A new method for plates with circular holes

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Outlines

- Introduction
- Problem statement
- Boundary integral formulation
- Degenerate kernels
- Linear algebraic system
- A numerical example
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Introduction

Engineering problems	Biharmonic problems	**Solid mechanics：plate problem**
		Fluid mechanics：Stokes' flow
	Laplace Problems	Torsion bar with circular holes, Steady state heat conduction of tube……etc.
	Helmholtz Problems	Electromagnetic wave, Membrane vibration, Water wave and Acoustic problems……etc.

Methods

In the past	Finite difference method (FDM), finite element method (FEM) and boundary element method (BEM)
Recently	Shen (2004) *et al.* have successfully applied this method to solve Laplace problems with circular holes
Purpose	Extending to biharmonic problem

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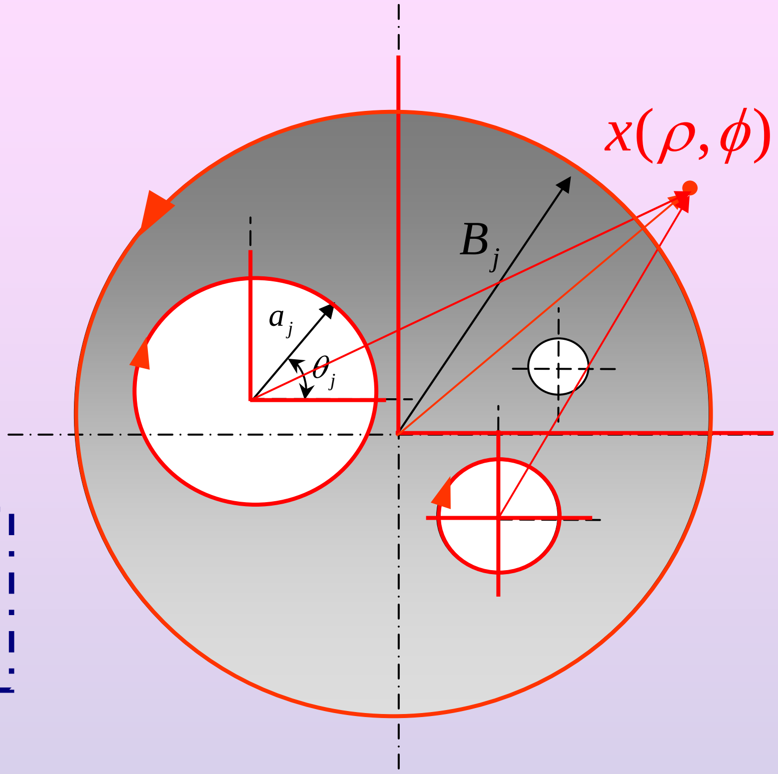
Problem statement



- G.E.： $\nabla^4 u(x) = 0, \quad x \in D$
- B.C.： $u(s)$, $\theta(s)$, $m(s)$, $v(s)$

$$a_{0j} + \sum_{n=1}^{M} (a_{nj} \cos n\theta_j + b_{nj} \sin n\theta_j), s \in B_j$$

where a_j is the radius of the circular hole.

B_j is the boundary of the circular domain.

$u(s)$, $\theta(s)$, $m(s)$, $v(s)$ are the displacement, slope, normal moment, and effective shear force on the boundary, respectively.

a_{0j}, a_{nj}, b_{nj} are the Fourier coefficients.

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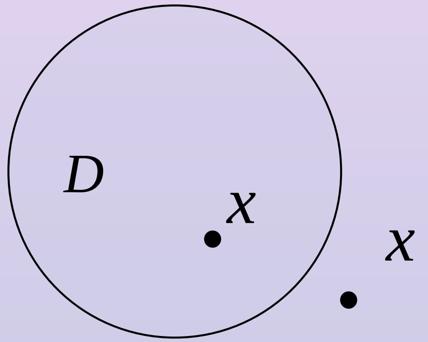
Boundary Integral formulation

