos k$	$\cos k$
	LM	$\sin k$	$\sin k$	$\sin k$
Anti-symmetric auxiliary system	*UT*	$\sin k$	$\sin k$	$\sin k$
	LM	$\cos k$	$\cos k$	$\cos k$

$$u(1)=\frac{\bar{n}}{\bar{m}+ik}\frac{\cos k}{\cos k},$$

the possible irregular values occur at the positions where k satisfies $\cos k=0$.

The fictitious frequencies for the Dirichlet, Neumann and mixed-type problems by using the dual BEMs are shown in Table 1.

3. Semi-analytical derivation of the fictitious frequencies for the two-dimensional exterior problem by using the degenerate kernels and Fourier series expansions in a continuous system

The governing equation for the two-dimensional eigenproblem is the Helmholtz equation,

$$\nabla^2 u(\mathbf{x})+k^2u(\mathbf{x})=0,\quad \mathbf{x}\in D, \tag{19}$$

where D is the domain of interest, $\mathbf{x}$ is the field point, k is the wave number and $u(\mathbf{x})$ is the acoustic potential, respectively. On the basis of the dual boundary integral formulation (Chen and Chen, 1998; Chen and Hong, 1999), we have

$$\alpha u(\mathbf{x})=\text{C.P.V.}\int_B T(\mathbf{s},\mathbf{x})u(\mathbf{s})\,\mathrm{d}B(\mathbf{s})-\text{R.P.V.}\int_B U(\mathbf{s},\mathbf{x})t(\mathbf{s})\,\mathrm{d}B(\mathbf{s}),\quad \mathbf{x}\in B, \tag{20}$$

$$\alpha t(\mathbf{x})=\text{H.P.V.}\int_B M(\mathbf{s},\mathbf{x})u(\mathbf{s})\,\mathrm{d}B(\mathbf{s})-\text{C.P.V.}\int_B L(\mathbf{s},\mathbf{x})t(\mathbf{s})\,\mathrm{d}B(\mathbf{s}),\quad \mathbf{x}\in B, \tag{21}$$

where $\mathbf{s}$ is the source point, $u(\mathbf{s})$ and $t(\mathbf{s})$ are the potential and its normal derivative on the boundary, respectively, R.P.V. denotes the Reimann principal value, C.P.V. denotes the Cauchy principal value, H.P.V. denotes the Hadamard principal value and α is the interior angle of the boundary at field point. The explicit forms of the four kernel functions $U(\mathbf{s},\mathbf{x})$, $T(\mathbf{s},\mathbf{x})$, $L(\mathbf{s},\mathbf{x})$ and $M(\mathbf{s},\mathbf{x})$ are summarized as follows:

$$U(\mathbf{s},\mathbf{x})=\frac{-i\pi H_0^{(1)}(k\mathbf{r})}{2}, \tag{22}$$

$$T(\mathbf{s},\mathbf{x})=\frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_s}=\frac{-ik\pi}{2}H_1^{(1)}(k\mathbf{r})\frac{y_in_i}{\mathbf{r}}, \tag{23}$$

$$L(\mathbf{s},\mathbf{x})=\frac{\partial U(\mathbf{s},\mathbf{x})}{\partial n_x}=\frac{ik\pi}{2}H_1^{(1)}(k\mathbf{r})\frac{y_i\bar{n}_i}{\mathbf{r}}, \tag{24}$$

$$M(\mathbf{s},\mathbf{x})=\frac{\partial^2 U(\mathbf{s},\mathbf{x})}{\partial n_s\partial n_x}=\frac{-ik\pi}{2}\left[-k\frac{H_2^{(1)}(k\mathbf{r})}{\mathbf{r}^2}y_iy_jn_i\bar{n}_j+\frac{H_1^{(1)}(k\mathbf{r})}{\mathbf{r}}n_i\bar{n}_i\right], \tag{25}$$

where $H_n^{(1)}(k\mathbf{r})$ is the nth order Hankel function of the first kind, $\mathbf{r} = |\mathbf{x} - \mathbf{s}|$, $y_i = s_i - x_i$, $i^2 = -1$, n_i and $\bar{n}_i$ are the ith component of the outer normal vectors at $\mathbf{s}$ and $\mathbf{x}$, respectively. The null-field integral equations (Matrin, 1980) corresponding to Eqs. (20) and (21) are

$$0 = \int_B T(\mathbf{s}, \mathbf{x}) u(\mathbf{s}) \, dB(\mathbf{s}) - \int_B U(\mathbf{s}, \mathbf{x}) t(\mathbf{s}) \, dB(\mathbf{s}), \quad \mathbf{x} \in D^c, \tag{26}$$

$$0 = \int_B M(\mathbf{s}, \mathbf{x}) u(\mathbf{s}) \, dB(\mathbf{s}) - \int_B L(\mathbf{s}, \mathbf{x}) t(\mathbf{s}) \, dB(\mathbf{s}), \quad \mathbf{x} \in D^c, \tag{27}$$

where D^c is the complementary domain.

The kernel functions in Eqs. (22)–(25) can be expanded into the following degenerate kernels:

$$U(\mathbf{s}, \mathbf{x}) = \begin{cases} U^i(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} J_m(k\rho) H_m^{(1)}(kR) \cos(m(\theta - \phi)), & R > \rho, \\ U^e(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} J_m(kR) H_m^{(1)}(k\rho) \cos(m(\theta - \phi)), & \rho > R, \end{cases} \tag{28}$$

$$T(\mathbf{s}, \mathbf{x}) = \begin{cases} T^i(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} k J_m(k\rho) H_m'^{(1)}(kR) \cos(m(\theta - \phi)), & R > \rho, \\ T^e(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} k J_m'(kR) H_m^{(1)}(k\rho) \cos(m(\theta - \phi)), & \rho > R, \end{cases} \tag{29}$$

$$L(\mathbf{s}, \mathbf{x}) = \begin{cases} L^i(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} k J_m'(k\rho) H_m^{(1)}(kR) \cos(m(\theta - \phi)), & R > \rho, \\ L^e(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} k J_m(kR) H_m'^{(1)}(k\rho) \cos(m(\theta - \phi)), & \rho > R, \end{cases} \tag{30}$$

$$M(\mathbf{s}, \mathbf{x}) = \begin{cases} M^i(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} k^2 J_m'(k\rho) H_m'^{(1)}(kR) \cos(m(\theta - \phi)), & R > \rho, \\ M^e(\mathbf{s}, \mathbf{x}) = \sum_{m=-\infty}^{\infty} \frac{-\pi i}{2} k^2 J_m'(kR) H'(1)_m(k\rho) \cos(m(\theta - \phi)), & \rho > R, \end{cases} \tag{31}$$

where (R, θ) and (ρ, ϕ) are the polar coordinates of $\mathbf{s}$ and $\mathbf{x}$, respectively. Note that $U = U^i$, $T = T^i$, $L = L^i$ and $M = M^i$ are for the exterior problem; $U = U^e$, $T = T^e$, $L = L^e$ and $M = M^e$ are for the interior problem.

