A Highly Accurate Solver for the Mixed-Boundary Potential Problem and Singular Problem in Arbitrary Plane Domain

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Abstract: A highly accurate new solver is developed to deal with interior and exterior mixed-boundary value problems for two-dimensional Laplace equation, including the singular ones. To promote the present study, we introduce a circular artificial boundary which is uniquely determined by the physical problem domain, and derive a Dirichlet to Robin mapping on that artificial circle, which is an exact boundary condition described by the first kind Fredholm integral equation. As a consequence, we obtain a modified Trefftz method equipped with a characteristic length factor, ensuring that the new solver is stable because the condition number can be greatly reduced. Then, the collocation method is used to derive a linear equations system to determine the Fourier coefficients. We find that the new method is powerful even for the problem with complex boundary shape and with adding random noise on the boundary data. It is also applicable to the singular problem of Motz type, resulting to a highly accurate result never seen before.

Keyword: Laplace equation, Artificial circle, DtR mapping, Modified Trefftz method, Mixed-boundary value problem, Singular problem

1 Introduction

The most widely used numerical methods for engineering computations are FDM, FEM and BEM. Despite its popularity, the BEM has some inherent drawbacks. To name a few, Lesnic, Elliott and Ingham (1998) have found that the BEM is weak to against the boundary data disturbance producing unstable solution. Chen, Lin and Chen (2005) have found that the degenerate scale prob-

lem and rank deficiency problem may occur for the BEM used in the Laplace equation. The other drawbacks are the requirement of meshing, evaluation of singular integrals, and slow convergence.

For a complicated shape of the problem domain the above methods usually require a large number of nodes and elements to match the geometrical shape. In order to get over these deficiencies, various numerical methods for solving the Laplace equation are rapidly developed in the last three decades. Recently, Young, Chen and Lee (2005) have proposed a novel meshless method for solving the Laplace equation in arbitrary domain through a rather complicated desingularization technique, and Chen, Shen and Chen (2006) utilized the null-field method to calculate the torsion problem with many holes. The meshless numerical methods were also proposed, which are meshes free and only boundary nodes are necessary [Atluri, Kim and Cho (1999); Atluri and Shen (2002)]. Under the same spirit, Liu (2007a) has developed a meshless regularized integral equation method for the Laplace equation in arbitrary plane domain, and then Liu (2007b) extended these results to the doubly-connected region.

On the other hand, the method of fundamental solutions (MFS), also called the F-Trefftz method, utilizes the fundamental solutions as basis functions to expand the solution, which is another popularly used meshless method [Cho, Golberg, Muleshkov and Li (2004)]. In order to tackle of the ill-posedness of MFS, Jin (2004) has proposed a new numerical scheme for the solution of the Laplace and biharmonic equations subjected to noisy boundary data. A regularized solution was obtained by using the truncated singular value decomposition, with the regularization parameter determined by the L-curve method. Tsai, Lin,

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Young and Atluri (2006) announced a practical procedure to locate the sources in the use of MFS for various time independent operators, including Laplacian operator, Helmholtz operator, modified Helmholtz operator, and biharmonic operator. The procedure is developed through some systematic numerical experiments for relations among the accuracy, condition number, and source positions in different shapes of computational domains. By numerical experiments, they found that good accuracy can be achieved when the condition number approaches the limit of equation solver. The MFS has a broad application in engineering computations, for example, Cho, Golberg, Muleshkov and Li (2004), Hong and Wei (2005), Young and Ruan (2005), Young, Fan, Tsai and Chen (2006) and Young, Tsai, Lin and Chen (2006).

It is known that the standard numerical methods are ineffective to treat the elliptic boundary value problems with singularities. The singularity often arises in engineering problems when there is a sudden change in boundary conditions or the boundary itself. In general it is difficult to obtain accurate approximation in the neighborhood of singular point by using the standard methods. In order to achieve a satisfactory solution near the singular point, some special techniques are required as that given by Li, Mathon and Sermer (1987), Li (1998), Georgiou, Olson and Smyrlis (1996), Georgiou, Boudouvis and Poullikkas (1997), Yosibash, Arad, Yakhot and Ben-Dor (1998), Arad, Yosibash, Ben-Dor and Yakhot (1998), Huang and Li (2003, 2006), Dosiyev (2004), and Lu, Hu and Li (2004).

One of the most efficient method to solve the singular problems is the boundary approximation method [Li, Mathon and Sermer (1987)], which is a special version of the spectral method. In engineering, it is recognized as the collocation Trefftz method (CTM) [Li, Lu, Huang and Cheng (2007)], which uses the linear combination of the T-complete functions to approximate the boundary conditions as best as possible, while the solution exactly satisfies the partial differential equation itself in the problem domain. This method is simple in that only the boundary and interface

conditions need to be fitted. Moreover, it exhibits an exponential rate of convergence.

Recently, Li, Lu, Huang and Cheng (2007) have given a fairly comprehensive comparison of the Trefftz, collocation and other boundary methods. They concluded that the CTM is the simplest algorithm and provides the most accurate solution with the best numerical stability. However, the conventional CTM may have a major drawback that the resulting linear equations system is extremely ill-conditioned. In order to obtain an accurate solution of the linear equations some special techniques, e.g., preconditioner and truncated SVD, are required.