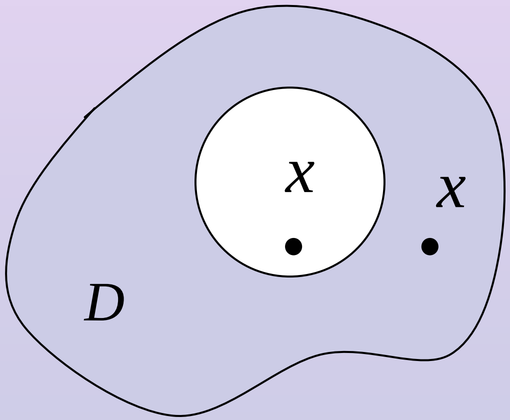
- The Rayleigh-Green identity ：

$$8\pi u(x)=\int_B\left\{-U(s,x)\boldsymbol{u}(s)+\Theta(s,x)m(s)-M(s,x)\theta(s)+V(s,x)u(s)\right\}dB(s),\,x\in D$$

$$0=\int_B\left\{-U(s,x)\boldsymbol{u}(s)+\Theta(s,x)m(s)-M(s,x)\theta(s)+V(s,x)u(s)\right\}dB(s),\,x\notin D$$



Interior problem



Exterior problem

- The boundary integral formulation of the domain point：

$$8\pi u(x)=\int_B\left\{-U(s,x)\boldsymbol{v}(s)+\Theta(s,x)m(s)-M(s,x)\theta(s)+V(s,x)u(s)\right\}dB(s),\ x\in D$$

$$8\pi\theta(x)=\int_B\left\{-U_\theta(s,x)\boldsymbol{v}(s)+\Theta_\theta(s,x)m(s)-M_\theta(s,x)\theta(s)+V_\theta(s,x)u(s)\right\}dB(s),\ x\in D$$

$$8\pi m(x)=\int_B\left\{-U_m(s,x)\boldsymbol{v}(s)+\Theta_m(s,x)m(s)-M_m(s,x)\theta(s)+V_m(s,x)u(s)\right\}dB(s),\ x\in D$$

$$8\pi\boldsymbol{v}(x)=\int_B\left\{-U_{\boldsymbol{v}}(s,x)\boldsymbol{v}(s)+\Theta_{\boldsymbol{v}}(s,x)m(s)-M_{\boldsymbol{v}}(s,x)\theta(s)+V_{\boldsymbol{v}}(s,x)u(s)\right\}dB(s),\ x\in D$$

- Null-field boundary integral formulation ：

$$0 = \int_B \left\{ -U(s,x)\boldsymbol{u}(s) + \Theta(s,x)m(s) - M(s,x)\theta(s) + V(s,x)u(s) \right\} dB(s), x \notin D$$

$$0 = \int_B \left\{ -U_\theta(s,x)\boldsymbol{u}(s) + \Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s) + V_\theta(s,x)u(s) \right\} dB(s), x \notin D$$

$$0 = \int_B \left\{ -U_m(s,x)\boldsymbol{u}(s) + \Theta_m(s,x)m(s) - M_m(s,x)\theta(s) + V_m(s,x)u(s) \right\} dB(s), x \notin D$$

$$0 = \int_B \left\{ -U_{\boldsymbol{u}}(s,x)\boldsymbol{u}(s) + \Theta_{\boldsymbol{u}}(s,x)m(s) - M_{\boldsymbol{u}}(s,x)\theta(s) + V_{\boldsymbol{u}}(s,x)u(s) \right\} dB(s), x \notin D$$

- The degenerate kernels for the sixteen kernel functions

$$
\begin{array}{rcccc}
 & \xrightarrow{K_{\theta,s}(\cdot)} & \xrightarrow{K_{m,s}(\cdot)} & \xrightarrow{K_{v,s}(\cdot)} & \\
 & U(s,x) & \Theta(s,x) & M(s,x) & V(s,x) \\
K_{\theta,x}(\cdot)\downarrow & & & & \\
 & U_{\theta}(s,x) & \Theta_{\theta}(s,x) & M_{\theta}(s,x) & V_{\theta}(s,x) \\
K_{m,x}(\cdot)\downarrow & & & & \\
 & U_{m}(s,x) & \Theta_{m}(s,x) & M_{m}(s,x) & V_{m}(s,x) \\
K_{v,x}(\cdot)\downarrow & & & & \\
 & U_{v}(s,x) & \Theta_{v}(s,x) & M_{v}(s,x) & V_{v}(s,x)
\end{array}
$$