For the exterior problem subject to the mixed-type boundary condition on the circular boundary as shown in Fig. 2, the null-field integral equation can be rewritten as

$$\begin{aligned} 0 &= \int_B [T^i(\mathbf{s}, \mathbf{x}) u(\mathbf{s}) - U^i(\mathbf{s}, \mathbf{x}) t(\mathbf{s})] \, dB(\mathbf{s}) \\ &= \int_{B_1} [T^i(\mathbf{s}, \mathbf{x}) u_1(\mathbf{s}) - U^i(\mathbf{s}, \mathbf{x}) t_1(\mathbf{s})] \, dB_1(\mathbf{s}) + \int_{B_2} [T^i(\mathbf{s}, \mathbf{x}) u_2(\mathbf{s}) - U^i(\mathbf{s}, \mathbf{x}) t_2(\mathbf{s})] \, dB_2(\mathbf{s}). \end{aligned} \tag{32}$$

Give the boundary conditions,

$$t_1(\theta) = \left(k \frac{H_{\mathcal{N}+1}^{(1)}(kR)}{H_{\mathcal{N}}^{(1)}(k\rho)} - \frac{\mathcal{N}}{R} \right) \cos \mathcal{N}\theta, \quad \mathbf{s} \in B_1, \tag{33}$$

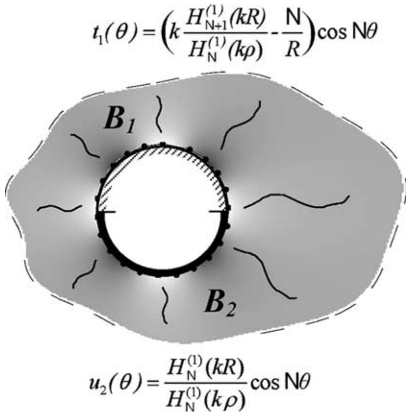

Fig. 2. The 2-D radiation problem subject to the mixed-boundary condition.

$$u_2(\theta)=\frac{H^{(1)}_{\mathcal{N}}(kR)}{H^{(1)}_{\mathcal{N}}(k\rho)}\cos\mathcal{N}\theta,\quad \mathbf{s}\in B_2, \tag{34}$$

as shown in Fig. 2, the designed analytical solution is

$$u(\theta)=\frac{H^{(1)}_{\mathcal{N}}(kR)}{H^{(1)}_{\mathcal{N}}(k\rho)}\cos\mathcal{N}\theta, \tag{35}$$

where $\mathcal{N}$ is a natural number.

Substituting the given boundary conditions into Eq. (32), it can be rewritten as

$$\begin{aligned}0=&\left[\int_{B_1}T^i(R,\theta;\rho,\phi)u_1(\theta)\,\mathrm{d}B_1(\theta)-\int_{B_2}U^i(R,\theta;\rho,\phi)t_2(\theta)\,\mathrm{d}B_2(\theta)\right]\\&+\left[\int_{B_2}T^i(R,\theta;\rho,\phi)\frac{H^{(1)}_{\mathcal{N}}(kR)}{H^{(1)}_{\mathcal{N}}(k\rho)}\cos\mathcal{N}\theta\,\mathrm{d}B_2(\theta)-\int_{B_1}U^i(R,\theta;\rho,\phi)\left(k\frac{H^{(1)}_{\mathcal{N}+1}(kR)}{H^{(1)}_{\mathcal{N}}(k\rho)}-\frac{\mathcal{N}}{R}\right)\right.\\&\left.\times\cos\mathcal{N}\theta\,\mathrm{d}B_1(\theta)\right].\end{aligned} \tag{36}$$

We can expand the unknown boundary densities in terms of Fourier series with the period π

$$u_1(\theta)=a_0+\sum_{n=1}^{\infty}(a_n\cos 2n\theta+b_n\sin 2n\theta),\quad 0<\theta<\pi,\ \mathbf{s}\in B_1, \tag{37}$$

$$t_2(\theta)=p_0+\sum_{n=1}^{\infty}(p_n\cos 2n\theta+q_n\sin 2n\theta),\quad \pi<\theta<2\pi,\ \mathbf{s}\in B_2, \tag{38}$$

where a_0, a_n, b_n, p_0, p_n and q_n are the undetermined Fourier coefficients.

Eq. (32) can be calculated as follows:

$$\begin{aligned}\int_{B_1} T^i(R,\theta;\rho,\phi)u_1(\mathbf{s})\,\mathrm{d}B_1 &= \int_0^{\pi}\left\{\frac{\pi k}{2}H_0'(kR)J_0(k\rho)+\pi k\sum_{m=1}^{\infty}H_m'(kR)J_m(k\rho)\cos[m(\theta-\phi)]\right\}\\ &\quad\times\left[a_0+\sum_{n=1}^{\infty}(a_n\cos 2n\theta+b_n\sin 2n\theta)\right]R\,\mathrm{d}\theta\\ &=\int_0^{\pi}\left\{\frac{\pi k a_0}{2}H_0'(kR)J_0(k\rho)+\pi k a_0\sum_{m=1}^{\infty}H_m'(kR)J_m(k\rho)\cos[m(\theta-\phi)]\right.\\ &\quad+\frac{\pi k}{2}H_0'(kR)J_0(k\rho)\sum_{n=1}^{\infty}(a_n\cos 2n\theta+b_n\sin 2n\theta)\\ &\quad\left.+\pi k\sum_{m=1}^{\infty}H_m'(kR)J_m(k\rho)\cos[m(\theta-\phi)]\sum_{n=1}^{\infty}(a_n\cos 2n\theta+b_n\sin 2n\theta)\right\}R\,\mathrm{d}\theta.\end{aligned} \tag{39}$$