In order to overcome these difficulties appeared in the conventional CTM, we are going to refine this method by taking the characteristic length of problem domain into the Trefftz functions, such that the condition number of the resulting linear equations system can be greatly reduced. The new method is essentially stable and has the exponential rate of convergence. Here we will use, in addition other examples, the Motz problem as a testing example of our method. The Motz problem is asserted as a benchmark of singularity problems. It is a Laplace mixed-boundary value problem with the mixed Dirichlet-Neumann conditions in a rectangle. The derivative of its solution has a strong singularity at the original point [Liu, Chen and Chang (2007)]. After using the new method one may appreciate its high accuracy, which is never seen before.

The other parts of present paper are arranged as follows. In Section 2 we derive the first kind Fredholm integral equation along a given artificial circle. This derivation naturally leads to a new set of T-complete functions. In Section 3 we consider a direct collocation method, the comparisons of which with the conventional CTM are made in Section 4. In Section 5 we use some examples, including the Motz problem, to test the new method, and then, we give conclusions in Section 6.

2   The Fredholm integral equation

In this paper we consider a new meshless method to solve the mixed-boundary value prob-

lem, which consists of the Laplace equation and a mixed-boundary condition at a non-circular boundary:

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \ \ r < \rho \tag{1}$$
$$\text{or} \ \ r > \rho, \ \ 0 \le \theta \le 2\pi,$$

$$\beta_D u(\rho,\theta) + \beta_N u_n(\rho,\theta) = h(\theta), \ \ 0 \le \theta \le 2\pi, \tag{2}$$

where $h(\theta)$ is a given function, and $r = \rho(\theta)$ is a given contour describing the boundary shape of the interior or exterior domain Ω. The contour Γ in the polar coordinates is described by $\Gamma = \{(r,\theta)|r = \rho(\theta), \ 0 \le \theta \le 2\pi\}$. β_D and β_N satisfy $\beta_D^2 + \beta_N^2 > 0$.

If the Dirichlet boundary condition is prescribed on the whole boundary, then we have $\beta_D = 1$ and $\beta_N = 0$ thereon. Conversely, if the Neumann boundary condition is prescribed on the whole boundary, then we have $\beta_D = 0$ and $\beta_N = 1$ thereon. Usually we have a mixed-boundary condition with u prescribed on a partial boundary Γ_1 and with u_n prescribed on the other remaining boundary $\Gamma_2 = \Gamma/\Gamma_1$. Therefore, we have $\beta_D = 1$ and $\beta_N = 0$ on Γ_1, while $\beta_D = 0$ and $\beta_N = 1$ on Γ_2.

Through some effort we can derive

$$u_n(\rho,\theta) = \frac{\partial u(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial \theta}, \tag{3}$$

where n is the outward-normal direction of the boundary.

We replace Eq. (2) by the following boundary condition:

$$u(R_0,\theta) = f(\theta), \ \ 0 \le \theta \le 2\pi, \tag{4}$$

where $f(\theta)$ is an unknown function to be determined, and R_0 is a given positive constant, such that the disk $D = \{(r,\theta)|r \le R_0, \ 0 \le \theta \le 2\pi\}$ can cover Ω for the interior problem, or for the exterior problem it is inside in the complement of Ω,

that is, $D \in \mathbb{R}^2/\overline{\Omega}$. Specifically, we may let

$$R_0 \ge \rho_{\max} = \max_{\theta \in [0,2\pi]} \rho(\theta) \ \text{(interior problem)}, \tag{5}$$

$$R_0 \le \rho_{\min} = \min_{\theta \in [0,2\pi]} \rho(\theta) \ \text{(exterior problem)}. \tag{6}$$

The basic idea is to replace the original complicated Robin boundary condition (2) on a complicated contour by a simpler Dirichlet boundary condition (4) on a specified circle. However, we require to derive a new equation to solve $f(\theta)$. If this task can be finished and if the function $f(\theta)$ is available, then the advantage of this formulation is that we have a Fourier series expansion of $u(r,\theta)$ satisfying Eqs. (1) and (4):

$$u(r,\theta) = a_0 +$$
$$\sum_{k=1}^{\infty}\left[a_k\left(\frac{R_0}{r}\right)^{\pm k}\cos k\theta + b_k\left(\frac{R_0}{r}\right)^{\pm k}\sin k\theta\right], \tag{7}$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)d\xi, \tag{8}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\cos k\xi \, d\xi, \tag{9}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin k\xi \, d\xi \tag{10}$$

are the Fourier coefficients of $f(\theta)$. In Eq. (7) the positive sign before k is used for exterior problem, and conversely the minus sign before k is used for interior problem.