- Operator：

$$K_{\theta,s}(\cdot) = \frac{\partial(\cdot)}{\partial n_s},$$

$$K_{m,s}(\cdot) = \nu\nabla^2(\cdot) + (1-\nu)\frac{\partial^2(\cdot)}{\partial n_s},$$

$$K_{v,s}(\cdot) = \frac{\partial\nabla^2(\cdot)}{\partial n_s} + (1-\nu)\frac{\partial}{\partial t_s}\left[\frac{\partial}{\partial n_s}\left(\frac{\partial(\cdot)}{\partial t_s}\right)\right],$$

where ν the is Poisson ratio .

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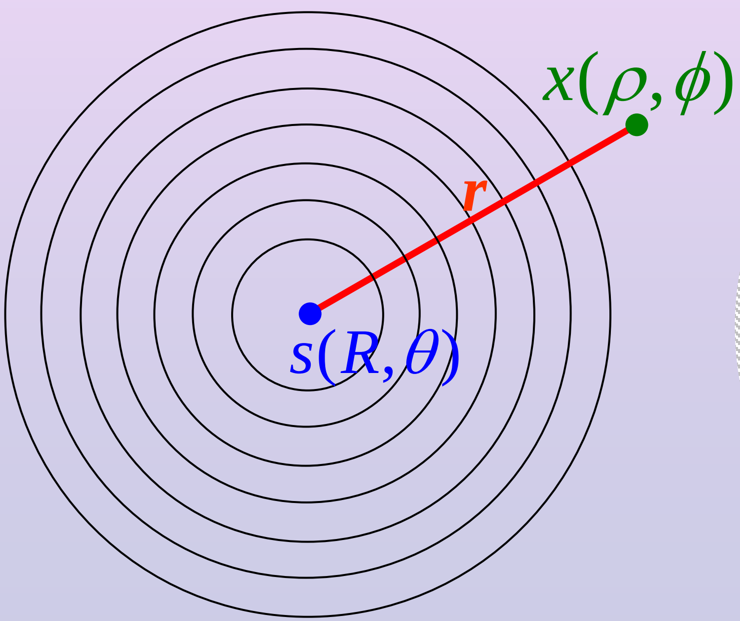
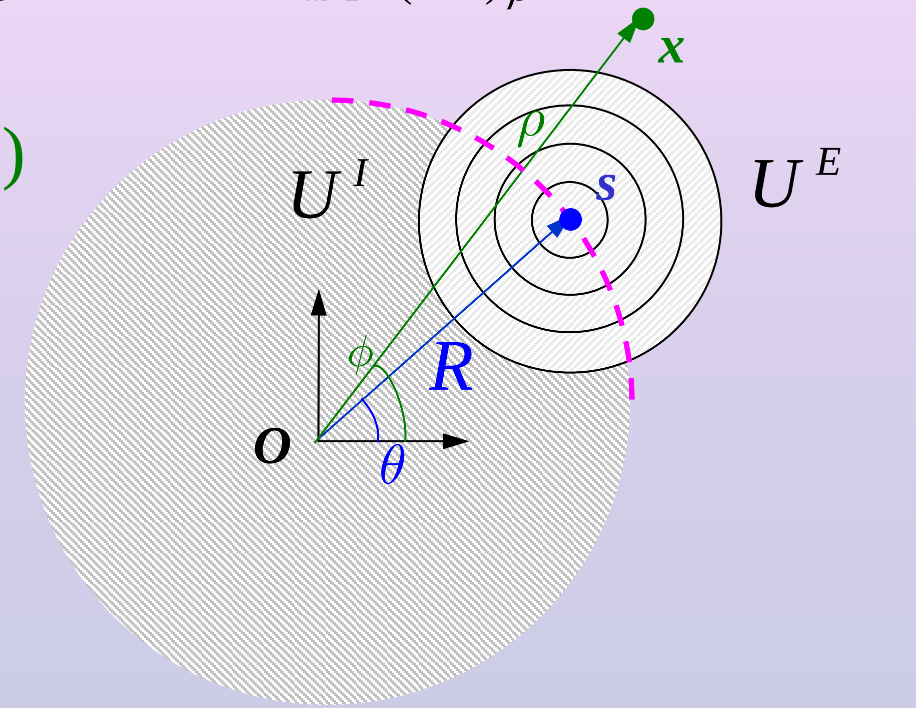
Degenerate kernels