By employing the relation of the trigonometric function and the theorem of orthogonality in an interval between 0 and π to Eq. (39), we have

$$\begin{aligned}&Rk\left\{\frac{\pi^2 a_0}{2}H_0'(kR)J_0(k\rho)+\pi a_0\sum_{m=1,3,5,\ldots}^{\infty}H_m'(kR)J_m(k\rho)\frac{2\sin m\phi}{m}+\frac{\pi^2}{2}\sum_{n=1}^{\infty}H_{2n}'(kR)J_{2n}(k\rho)(a_n\cos 2n\phi\right.\\ &\quad\left.+b_n\sin 2n\phi)+\pi\sum_{n=1}^{\infty}\sum_{m=1,3,5,\ldots}^{\infty}H_m'(kR)J_m(k\rho)\left(\frac{2m\sin m\phi}{m^2-4n^2}a_n+\frac{4n\cos m\phi}{m^2-4n^2}b_n\right)\right\}\\ &=\pi Rk\left\{\frac{\pi a_0}{2}H_0'(kR)J_0(k\rho)+a_0\sum_{m=0}^{\infty}H_{2m+1}'(kR)J_{2m+1}(k\rho)\frac{2\sin(2m+1)\phi}{2m+1}\right.\\ &\quad+\frac{\pi}{2}\sum_{n=1}^{\infty}H_{2n}'(kR)J_{2n}(k\rho)(a_n\cos 2n\phi+b_n\sin 2n\phi)\\ &\quad\left.+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}H_{2m+1}'(kR)J_{2m+1}(k\rho)\left(\frac{2(2m+1)\sin(2m+1)\phi}{(2m+1)^2-4n^2}a_n+\frac{4n\cos(2m+1)\phi}{(2m+1)^2-4n^2}b_n\right)\right\}\\ &=\pi kR\sum_{n=0}^{\infty}\left\{\frac{\pi}{2}H_{2n}'(kR)J_{2n}(k\rho)(a_n\cos 2n\phi+b_n\sin 2n\phi)\right.\\ &\quad\left.+\sum_{m=0}^{\infty}H_{2m+1}'(kR)J_{2m+1}(k\rho)\left(\frac{2(2m+1)\sin(2m+1)\phi}{(2m+1)^2-4n^2}a_n+\frac{4n\cos(2m+1)\phi}{(2m+1)^2-4n^2}b_n\right)\right\}\\ &=\pi Rk\sum_{n=0}^{\infty}\left\{\left[\frac{\pi}{2}H_{2n}'(kR)J_{2n}(k\rho)\cos 2n\phi+\sum_{m=0}^{\infty}\frac{2(2m+1)}{(2m+1)^2-4n^2}H_{2m+1}'(kR)J_{2m+1}(k\rho)\sin(2m+1)\phi\right]a_n\right.\\ &\quad\left.+\left[\frac{\pi}{2}H_{2n}'(kR)J_{2n}(k\rho)\sin 2n\phi+\sum_{m=0}^{\infty}\frac{4n}{(2m+1)^2-4n^2}H_{2m+1}'(kR)J_{2m+1}(k\rho)\cos(2m+1)\phi\right]b_n\right\}.\end{aligned} \tag{40}$$

Similarly,

$$\int_{B_2} U^i(R,\theta;\rho,\phi)t_2(\mathbf{s})\,\mathrm{d}B_2 = \pi R\sum_{n=0}^{\infty}\Bigg\{\Bigg[\frac{\pi}{2}H_{2n}(kR)J_{2n}(k\rho)\cos 2n\phi - \sum_{m=0}^{\infty}\frac{2(2m+1)}{(2m+1)^2-4n^2}H_{2m+1} \times (kR)J_{2m+1}(k\rho)\sin(2m+1)\phi\Bigg]p_n + \Bigg[\frac{\pi}{2}H_{2n}(kR)J_{2n}(k\rho)\sin 2n\phi - \sum_{m=0}^{\infty}\frac{4n}{(2m+1)^2-4n^2}H_{2m+1}(kR)J_{2m+1}(k\rho)\cos(2m+1)\phi\Bigg]q_n\Bigg\}. \tag{41}$$

Furthermore,

$$\int_0^{\pi}\left\{\frac{\pi k}{2}H_0'^{(1)}(kR)J_0(k\rho) + \pi k\sum_{m=1}^{\infty}H_m'^{(1)}(kR)J_m(k\rho)\cos[m(\theta-\phi)]\right\}\frac{H_{\mathcal{N}}^{(1)}(kR)}{H_{\mathcal{N}}^{(1)}(k\rho)}\cos\mathcal{N}\theta R\,\mathrm{d}\theta$$
$$-\int_{\pi}^{2\pi}\left\{\frac{\pi}{2}H_0^{(1)}(kR)J_0(k\rho) + \pi\sum_{m=1}^{\infty}H_m^{(1)}(kR)J_m(k\rho)\cos[m(\theta-\phi)]\right\}\left(k\frac{H_{\mathcal{N}+1}^{(1)}(kR)}{H_{\mathcal{N}}^{(1)}(k\rho)} - \frac{\mathcal{N}}{R}\right)\cos\mathcal{N}\theta R\,\mathrm{d}\theta$$
$$= \frac{\pi^2 kR}{2}\frac{H_{\mathcal{N}}^{(1)}(kR)}{H_{\mathcal{N}}^{(1)}(k\rho)}H_{\mathcal{N}}'^{(1)}(kR)J_{\mathcal{N}}(k\rho)\cos\mathcal{N}\phi + \frac{\pi^2 R}{2}\left(k\frac{H_{\mathcal{N}+1}^{(1)}(kR)}{H_{\mathcal{N}}^{(1)}(k\rho)} - \frac{\mathcal{N}}{R}\right)H_{\mathcal{N}}^{(1)}(kR)J_{\mathcal{N}}(k\rho)\cos\mathcal{N}\phi.$$

Then Eq. (32) is rewritten as

$$0 = \pi kR\sum_{n=0}^{\infty}\Bigg\{\Bigg[\frac{\pi}{2}H_{2n}'^{(1)}(kR)J_{2n}(k\rho)\cos 2n\phi + \sum_{m=0}^{\infty}\frac{2(2m+1)}{(2m+1)^2-4n^2}H_{2m+1}'^{(1)}(kR)J_{2m+1}(k\rho)\sin(2m+1)\phi\Bigg]a_n + \Bigg[\frac{\pi}{2}H_{2n}'^{(1)}(kR)J_{2n}(k\rho)\sin 2n\phi + \sum_{m=0}^{\infty}\frac{4n}{(2m+1)^2-4n^2}H_{2m+1}'^{(1)}(kR)J_{2m+1}(k\rho)\cos(2m+1)\phi\Bigg]b_n\Bigg\}$$
$$- \pi R\sum_{n=0}^{\infty}\Bigg\{\Bigg[\frac{\pi}{2}H_{2n}^{(1)}(kR)J_{2n}(k\rho)\cos 2n\phi - \sum_{m=0}^{\infty}\frac{2(2m+1)}{(2m+1)^2-4n^2}H_{2m+1}^{(1)}(kR)J_{2m+1}(k\rho)\sin(2m+1)\phi\Bigg]p_n + \Bigg[\frac{\pi}{2}H_{2n}^{(1)}(kR)J_{2n}(k\rho)\sin 2n\phi - \sum_{m=0}^{\infty}\frac{4n}{(2m+1)^2-4n^2}H_{2m+1}^{(1)}(kR)J_{2m+1}(k\rho)\cos(2m+1)\phi\Bigg]q_n\Bigg\}$$
$$+ \frac{\pi^2 kR}{2}\frac{H_{\mathcal{N}}^{(1)}(kR)}{H_{\mathcal{N}}^{(1)}(k\rho)}H_{\mathcal{N}}'^{(1)}(kR)J_{\mathcal{N}}(k\rho)\cos\mathcal{N}\phi + \frac{\pi^2 R}{2}\left(k\frac{H_{\mathcal{N}+1}^{(1)}(kR)}{H_{\mathcal{N}}^{(1)}(k\rho)} - \frac{\mathcal{N}}{R}\right)H_{\mathcal{N}}^{(1)}(kR)J_{\mathcal{N}}(k\rho)\cos\mathcal{N}\phi. \tag{42}$$