From Eqs. (3) and (7) it follows that

$$u_n(\rho,\theta) = \sum_{k=1}^{\infty}\gamma^k\left[\left\{\mp\frac{ka_k}{\rho} - \frac{kb_k\rho'}{\rho^2}\right\}\cos k\theta\right.$$
$$\left. + \left\{\frac{ka_k\rho'}{\rho^2} \mp \frac{kb_k}{\rho}\right\}\sin k\theta\right], \tag{11}$$

where

$$\gamma(\theta) := \left(\frac{R_0}{\rho(\theta)}\right)^{\pm 1}. \tag{12}$$

By imposing the condition (2) on Eq. (7) and utilizing Eq. (11) we obtain

$$a_0 \beta_D + \sum_{k=1}^{\infty} [a_k E_k(\theta) + b_k F_k(\theta)] = h(\theta), \qquad (13)$$

where

$$E_k(\theta) :=$$
$$\gamma^k \left[\beta_D \cos k\theta \mp \beta_N \frac{k}{\rho} \cos k\theta + \beta_N \frac{k\rho'}{\rho^2} \sin k\theta \right],$$
$$(14)$$

$$F_k(\theta) :=$$
$$\gamma^k \left[\beta_D \sin k\theta \mp \beta_N \frac{k}{\rho} \sin k\theta - \beta_N \frac{k\rho'}{\rho^2} \cos k\theta \right].$$
$$(15)$$

Substituting Eqs. (8)-(10) for a_0, a_k and b_k into Eq. (13) leads to the first kind Fredholm integral equation:

$$\int_0^{2\pi} K(\theta, \xi) f(\xi) d\xi = h(\theta), \qquad (16)$$

where

$$K(\theta, \xi) = \frac{\beta_D}{\pi}$$
$$+ \frac{1}{\pi} \sum_{k=1}^{\infty} [E_k(\theta) \cos k\xi + F_k(\theta) \sin k\xi] \quad (17)$$

is a kernel function.

Eq. (16) is an exact boundary condition, providing a mapping from the Dirichlet boundary condition on a simple artificial circle to the Robin boundary condition on the original complicated boundary. This mapping is named the Dirichlet to Robin mapping, abbreviated as the DtR mapping. However, it is difficult to directly inverse Eq. (16) to obtain the exact boundary data $f(\theta)$. Liu (2007a, 2007c, 2007d) has applied the regularization integral equation method to solve Eq. (16) for the Dirichlet boundary value problems. But in this paper, we are going to directly solve Eq. (7) to obtain the Fourier coefficients a_k and b_k as simple as possible.

3   The collocation method

We consider a mixed-boundary condition with u prescribed on a partial boundary Γ_1 and with u_n prescribed on the other boundary $\Gamma_2 = \Gamma / \Gamma_1$. Therefore, we have $\beta_D = 1$ and $\beta_N = 0$ on Γ_1, while $\beta_D = 0$ and $\beta_N = 1$ on Γ_2, i.e.,

$$u(\rho, \theta) = h_D(\theta), \qquad (\rho, \theta) \in \Gamma_1, \qquad (18)$$
$$u_n(\rho, \theta) = h_N(\theta), \qquad (\rho, \theta) \in \Gamma_2. \qquad (19)$$

The series expansions in Eqs. (7) and (11) are well suited to the entire solution domain. Hence, the admissible functions with finite terms can be used to determine the unknown coefficients a_k and b_k,

$$u(\rho, \theta) = a_0 + \sum_{k=1}^{m} [A_k a_k + B_k b_k], \qquad (20)$$

$$u_n(\rho, \theta) = \sum_{k=1}^{m} [C_k a_k + D_k b_k], \qquad (21)$$

where

$$A_k(\theta) := \gamma^k \cos(k\theta), \qquad (22)$$
$$B_k(\theta) := \gamma^k \sin(k\theta), \qquad (23)$$
$$C_k(\theta) := \mp \frac{k}{\rho} A_k + \frac{k\rho'}{\rho^2} B_k, \qquad (24)$$
$$D_k(\theta) := \mp \frac{k}{\rho} B_k - \frac{k\rho'}{\rho^2} A_k. \qquad (25)$$

Here we consider the collocation method. Eqs. (20) and (21) are imposed at different collocated points on two different boundaries with $[\rho(\theta_i), \theta_i] \in \Gamma_1$ and $[\rho(\overline{\theta}_j), \overline{\theta}_j] \in \Gamma_2$,

$$a_0 + \sum_{k=1}^{m} [A_k(\theta_i) a_k + B_k(\theta_i) b_k] = h_D(\theta_i), \qquad (26)$$

$$\sum_{k=1}^{m} [C_k(\overline{\theta}_j) a_k + D_k(\overline{\theta}_j) b_k] = h_N(\overline{\theta}_j). \qquad (27)$$

When the indices i and j in Eqs. (26) and (27) both run from 1 to m we obtain a linear equations

system with dimensions $n = 2m+1$:

$$
\begin{bmatrix}
1 & A_1(\theta_0) & B_1(\theta_0) & \ldots & A_m(\theta_0) & B_m(\theta_0) \\
1 & A_1(\theta_1) & B_1(\theta_1) & \ldots & A_m(\theta_1) & B_m(\theta_1) \\
0 & C_1(\overline{\theta}_1) & D_1(\overline{\theta}_1) & \ldots & C_m(\overline{\theta}_1) & D_m(\overline{\theta}_1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & A_1(\theta_m) & B_1(\theta_m) & \ldots & A_m(\theta_m) & B_m(\theta_m) \\
0 & C_1(\overline{\theta}_m) & D_1(\overline{\theta}_m) & \ldots & C_m(\overline{\theta}_m) & D_m(\overline{\theta}_m)
\end{bmatrix}
$$

$$
\cdot
\begin{bmatrix}
a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_m \\ b_m
\end{bmatrix}
=
\begin{bmatrix}
h_D(\theta_0) \\ h_D(\theta_1) \\ h_N(\overline{\theta}_1) \\ \vdots \\ h_D(\theta_m) \\ h_N(\overline{\theta}_m)
\end{bmatrix}. \quad (28)
$$

In the above θ_0 is an extra collocated point on the boundary Γ_1 used to supplement another equation to determine the n unknowns.