$$U(s,x)=r^2\ln r=\begin{cases} U^I(s,x)=\rho^2(1+\ln R)+R^2\ln R-R\rho(1+2\ln R)\cos(\theta-\phi) \\ \quad -\sum_{m=1}^{\infty}\frac{1}{m(m+1)}\frac{\rho^{m+2}}{R^m}\cos[m(\theta-\phi)]+\sum_{m=2}^{\infty}\frac{1}{m(m-1)}\frac{\rho^m}{R^{m-2}}\cos[m(\theta-\phi)],\ R>\rho \\ U^E(s,x)=R^2(1+\ln\rho)+\rho^2\ln\rho-\rho R(1+2\ln\rho)\cos(\theta-\phi) \\ \quad -\sum_{m=1}^{\infty}\frac{1}{m(m+1)}\frac{R^{m+2}}{\rho^m}\cos[m(\theta-\phi)]+\sum_{m=2}^{\infty}\frac{1}{m(m-1)}\frac{R^m}{\rho^{m-2}}\cos[m(\theta-\phi)],\ \rho>R \end{cases}$$





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Linear algebraic system

- By using the null-field integral equations for $u(x)$ and $\theta(x)$ we have：

$$\begin{bmatrix} U(s,x) & \Theta(s,x) \\ U_\theta(s,x) & \Theta_\theta(s,x) \end{bmatrix} \begin{Bmatrix} a_{0j} \\ \vdots \\ a_{nj} \\ \vdots \\ b_{nj} \end{Bmatrix} = \begin{bmatrix} M(s,x) & V(s,x) \\ M_\theta(s,x) & V_\theta(s,x) \end{bmatrix} \begin{Bmatrix} p_{0j} \\ \vdots \\ p_{nj} \\ \vdots \\ q_{nj} \end{Bmatrix}$$

■ Flowchart of present method

$$0=\int_B\{-U(s,x)v(s)+\Theta(s,x)m(s)-M(s,x)\theta(s)+V(s,x)u(s)\}\,dB(s),\ x\notin D$$

$$0=\int_B\{-U_\theta(s,x)v(s)+\Theta_\theta(s,x)m(s)-M_\theta(s,x)\theta(s)+V_\theta(s,x)u(s)\}\,dB(s),\ x\notin D$$

$$0=\int_B\{-U_m(s,x)v(s)+\Theta_m(s,x)m(s)-M_m(s,x)\theta(s)+V_m(s,x)u(s)\}\,dB(s),\ x\notin D$$

Degenerate kernels

Fourier series

$$0=\int_B\{-U_v(s,x)v(s)+\Theta_v(s,x)m(s)-M_v(s,x)\theta(s)+V_v(s,x)u(s)\}\,dB(s),\ x\notin D$$