3.1. Semi-analytical approach—reduction to linear algebraic system

To solve the unknown coefficients in Eq. (42) is not an easy work, a semi-analytical approach is adopted by choosing collocation points on the boundary instead of deriving the Fourier coefficients of the boundary densities exactly. Eq. (42) can be simplified to

$$0 = \pi R\left[\sum_{n=0}^{\infty}(kf'_{UT(\phi,n)}a_n + kg'_{UT(\phi,n)}b_n - f_{UT(\phi,n)}p_n - g_{UT(\phi,n)}q_n) + h_{UT}\cos\mathcal{N}\phi\right], \tag{43}$$

where the symbols, $f'_{UT(\phi,n)}$, $g'_{UT(\phi,n)}$, $f_{UT(\phi,n)}$, $g_{UT(\phi,n)}$ and h_{UT}, denote the influence coefficients of a_n, b_n, p_n, q_n and $\cos\mathscr{N}\phi$, respectively, as follows:

$$f'_{UT(\phi,n)}=\frac{\pi}{2}H'^{(1)}_{2n}(kR)J_{2n}(k\rho)\cos 2n\phi+\sum_{m=0}^{\infty}\frac{2(2m+1)}{(2m+1)^2-4n^2}H'^{(1)}_{2m+1}(kR)J_{2m+1}(k\rho)\sin(2m+1)\phi, \tag{44}$$

$$g'_{UT(\phi,n)}=\frac{\pi}{2}H'^{(1)}_{2n}(kR)J_{2n}(k\rho)\sin 2n\phi+\sum_{m=0}^{\infty}\frac{4n}{(2m+1)^2-4n^2}H'^{(1)}_{2m+1}(kR)J_{2m+1}(k\rho)\cos(2m+1)\phi, \tag{45}$$

$$f_{UT(\phi,n)}=\frac{\pi}{2}H^{(1)}_{2n}(kR)J_{2n}(k\rho)\cos 2n\phi-\sum_{m=0}^{\infty}\frac{2(2m+1)}{(2m+1)^2-4n^2}H^{(1)}_{2m+1}(kR)J_{2m+1}(k\rho)\sin(2m+1)\phi, \tag{46}$$

$$g_{UT(\phi,n)}=\frac{\pi}{2}H^{(1)}_{2n}(kR)J_{2n}(k\rho)\sin 2n\phi-\sum_{m=0}^{\infty}\frac{4n}{(2m+1)^2-4n^2}H^{(1)}_{2m+1}(kR)J_{2m+1}(k\rho)\cos(2m+1)\phi, \tag{47}$$

$$h_{UT}=\frac{\pi kH^{(1)}_{\mathscr{N}}(kR)}{2H^{(1)}_{\mathscr{N}}(k\rho)}H'^{(1)}_{\mathscr{N}}(kR)J_{\mathscr{N}}(k\rho)+\left(\frac{\pi kH^{(1)}_{\mathscr{N}+1}(kR)}{2H^{(1)}_{\mathscr{N}}(k\rho)}-\frac{\mathscr{N}\pi}{2R}\right)H^{(1)}_{\mathscr{N}}(kR)J_{\mathscr{N}}(k\rho). \tag{48}$$

To match the number of Fourier coefficients, $4n+2$ collocation points are collocated and Eq. (43) can be extended to an algebraic equation with matrix $[\mathbf{W_{UT}}]$ and vectors, $\{\mathbf{v}\}$ and $\{\mathbf{H_{UT}}\}$, as follows:

$$0=[\mathbf{W_{UT}}]_{(4n+2)\times(4n+2)}\{\mathbf{v}\}_{(4n+2)\times 1}+\{\mathbf{H_{UT}}\}_{(4n+2)\times 1}, \tag{49}$$

$$0=[\,\mathbf{A}\quad \mathbf{B}\quad -\mathbf{P}\quad -\mathbf{Q}\,]_{(4n+2)\times(4n+2)}\begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\\ \mathbf{p}\\ \mathbf{q}\end{Bmatrix}_{(4n+2)\times 1}+h_{UT}\begin{Bmatrix}\cos\mathscr{N}\phi_1\\ \cos\mathscr{N}\phi_2\\ \vdots\\ \cos\mathscr{N}\phi_{4n+2}\end{Bmatrix}_{(4n+2)\times 1}, \tag{50}$$

where

$$[\mathbf{A}]=\pi kR\begin{bmatrix} f'_{UT(\phi_1,0)} & f'_{UT(\phi_1,1)} & \cdots & f'_{UT(\phi_1,n)}\\ f'_{UT(\phi_2,0)} & f'_{UT(\phi_2,1)} & \cdots & f'_{UT(\phi_2,n)}\\ \vdots & \vdots & \ddots & \vdots\\ f'_{UT(\phi_{4n+2},0)} & f'_{UT(\phi_{4n+2},1)} & \cdots & f'_{UT(\phi_{4n+2},n)}\end{bmatrix}_{(4n+2)\times(n+1)},$$

$$[\mathbf{B}]=\pi kR\begin{bmatrix} g'_{UT(\phi_1,1)} & g'_{UT(\phi_1,2)} & \cdots & g'_{UT(\phi_1,n)}\\ g'_{UT(\phi_2,1)} & g'_{UT(\phi_2,2)} & \cdots & g'_{UT(\phi_2,n)}\\ \vdots & \vdots & \ddots & \vdots\\ g'_{UT(\phi_{4n+2},1)} & g'_{UT(\phi_{4n+2},2)} & \cdots & g'_{UT(\phi_{4n+2},n)}\end{bmatrix}_{(4n+2)\times n},$$