We denote the above equation by

$$\mathbf{Rc} = \mathbf{b}_1,$$

where $\mathbf{c} = [a_0, a_1, b_1, \cdots, a_m, b_m]^{\mathrm{T}}$ is the vector of unknown coefficients. The conjugate gradient method can be used to solve the following normal equation:

$$\mathbf{Ac} = \mathbf{b}, \quad (29)$$

where

$$\mathbf{A} := \mathbf{R}^{\mathrm{T}}\mathbf{R}, \quad \mathbf{b} := \mathbf{R}^{\mathrm{T}}\mathbf{b}_1. \quad (30)$$

Inserting the calculated $\mathbf{c}$ into Eq. (7) we thus have a semi-analytical solution of $u(r, \theta)$:

$$u(r, \theta) = c_1 +$$
$$\sum_{k=1}^{m} \left[c_{2k} \left(\frac{R_0}{r}\right)^{\pm k} \cos k\theta + c_{2k+1} \left(\frac{R_0}{r}\right)^{\pm k} \sin k\theta \right], \quad (31)$$

where $(c_1, \ldots, c_{2m+1})$ are the components of $\mathbf{c}$.

4 Comments on the new method

It is known that for the Laplace equation in the two-dimensional domain the set

$$\left\{ 1, r^{\mp k} \cos k\theta, r^{\mp k} \sin k\theta, \ k = 1, 2, \ldots \right\} \quad (32)$$

forms the T-complete functions, and the solution can be expanded by these bases [Kita and Kamiya (1995); Li, Lu, Huang and Cheng (2007)]:

$$u(r, \theta) = a_0 + \sum_{k=1}^{\infty} [a_k r^{\mp k} \cos k\theta + b_k r^{\mp k} \sin k\theta]. \quad (33)$$

It is simply a direct consequence of Eq. (7) by inserting $R_0 = 1$. For exterior problem one takes the minus sign, while for interior problem one takes the positive sign.

Our starting point in Eq. (7) is quite similar to the Trefftz method, which is designed to satisfy the governing equation and leaves the unknown coefficients determined by satisfying the boundary conditions in some manners as by means of the collocation, the least square or the Galerkin method [Kita and Kamiya (1995)].

The present modification is suggested to use a new set of T-complete bases by

$$\left\{ 1, \left(\frac{R_0}{r}\right)^{\pm k} \cos k\theta, \left(\frac{R_0}{r}\right)^{\pm k} \sin k\theta, \ k = 1, 2, \ldots \right\}. \quad (34)$$

As seen in Section 2, the above set is a very natural result from the concept of the artificial circle with a radius R_0, where an exact boundary condition can be established by solving Eq. (16). This factor of R_0 indeed plays a major role to stabilize the conventional Trefftz method.

Liu (2007e) has employed the same idea to modify the direct Trefftz method for the two-dimensional potential problem, and Liu (2007f) used this idea to develop a highly accurate numerical method to calculate the Laplace equation in the doubly-connected domain.

For the Trefftz method the numerical instability is an inherent property, which uses the power functions r^k in the bases for interior problem and $(1/r)^k$ in the bases for exterior problem. It is a main reason to cause the numerical instability, because r may be greater than 1 for interior problem and r may be smaller than 1 for exterior problem. When the interior problem domain has a larger size with its largest distance of the boundary points to the origin being greater than 1, the

powers r^k are divergent. Similarly, when the exterior problem domain with its complement has a smaller size with its smallest distance of the boundary points to the origin being smaller than 1, the powers $(1/r)^k$ are divergent. They are thus inevitably led to numerical instability.

But in our modification the situation is drastically different. For the interior problem the power functions $(r/R_0)^k$ in Eq. (34) are always smaller than 1 because of definition (5). Similarly, for the exterior problem the power functions $(R_0/r)^k$ in Eq. (34) are always smaller than 1 because of definition (6).

It will be clear in the next section that the use of characteristic length ensures the stability of the modified Trefftz method. Through this new modification the condition number of the linear equations system can be greatly reduced. To our best knowledge, the new concept does not appear in the literatue of the Trefftz method; see, e.g., Kita and Kamiya (1995) and Li, Lu, Huang and Cheng (2007).

5 Numerical examples

Before embarking numerical study of the new method, we are concerned with its stability in the case when the boundary data are contaminated by random noise, which is investigated by adding the different levels of random noise on the boundary data. We use the function RANDOM_NUMBER given in Fortran to generate the noisy data $R(i)$, where $R(i)$ are random numbers in $[-1,1]$. Hence we can use the simulated noisy data as the input on our calculations,

$$\hat{h}(\theta_i) = h(\theta_i) + sR(i), \qquad (35)$$

where $\theta_i = 2i\pi/m$, $i = 0, 1, \ldots, m$, and s is the level of additive noise on the data.