$$C_2^4=6\ options$$

Linear algebraic system → Fourier coefficients → BIE of domain point → Potential

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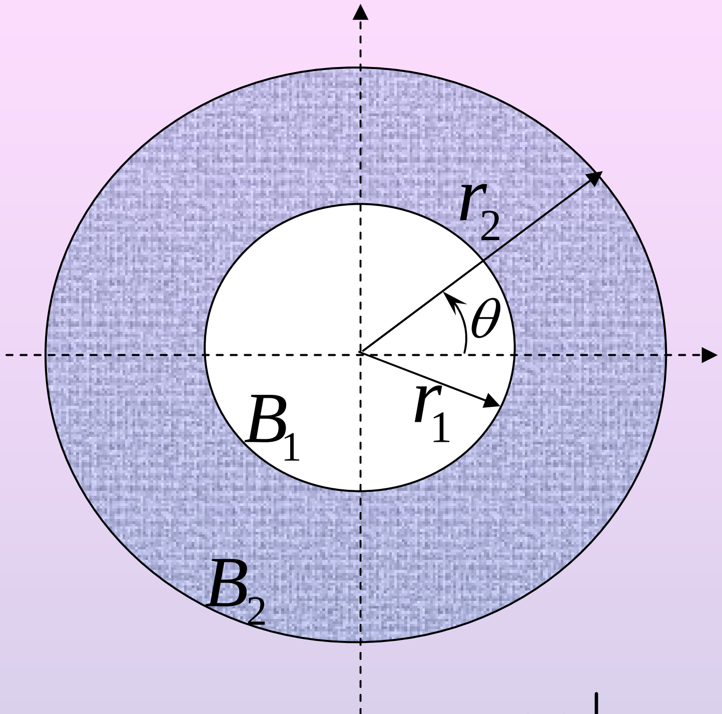
A numerical example



Exact solution：

$$u(\rho,\phi)=(\rho-\frac{4}{\rho})\sin\phi$$

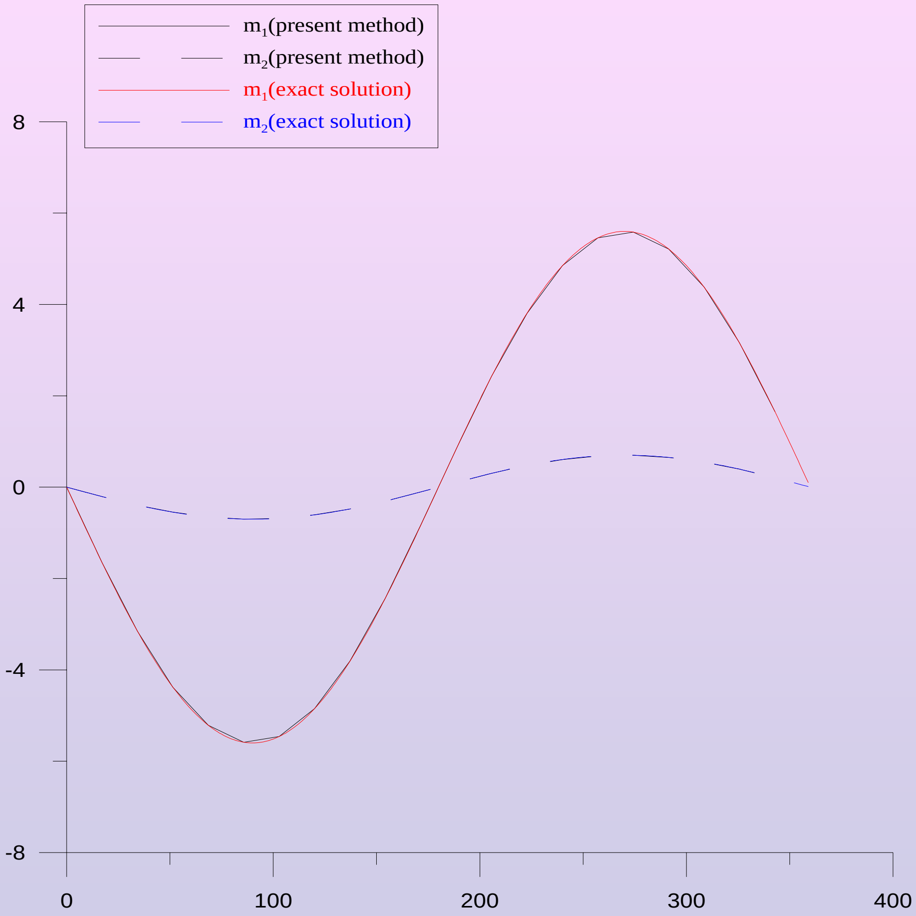
- Given： $r_1=1,\quad u(s)\Big|_{s\in B_1}=-3\sin\theta,\quad \theta(s)\Big|_{s\in B_1}=-5\sin\theta$