$$[\mathbf{P}]=\pi R\begin{bmatrix} f_{UT(\phi_1,0)} & f_{UT(\phi_1,1)} & \cdots & f_{UT(\phi_1,n)}\\ f_{UT(\phi_2,0)} & f_{UT(\phi_2,1)} & \cdots & f_{UT(\phi_2,n)}\\ \vdots & \vdots & \ddots & \vdots\\ f_{UT(\phi_{4n+2},0)} & f_{UT(\phi_{4n+2},1)} & \cdots & f_{UT(\phi_{4n+2},n)}\end{bmatrix}_{(4n+2)\times(n+1)},$$

$$[\mathbf{Q}] = \pi R \begin{bmatrix} g_{UT(\phi_1,1)} & g_{UT(\phi_1,2)} & \cdots & g_{UT(\phi_1,n)} \\ g_{UT(\phi_2,1)} & g_{UT(\phi_2,2)} & \cdots & g_{UT(\phi_2,n)} \\ \vdots & \vdots & \ddots & \vdots \\ g_{UT(\phi_{4n+2},1)} & g_{UT(\phi_{4n+2},2)} & \cdots & g_{UT(\phi_{4n+2},n)} \end{bmatrix}_{(4n+2)\times n},$$

$$\mathbf{a} = \begin{Bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix}, \quad \mathbf{p} = \begin{Bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{Bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix}.$$

Based on the same procedure using the singular formulation, the coefficient matrix $[\mathbf{W_{UT}}]$ of Eq. (49) is replaced by $[\mathbf{W_{LM}}]$ for hypersingular one. The coefficients of Eq. (42) for the hypersingular formulation can be calculated as follows:

$$f'_{LM(\phi,n)} = \frac{\pi}{2} H^{(1)}_{2n}(kR) J'_{2n}(k\rho)\cos 2n\phi + \sum_{m=0}^{\infty} \frac{2(2m+1)}{(2m+1)^2 - 4n^2} H^{(1)}_{2m+1}(kR) J'_{2m+1}(k\rho)\sin(2m+1)\phi, \tag{51}$$

$$g'_{LM(\phi,n)} = \frac{\pi}{2} H^{(1)}_{2n}(kR) J'_{2n}(k\rho)\sin 2n\phi + \sum_{m=0}^{\infty} \frac{4n}{(2m+1)^2 - 4n^2} H^{(1)}_{2m+1}(kR) J'_{2m+1}(k\rho)\cos(2m+1)\phi, \tag{52}$$

$$f_{LM(\phi,n)} = \frac{\pi}{2} H'^{(1)}_{2n}(kR) J'_{2n}(k\rho)\cos 2n\phi - \sum_{m=0}^{\infty} \frac{2(2m+1)}{(2m+1)^2 - 4n^2} H'^{(1)}_{2m+1}(kR) J'_{2m+1}(k\rho)\sin(2m+1)\phi, \tag{53}$$

$$g_{LM(\phi,n)} = \frac{\pi}{2} H'^{(1)}_{2n}(kR) J'_{2n}(k\rho)\sin 2n\phi - \sum_{m=0}^{\infty} \frac{4n}{(2m+1)^2 - 4n^2} H'^{(1)}_{2m+1}(kR) J'_{2m+1}(k\rho)\cos(2m+1)\phi, \tag{54}$$

$$h_{LM} = \frac{\pi k H^{(1)}_{\mathcal{N}}(kR)}{2H^{(1)}_{\mathcal{N}}(k\rho)} H'^{(1)}_{\mathcal{N}}(kR) J'_{\mathcal{N}}(k\rho) + \left(\frac{\pi k H^{(1)}_{\mathcal{N}+1}(kR)}{2H^{(1)}_{\mathcal{N}}(k\rho)} - \frac{\mathcal{N}\pi}{2R} \right) H^{(1)}_{\mathcal{N}}(kR) J'_{\mathcal{N}}(k\rho). \tag{55}$$

The rank-deficiency of matrices, $[\mathbf{W_{UT}}]$ and $[\mathbf{W_{LM}}]$, due to fictitious frequencies for non-trivial solutions is checked by using the concept of the minimum singular value.

4. Formulation of the two-dimensional exterior problem by discretizing the BIEs into a discrete system—boundary element method

Discretizing the boundary B into N boundary elements in Eqs. (26) and (27), we obtain the dual algebraic system as follows:

$$[U]\{t\} = [T]\{u\}, \tag{56}$$

$$[L]\{t\} = [M]\{u\}. \tag{57}$$

For the problem with mixed-type boundary conditions, Eq. (56) can be decomposed into

$$\begin{bmatrix} U_L & \vdots & U_R \end{bmatrix}_{N\times N} \begin{Bmatrix} t \\ \bar{t} \end{Bmatrix}_{N\times 1} = \begin{bmatrix} T_L & \vdots & T_R \end{bmatrix}_{N\times N} \begin{Bmatrix} \bar{u} \\ u \end{Bmatrix}_{N\times 1}, \tag{58}$$

By collecting the given and unknown sets, we rearrange the influences matrices into

$$\left[[U_L] \ \vdots \ -T_R \right]_{N\times N} \left\{ \begin{array}{c} t \\ u \end{array} \right\}_{N\times 1} = \left[T_L \ \vdots \ -U_R \right]_{N\times N} \left\{ \begin{array}{c} \bar{u} \\ \bar{t} \end{array} \right\}_{N\times 1}. \tag{59}$$

Therefore, Eq. (56) can be simplified to

$$[A]_{N\times N}\{p\}_{N\times 1} = [B]_{N\times N}\{q\}_{N\times 1}, \tag{60}$$

where

$$[A] = \left[U_L \ \vdots \ -T_R \right], \quad [B] = \left[T_L \ \vdots \ -U_R \right], \tag{61}$$

$$\{p\} = \left\{ \begin{array}{c} t \\ u \end{array} \right\} \quad \text{and} \quad \{q\} = \left\{ \begin{array}{c} \bar{t} \\ \bar{u} \end{array} \right\}. \tag{62}$$

Similarly, Eq. (57) yields

$$[C]_{N\times N}\{p\}_{N\times 1} = [D]_{N\times N}\{q\}_{N\times 1}. \tag{63}$$