5.1 Example 1 (interior problem)

In this example we consider an epitrochoid boundary shape

$$\rho(\theta) = \sqrt{(a+b)^2 + 1 - 2(a+b)\cos(a\theta/b)}, \qquad (36)$$

$$x(\theta) = \rho\cos\theta, \quad y(\theta) = \rho\sin\theta \qquad (37)$$

with $a = 4$ and $b = 1$; see the inset in Fig. 1. The analytical solution is supposed to be

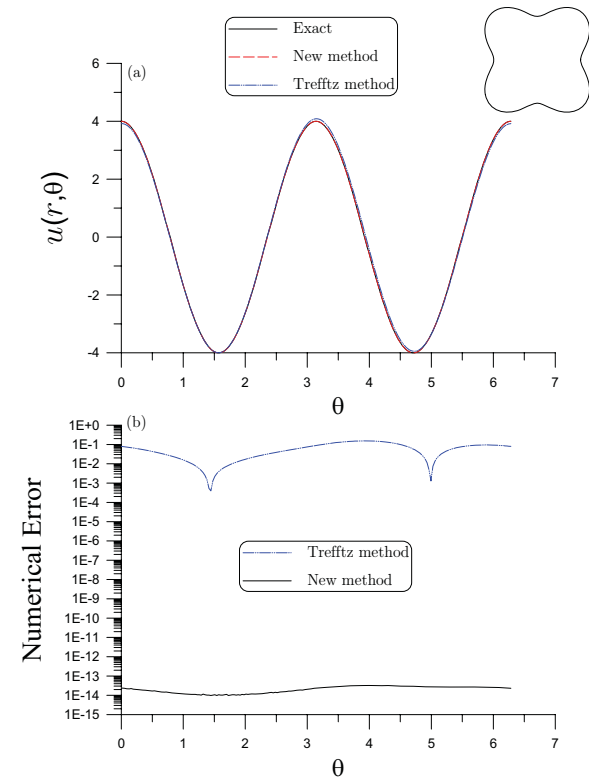$$u(x,y) = x^2 - y^2. \qquad (38)$$

Figure 1: (a) comparing the exact solution and numerical solutions, and (b) plotting the numerical errors.

The exact boundary data can be easily derived by inserting Eqs. (36) and (37) into the above equation and Eq. (3) with $\Gamma_1 = \{(r,\theta) \mid r = \rho, 0 \le \theta < \pi\}$ and $\Gamma_2 = \{(r,\theta) \mid r = \rho, \pi \le \theta < 2\pi\}$.

In the numerical computation we have fixed $R_0 = \rho_{max} = 6$ and $m = 12$. In Fig. 1(a) we compare the numerical solution with exact solution along a circle with a radius $r = 2$. It can be seen that the numerical result are almost coincident with the exact one. The accurcay of the numerical solution is found to be very good with the absolute error plotted in Fig. 1(b) in the order of 10^{-13}.

In Section 4 we have mentioned the similarity between the present method and the Trefftz method. However, when we apply the Trefftz method on this problem by using $R_0 = 1$ and $m \geq 13$, we find that the solution is unstable. Therefore we adjust the m to $m = 12$ such that the Trefftz method is applicable, whose solution is shown in Fig. 1(a) by the dashed-dotted line.

As one has been convinced by Li, Lu, Huang and Cheng (2007), that the CTM is the simplest, accurate and stable method, one may surprise that the Trefftz method is very inaccurate; as can be seen the numerical solution shown in Fig. 1(a) for the CTM is far from an accurate solution, where one may expect that the numerical solution for the direct problem is always accurate. What happens for the CTM.

The inaccuracy of CTM may result from its numerical instability. In order to observe this phenomenon we plot the condition number of $\mathbf{A}$ with different number of bases in Fig. 2, which is defined by

$$\text{Cond}(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|. \tag{39}$$

The norm used for $\mathbf{A}$ is the Frobenius norm. Therefore, we have

$$\frac{1}{n}\text{Cond}(\mathbf{A}) \leq \frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})} \leq \text{Cond}(\mathbf{A}). \tag{40}$$

where λ is the eigenvalue of $\mathbf{A}$. Conventionally, $\lambda_{\max}(\mathbf{A})/\lambda_{\min}(\mathbf{A})$ is used to define the condition number of $\mathbf{A}$. For the present study we use Eq. (39) to define the condition number of $\mathbf{A}$.

As mentioned by Kita and Kamiya (1995) when one uses the Trefftz boundary type method, the condition number may increase fast as the numer of elements increases. It can be seen that the present method can greatly reduce the condition number about thirty orders under $m = 20$ as shown in Fig. 2. No matter which m is selected, the condition number of the presently modified Trefftz method is always much smaller than that of the original Trefftz method. Therefore, when the new method is very accurate, the Trefftz method leads to a bad numerical result as shown in Fig. 1(a), whose error as shown in Fig. 1(b) may be large up to 0.1.
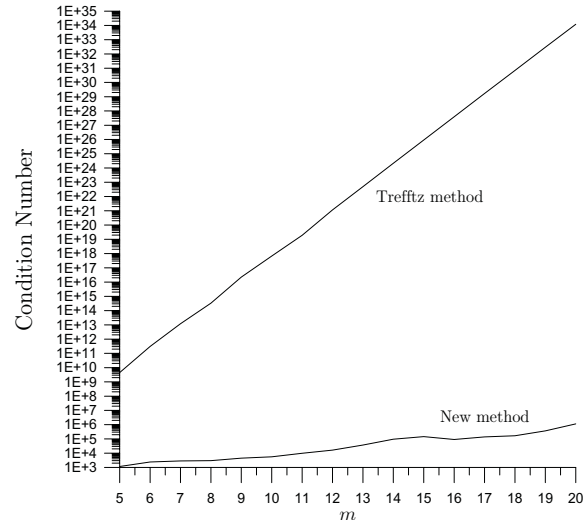

Figure 2: Comparing the condition numbers with respect to m for the Trefftz method and new method for Example 1.