  $r_2=2,\quad u(s)\Big|_{s\in B_2}=0,\quad \theta(s)\Big|_{s\in B_2}=2\sin\theta$
- Unknown：

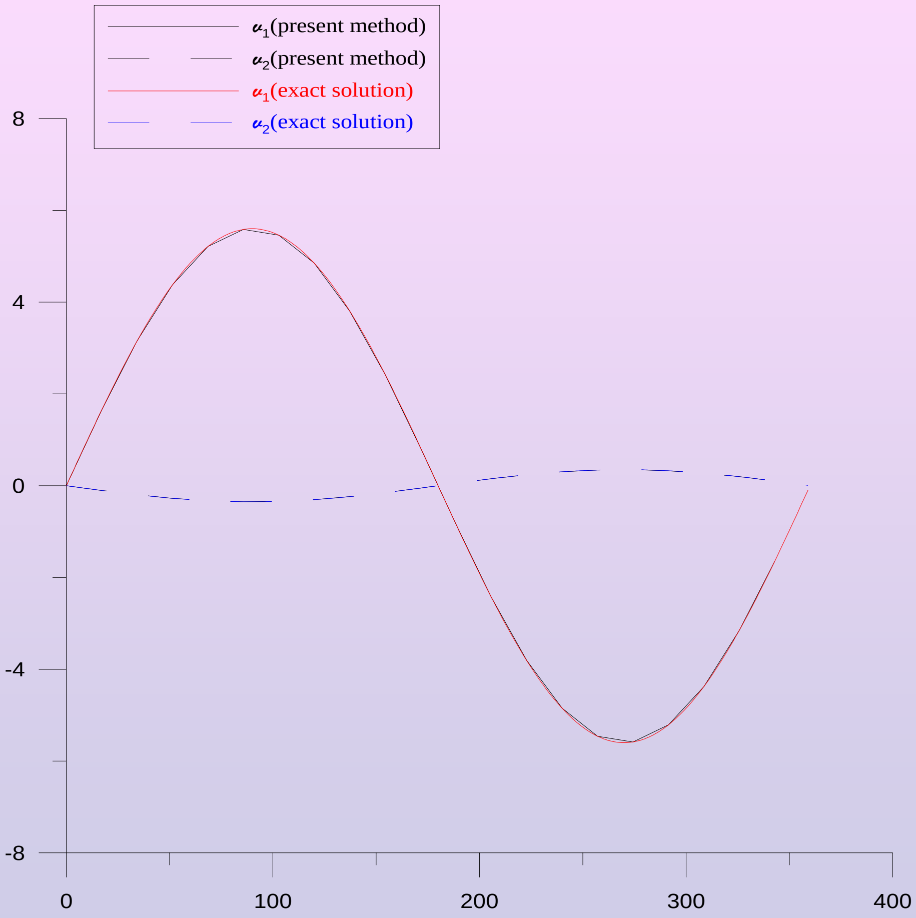
  $m(s)\Big|_{s\in B_1},\quad v(s)\Big|_{s\in B_1},\quad m(s)\Big|_{s\in B_2},\quad v(s)\Big|_{s\in B_2}$