Substituting the given boundary condition Eqs. (33) and (34) into Eq. (63), we have the column vector below

$$\{q\} = \left\{ \begin{array}{c} \left(k\frac{H^{(1)}_{\mathcal{N}+1}(k)}{H^{(1)}_{\mathcal{N}}(k)} - \mathcal{N}\right) \left\{ \begin{array}{c} \cos\mathcal{N}\frac{(2+N)\pi}{2N} \\ \cos\mathcal{N}\left(\frac{(2+N)\pi}{2N} + \frac{2\pi}{N}\right) \\ \vdots \\ \cos\mathcal{N}\left(\frac{(2+N)\pi}{2N} + \frac{2\pi}{N}(N-1)\right) \end{array} \right\} \\ \frac{H^{(1)}_{\mathcal{N}}(k)}{H^{(1)}_{\mathcal{N}}(k)} \left\{ \begin{array}{c} \cos\mathcal{N}\frac{(2+N)\pi}{2N} \\ \cos\mathcal{N}\left(\frac{(2+N)\pi}{2N} + \frac{2\pi}{N}\right) \\ \vdots \\ \cos\mathcal{N}\left(\frac{(2+N)\pi}{2N} + \frac{2\pi}{N}(N-1)\right) \end{array} \right\} \end{array} \right\}. \tag{64}$$

5. Review of the treatments for the fictitious frequency

5.1. Burton and Miller method

For the Dirichlet or Neumann problem, Burton and Miller (1971) utilized the product of hypersingular equation with an imaginary constant to the singular equation,

$$(ik[U] + [L])\{t\} = (ik[T] + [M])\{u\} \tag{65}$$

to deal with the problem of fictitious frequency which result in the non-uniqueness solution. The mixed-type boundary condition problem can be solved by the same concept for the equation

$$(ik[A] + [C])\{p\} = (ik[B] + [D])\{q\}, \tag{66}$$

where $[A]$, $[B]$, $[C]$ and $[D]$ can be found in Eqs. (60) and (63).

5.2. CHIEF method

In order to remove the fictitious frequencies, Schenck (1968) and Benthien and Schenck (1997) used the CHIEF method, which employed the boundary integral equations by collocating the interior point as an auxiliary constraint to promote the rank of influence matrix. By using the concept, combination of the integral equations for the boundary points and those for the points in the complementary domain with the mixed-type boundary conditions yields an over-determined equation system,

$$\begin{bmatrix} A^{\mathrm{B}}_{N\times N} \\ A^{\mathrm{c}}_{a\times N} \end{bmatrix}\{p\} = \begin{bmatrix} B^{\mathrm{B}}_{N\times N} \\ B^{\mathrm{c}}_{a\times N} \end{bmatrix}\{q\}, \tag{67}$$

where the superscript B denotes collocation on the boundary, the superscript c denotes collocation on the complementary domain and a is the number of additional interior points.

6. Numerical experiments and discussions

Case I: A semi-infinite rod. A semi-infinite rod is considered. The boundary condition is shown in Fig. 1. The fictitious frequencies for the mixed-type boundary conditions using different auxiliary systems (symmetric and anti-symmetric) are compared with those of Dirichlet and Neumann problems in Table 1. The fictitious frequencies satisfy $\cos k = 0$ for the singular formulation and $\sin k = 0$ for the hypersingular formulation by using the symmetric auxiliary system with Neumann boundary condition. The fictitious frequencies satisfy $\sin k = 0$ for the singular formulation and $\cos k = 0$ for the hypersingular formulation by using the anti-symmetric auxiliary system with Dirichlet boundary condition. The fictitious frequencies appear as shown in Figs. 3 and 4 with the potential versus the wave number for symmetric and anti-symmetric auxiliary system, respectively. The irregular values occur at the zeros of the trigonometric function, $\sin k$ or $\cos k$, that match with those of the Dirichlet and Neumann problems.

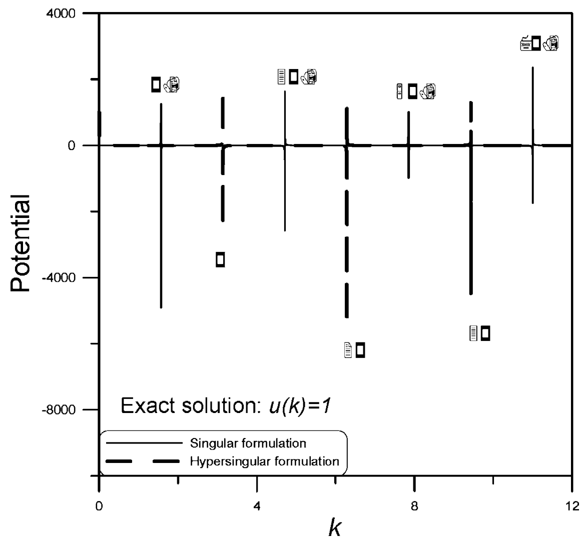

Fig. 3. The potential versus the wave number using the singular and hypersingular formulations by using symmetric auxiliary system.

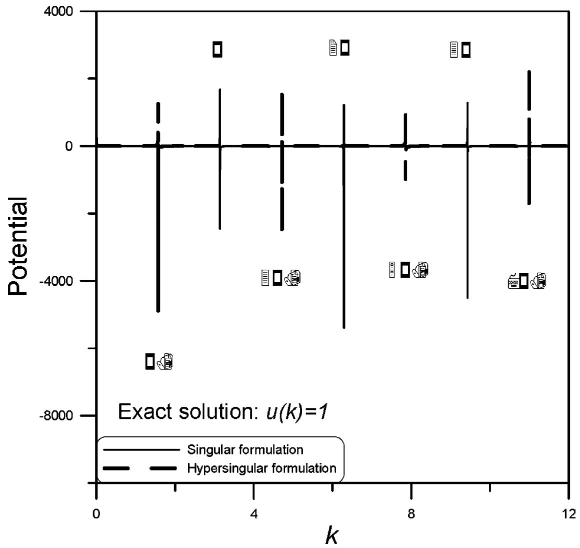

Fig. 4. The potential versus the wave number using the singular and hypersingular formulations by using anti-symmetric auxiliary system.

Case II: A circular radiator. A circular radiator with a radius 1 m is considered. The boundary condition and the number of collocating points are shown in Fig. 2. A semi-analytical approach is adopted instead of calculating the Fourier coefficients of the boundary densities exactly. The number of the terms for Fourier series and the order of the Bessel functions are $n = 5$ and $m = 20$ in Eqs. (44)–(47) and Eqs. (51)–(54), respectively. Twenty two $(4n + 2)$ collocation points on the boundary are used. The fictitious frequencies appear as shown in Figs. 5 and 6 by using the singular and hypersingular formulations, respectively, where J_m^n denotes the nth zero for J_m function.