5.2 Example 2 (Motz problem)

Among the many singular problems, the Motz problem was first studied by Motz (1946) for the relaxation method. Since there is a strong singularity $O(r^{\frac{1}{2}})$ at the original point, the Motz problem has become a benchmark of singularity problems [Li (1998)], which solves the Laplace equation in a rectangle with width 2 on the horizontal sides and with height 1 on the vertical sides. The boundary conditions are given by

$$
\begin{align}
u(1, y) &= 500, & 0 \leq y \leq 1, \tag{41} \\
u_y(x, 1) &= 0, & -1 \leq x \leq 1, \tag{42} \\
u_x(-1, y) &= 0, & 0 \leq y \leq 1, \tag{43} \\
u(x, 0) &= 0, & -1 \leq x < 0, \tag{44} \\
u_y(x, 0) &= 0, & 0 < x \leq 1. \tag{45}
\end{align}
$$

To solve this problem, the most researchers [see, e.g., Lu, Hu and Li (2004) and references therein] use the following approximation:

$$u(r, \theta) = \sum_{k=1}^{N} D_k r^{k-1/2} \cos\left(k - \frac{1}{2}\right)\theta, \tag{46}$$

which satisfies the Laplace equation and boundary conditions (44) and (45) automatically. Then, the coefficients D_k are determined by other boundary conditions.

An extremely accurate result is obtained by Li, Mathon and Sermer (1987), of which the maximal absolute error of boundary value on $x = 1$ is 5.47×10^{-9}. The least-square method used by Li, Mathon and Sermer (1987) led to an ill-conditioned linear equations system with the condition number 3.97×10^7. In order to decrease the condition number, Li, Mathon and Sermer (1987) have considered different subdivisions and used different numbers of particular solutions. However, even when the best combination is used, the accuracy is in the order of 10^{-6}.

As suggested in Section 3, we can greatly reduce the condition number by considering the following approximation:

$$u(r,\theta) = \sum_{k=1}^{N} C_k \left(\frac{r}{R_0} \right)^{k-1/2} \cos\left(k - \frac{1}{2} \right)\theta, \quad (47)$$

where R_0 can be used as an extra parameter, which is adjustable to provide the best result. However, in order to meet Eq. (5) for the interior problem, we need $R_0 \geq \sqrt{2}$.

To find the unknown coefficients C_k, we can use the following information to construct the linear equations by the collocation method as that done in Section 3:

$$\rho(\theta) = \begin{cases} \frac{1}{\cos\theta} & 0 \leq \theta < \pi/4, \\ \frac{1}{\sin\theta} & \pi/4 \leq \theta < 3\pi/4, \\ \frac{-1}{\cos\theta} & 3\pi/4 \leq \theta < \pi, \end{cases} \quad (48)$$

$$\rho'(\theta) = \begin{cases} \frac{\sin\theta}{\cos^2\theta} & 0 \leq \theta < \pi/4, \\ \frac{-\cos\theta}{\sin^2\theta} & \pi/4 \leq \theta < 3\pi/4, \\ \frac{-\sin\theta}{\cos^2\theta} & 3\pi/4 \leq \theta < \pi, \end{cases} \quad (49)$$

$$(\beta_D, \beta_N) = \begin{cases} (1,0) & 0 \leq \theta < \pi/4, \\ (0,1) & \pi/4 \leq \theta < \pi. \end{cases} \quad (50)$$

In the numerical calculation of this example we have employed $N = 60$ and $R_0 = 1.71$, and the numerical error of the boundary data on $x = 1$ is shown in Fig. 3. Here we are imposed $N/2$ collocated points on $x = 1$ and $N/2$ collocated points on other sides. It can be seen that the maximum error 1.84×10^{-9} is better than that calculated by Li, Mathon and Sermer (1987). Then, in Table 1 we compare our numerical results at different in-

ner points (x_i, y_i) in the rectangle with the numerical results calculated by Li, Mathon and Sermer (1987) and Dosiyev and Cival (2004). The three methods provided almost the same results up to the fifth decimial point.
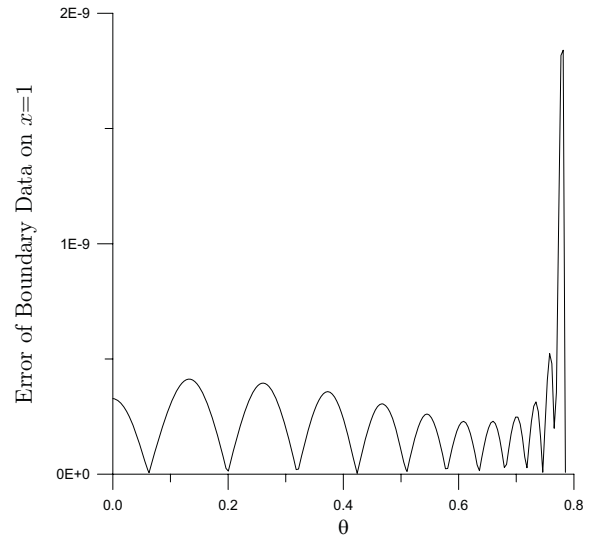

Figure 3: For Example 2 of the Motz problem the new method leads to a highly accurate boundary data at $x = 1$.