  are expanded by Fourier series

Comparison

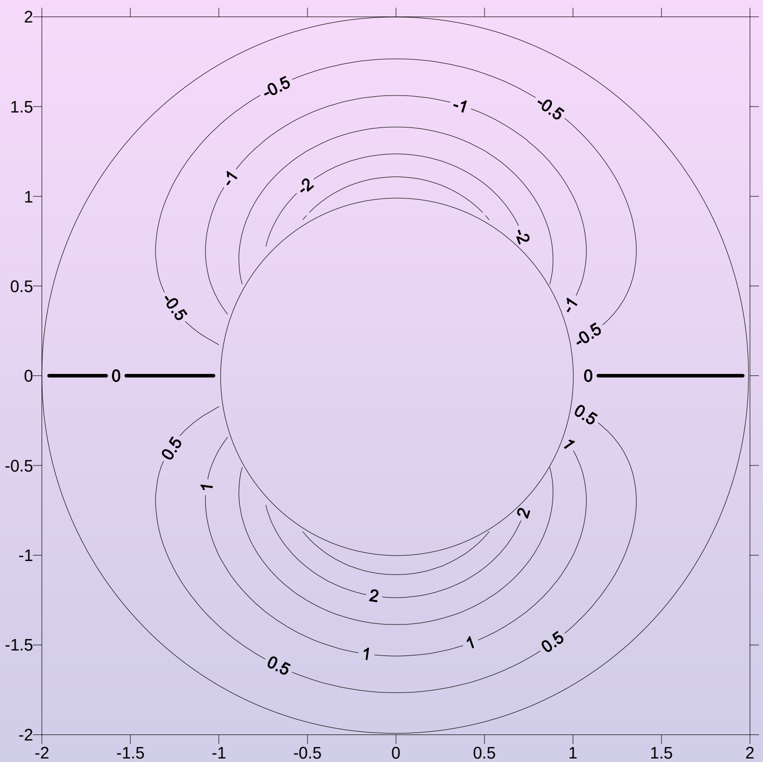


m_1(present method)
m_2(present method)
m_1(exact solution)
m_2(exact solution)
8
4
0
-4
-8
0
100
200
300
400

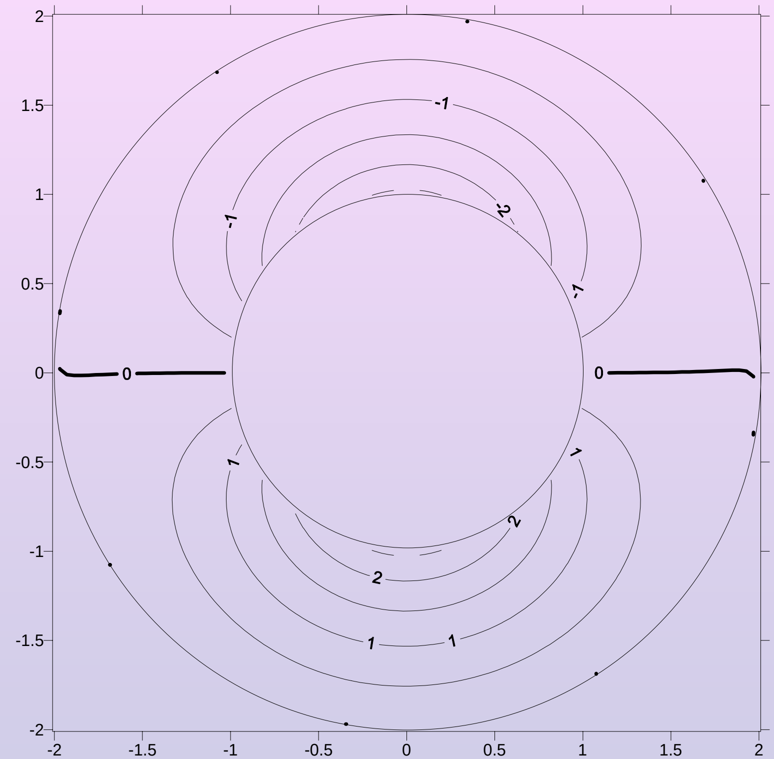


u_1(present method)
u_2(present method)
u_1(exact solution)
u_2(exact solution)
8
4
0
-4
-8
0
100
200
300
400

Exact solution



Present method



M=10

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Conclusions

- We have expanded Laplace problems to the biharmonic problems successfully.
- Numerical results agree very well with the exact solution.
- It can be extended to plate problems with multiple circular holes.

Thank you for your kind attention!