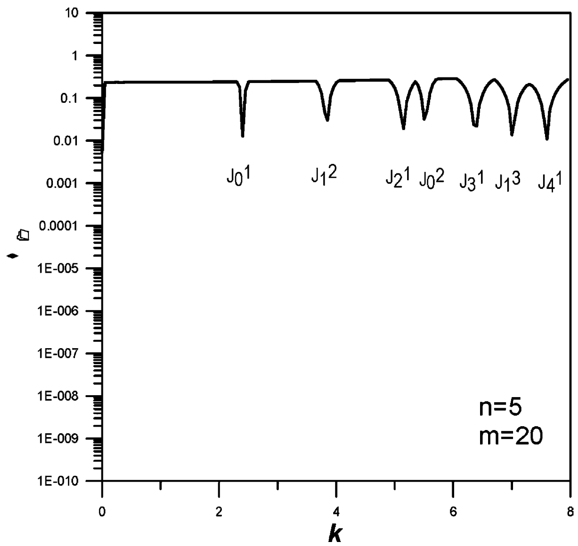

Fig. 5. The minimum singular value versus k using the singular formulation for the semi-analytical approach.

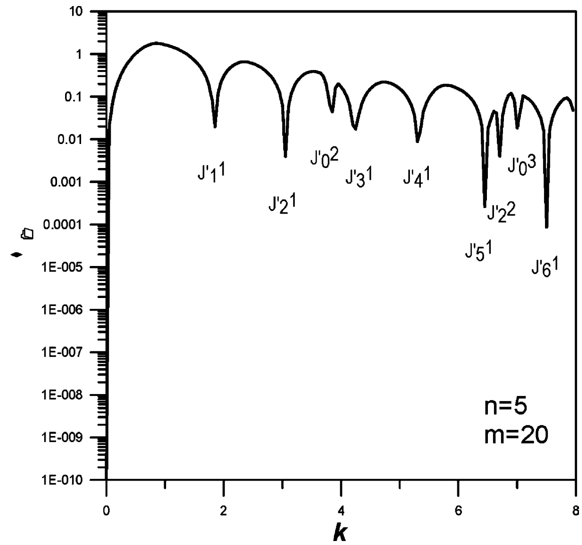

Fig. 6. The minimum singular value versus k using the hypersingular formulation for the semi-analytical approach.

The number of boundary elements are 30. Taking $\mathcal{N} = 4$ for our numerical experiment in Eq. (64), the BEM is utilized to solve the radiation problem. The CHIEF technique and Burton and Miller method are applied to filter out the irregular values in the numerical experiments. To demonstrate that, we show the potential and the flux at $(R, \theta) = (1, 0)$ and $(R, \theta) = (1, \pi)$, respectively, where the CHIEF point locates at $(x, y) = (0.5, 0.5)$. The irregular values of the potential and the treatments for them are plotted and compared with the analytical solution, Eq. (35), in Figs. 7 and 8, respectively. The irregular values of the flux

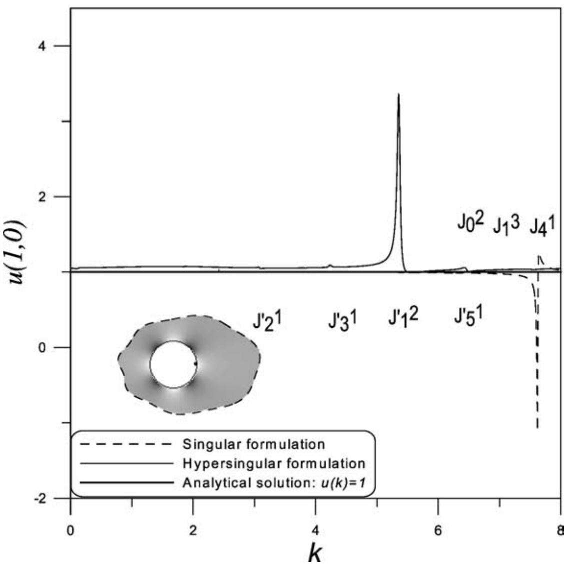

Fig. 7. The potential $u(1, 0)$ versus k using the singular and hypersingular formulations for the BEM.

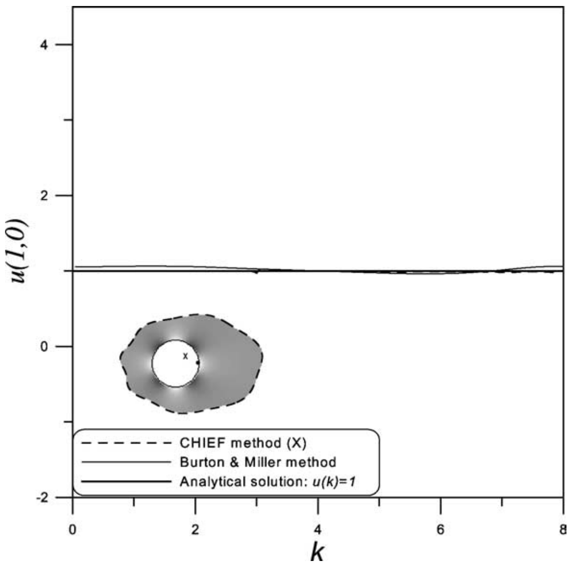

Fig. 8. The potential $u(1,0)$ versus k using the Burton and Miller method and CHIEF technique for the BEM.

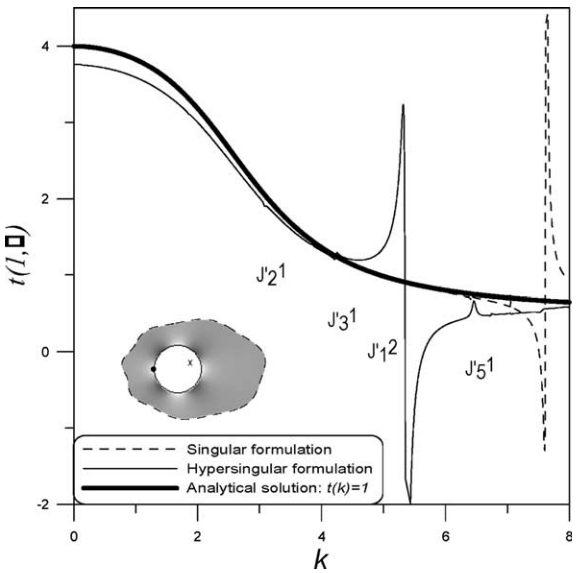

Fig. 9. The flux $t(1,\pi)$ versus k by using the singular and hypersingular formulations for the BEM.

and the treatments for them are plotted and compared with the analytical solution, Eq. (35), in Figs. 9 and 10, respectively. The irregular values locate at the zeros of $J_n(k)$ by using the singular formulation (*UT* formulation), while locate at the zeros of $J_n'(k)$ by using the hypersingular formulation (*LM* formulation). The contour of BEM result and the exact solutions are shown in Figs. 11 and 12, respectively. The numerical experiments match well with our semi-analytical results.