As pointed out by Liu, Chen and Chang (2007), the u_y may have a large variation from a very large value to zero, when (x,y) passes the singular point $(0,0)$. This fact makes the Motz problem not easy to handle by the conventional numerical methods. In our approach by applying the collocation method on Eq. (47), we can greatly reduce the condition number by selecting a suitable R_0. For example, in Fig. 4 we fix $N = 30$ and let R_0 vary from 1 to 2. The condition number is plotted with respect to R_0, of which we can see that there exists a best R_0 making the condition number smallest. However, if one uses Eq. (46) as a numerical solution of the Motz problem, the condition number may be large up to 10^{11} as reading from Fig. 4 with $R_0 = 1$.

Table 1: The comparison of present method with other methods for the Motz problem

| (x_i, y_i)	Dosiyev and Cival (2004)	Li, Mathon and Sermer (1987)	Present method
$(\frac{-2}{7}, \frac{2}{7})$	78.559447392	78.559230394	78.5591752835
$(0, \frac{2}{7})$	141.559500712	141.559519337	141.559480268
$(\frac{2}{7}, \frac{2}{7})$	243.811877860	243.811824635	243.811791504
$(0, \frac{1}{7})$	103.768314085	103.768301476	103.768277794
$(\frac{-1}{28}, \frac{1}{28})$	33.591514936	33.591507894	33.591499663
$(0, \frac{1}{28})$	53.186268406	53.186260997	53.186250870
$(\frac{1}{28}, \frac{1}{28})$	83.671280478	83.671271165	83.671255073
$(\frac{1}{28}, 0)$	76.408294083	76.408285701	76.408273111
$(\frac{3}{28}, 0)$	134.446993973	134.446980917	134.446959522
$(\frac{1}{7}, 0)$	156.482452033	156.482436984	156.482416783
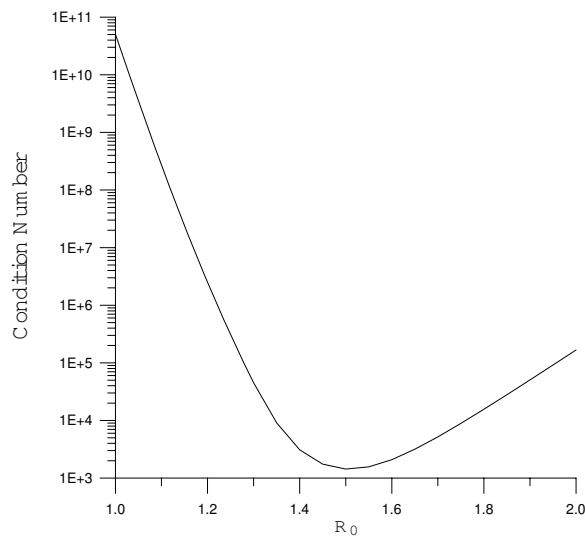
Figure 4: For Example 2 of the Motz problem we plotting the condition number of the new method with respect to R_0.

5.3 Example 3 (exterior problem)

In this example we investigate a mixed-boundary condition on a circle with a radius equal to a:

$$h(\theta) = \begin{cases} u(a,\theta) \\ u_r(a,\theta) \end{cases}$$

$$= \begin{cases} \frac{2}{\pi} \arctan\left(\frac{2a\sin\theta}{a^2-1}\right) & 0 \le \theta < \pi, \\ \frac{-4}{\pi} \frac{(a^2+1)\sin\theta}{(a^2-1)^2+4a^2\sin^2\theta} & \pi \le \theta < 2\pi. \end{cases} \quad (51)$$

For this example an exact solution is given by

$$u(r,\theta) = \frac{2}{\pi} \arctan\left(\frac{2r\sin\theta}{r^2-1}\right). \quad (52)$$

We apply the new method on this example by fixing $a = 2$, $R_0 = 2$ and $m = 25$. In Fig. 5(a) we compare the exact solution with the numerical solutions with $s = 0, 0.01$ along a circle with radius 3. It can be seen that the numerical solutions are very close to the exact solution. Furthermore, the numerical errors were plotted in Fig. 5(b), of which it can be seen that the present method is very robust to against the noise, whose level was taken up to 1%, but the numerical error is still smaller than 0.01.

5.4 Example 4 (exterior problem)

In this example we consider a complex amoeba-like irregular shape as shown in the inset of Fig. 6. The boundary conditions are given by

$$u(r,\theta) = \exp\left(\frac{\cos\theta}{r}\right) \cos\left(\frac{\sin\theta}{r}\right), \quad (53)$$

$$u = u(\rho,\theta), \quad (r,\theta) \in \Gamma_1, \quad (54)$$

$$u_n = u_\rho(\rho,\theta) - \frac{\rho'}{\rho^2} u_\theta(\rho,\theta), \quad (r,\theta) \in \Gamma_2, \quad (55)$$

$$\rho = \exp(\sin\theta)\sin^2(2\theta) + \exp(\cos\theta)\cos^2(2\theta), \quad (56)$$

where $\Gamma_1 = \{(r,\theta) \mid r = \rho, 0 \le \theta < \pi\}$ and $\Gamma_2 = \{(r,\theta) \mid r = \rho, \pi \le \theta < 2\pi\}$.