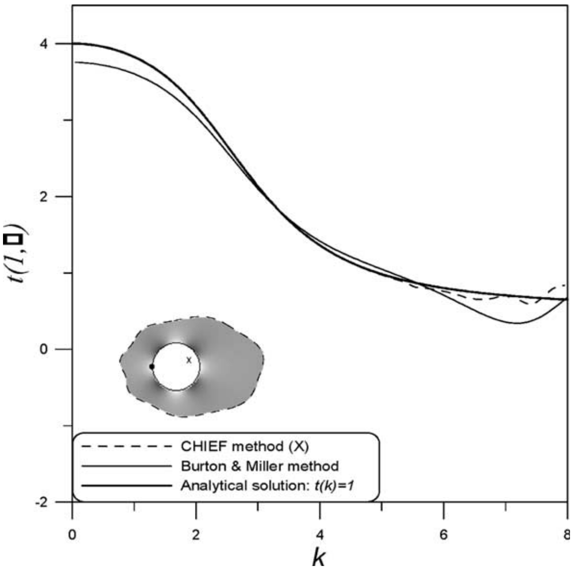

Fig. 10. The flux $t(1, \pi)$ versus k by using the Burton and Miller method and CHIEF technique for the BEM.

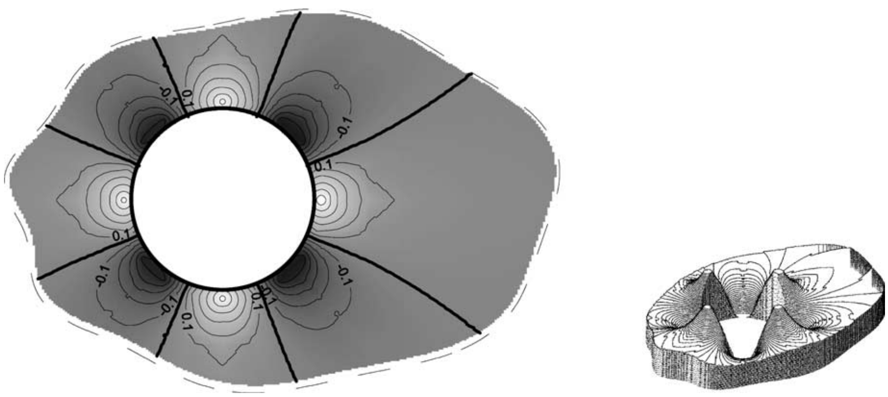

Fig. 11. The BEM solution for radiation problem subject to the mixed-type boundary condition ($k = 1$) for the BEM.

7. Conclusions

The study presented in this article has focused on the occurrence of fictitious frequencies for the problems subject to the mixed-type boundary condition. How to overcome the fictitious frequencies were also addressed. The derivations of irregular values for a semi-infinite rod were implemented analytically and numerically. The irregular values were derived semi-analytically and numerically for circular problems through null-field integral equations and BEM, respectively. The fictitious frequencies of circular radiator subject to the mixed-type boundary condition were studied by using the BEM. The irregular values were treated by using the CHIEF technique and Burton and Miller method. It was verified that the occurrence of fictitious frequencies depend on the integral representation (singular or hypersingular formulation) no matter what the given type of boundary condition for the problem is. A good agreement is made.

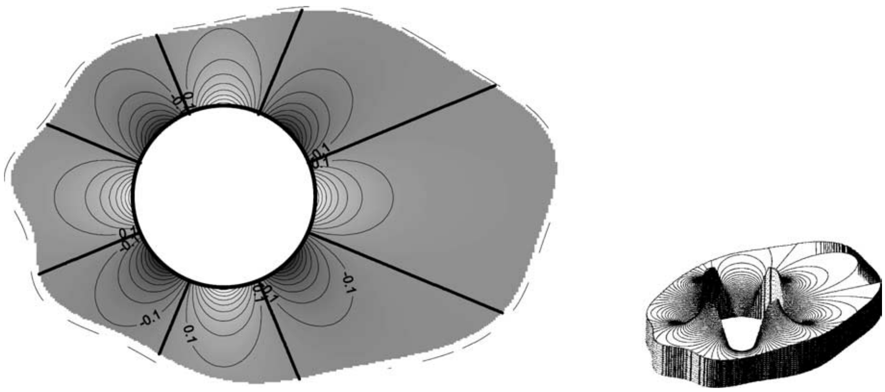

Fig. 12. The exact solution for the radiation problem subject to the mixed-type boundary condition ($k = 1$) for the BEM.

References

Benthien, G.W., Schenck, H.A., 1997. Comput. Struct. 65, 295.
Burton, A.J., Miller, G.F., 1971. Proc. Roy. Soc. Lon. Ser. A 323, 201.
Chen, J.T., 1998. Mech. Res. Commun. 25 (5), 529.
Chen, J.T., Chen, K.H., 1998. Engng. Analysis Bound. Elem. 21, 105.
Chen, J.T., Hong, H.-K., 1999. Appl. Mech. Rev. 52, 17–33.
Chen, J.T., Kuo, S.R., 2000. Mech. Res. Commun. 27 (1), 49.
Chen, J.T., Chen, C.T., Chen, K.H., Chen, I.L., 2000. Mech. Res. Commun. 27 (6), 685.
Chen, J.T., Chen, K.H., Chen, I.L., Liu, L.W., 2003. Mech. Res. Commun. 30 (6), 161.
Dokumaci, E., 1990. J. Sound Vib. 139 (1), 83.
Juhl, P., 1994. J. Sound Vib. 175 (1), 39.
Kress, R., Roach, G.F., 1977. Proc. Roy. Soc. Edinburgh 77A, 65.
Lee, C.H., Sclavounos, P.D., 1989. J. Fluid Mech. 207, 393.
Lee, C.H., Newmn, J.N., Zhu, X., 1996. Int. J. Numer. Methods Fluids 23, 637.
Malenica, S., Chen, X.B., 1998. Int. J. Offshore Polar Engng. 8 (2), 110.
Matrin, P.A., 1980. Quart. J. Mech. Appl. Math. 27, 386.
Ohmatsu, S., Papers of Ship Research Institute, 1983, p. 70.
Poulin, S., Ph.D. Dissertation of Department of Hydrodynamics and Water Resources, Technical University of Denmark, Lyngby, 1997.
Schenck, H.A., 1968. J. Acoust. Soc. Am. 44 (1), 41.
Seybert, A.F., Rengarajan, T.K., 1989. J. Acoust. Soc. Am. 81 (5), 1299.
Ursell, F., 1981. J. Fluid Mech. 143, 143.