We apply the new method on this example by fixing $R_0 = \rho_{min}$ and $m = 15$. In Fig. 6(a) we compare the exact solution with numerical solutions along a circle with radius 3. It can be seen that the numerical solution is almost coincident with the exact solution, of which the absolute error as
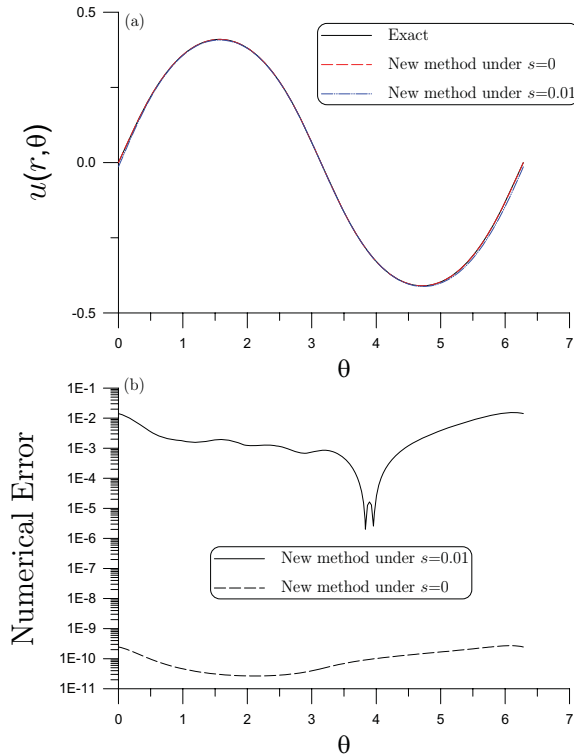
Figure 5: For Example 3: (a) comparing the exact solution and numerical solutions calculated by the new method under $s = 0$ and 0.01, and (b) plotting the numerical errors.
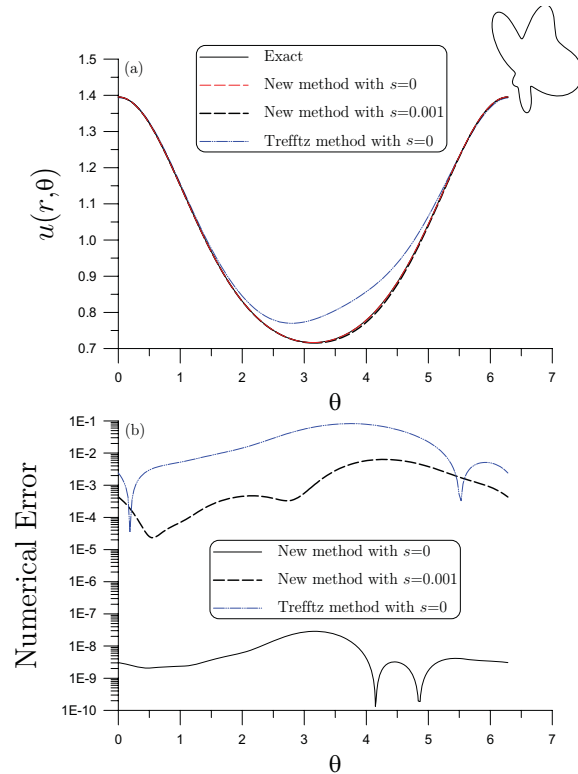

Figure 6: For Example 4: (a) comparing the exact solution and numerical solutions calculated by the new method under $s = 0$ and 0.001 and the Trefftz method under $s = 0$, and (b) plotting the numerical errors.

plotted in Fig. 6(b) is in the order of 10^{-8}. Also we are imposed a random noise with $s = 0.001$, of which the new method as can be seen is robust to against the disturbance on the boundary data. When we apply the Trefftz method to this example by fixing $R_0 = 1$ and $m = 15$ and without considering noise, its solution as shown in Fig. 6(a) by the dashed-dotted line is different much from the exact solution. The numerical error as shown in Fig. 6(a) is in the order of 0.1. When the noise is added, the Trefftz method produces a fully incorrect result due to its essential instability.

6   Conclusions

In this paper we have proposed a new collocation Trefftz method to calculate the solutions of Laplace equation in arbitrary plane domains under mixed-boundary condition. In order to tackle of the ill-conditioning of the Trefftz method, we

have employed a characteristic length factor into the basis functions. This type formulation is a very natural result in terms of the concept of artificial circle. The numerical examples show that the effectiveness of the new method and the accuracy is very good. Even under a large noise polluting on the boundary data, the numerical solutions are also good and stable without needing of extra treatment. The new method possesses several advantages than the conventional boundary-type solution methods, which including mesh-free, singularity-free, non-illposedness, semi-analyticity, efficiency, accuracy and stability. When the same idea is applied to solve the Motz problem, we have obtained a highly accurate numerical solution, of which the nonzero boundary value part can be matched very well with three times accuracy than before.

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