A STUDY ON LAPLACE PROBLEMS OF INFINITE PLANE WITH MULTIPLE CIRCULAR HOLES

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Abstract This paper describes a numerical procedure for solving the Laplace problems of infinite plane with multiple circular holes by using the null-field integral equation, Fourier series and degenerate kernels. The unknown boundary potential and flux are approximated by using the truncated Fourier series. Degenerate kernels are utilized in the null-field integral equation. A linear algebraic system is obtained without boundary discretization. The present method is verified through one example with the exact solution derived by Lebedev et al.

Keywords: infinite plane, laplace problem, null-field integral equation, degenerate kernel, fourier series.

1. INTRODUCTION

A number of problems in engineering involving infinite and half-plane domains, e.g., the soil-structure interaction, tunnel and concrete pipe design, have been studied by using finite element method (FEM), boundary element method (BEM), method of fundamental solution (MFS) and the Trefftz method. As we know, FEM is a very efficient method in solving finite-domain problems, but it is not convenient to deal with infinite-domain problems. In this aspect, BEM is an efficient alternative which has been extensively used for solving infinite and half-plane problems[1–3].

To tackle the exterior problems containing circular holes by using FEM, special treatment should be addressed for truncating the unbounded domain.

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Due to this reason, BEM is a more efficient method to solve an infinite or half-plane problem[4–6].

In this paper, the boundary integral equation method (BIEM) is utilized to solve infinite-plane problems with multiple circular holes. The unknown boundary potential and flux are approximated by using the truncated Fourier series[7,8]. The Fourier coefficients can be determined by substituting the degenerate kernels in the null-field integral equation. Numerical results of the Dirichlet problem is given to illustrate the validity of the present approach. The accuracy and efficiency for the present method are also examined.

2. PROBLEM STATEMENT AND INTEGRAL FORMULATION

Consider an infinite plane containing N randomly distributed circular holes centered at position vector $\underset{\sim}{c}_j$ $(j = 1, 2, \ldots, N)$ as shown in Figure 1. Let a_j and B_j denote the radius and boundary of the *ith* hole. We use the truncated Fourier series expansions to approximate the potential u and its normal flux t on the boundary

$$u_j(\underset{\sim}{s}) = f(\theta_j) = a_{0j} + \sum_{n=1}^{M}(a_{nj}\cos n\theta_j + b_{nj}\sin n\theta_j), \underset{\sim}{s} \in B_j, \quad (1)$$

$$t_j(\underset{\sim}{s}) = g(\theta_j) = p_{0j} + \sum_{n=1}^{M}(p_{nj}\cos n\theta_j + q_{nj}\sin n\theta_j), \underset{\sim}{s} \in B_j, \quad (2)$$

where the coefficients a_{0j}, a_{nj} and b_{nj} are specified once $f(\theta_j)$ is given, p_{0j}, p_{nj} and q_{nj} are the undetermined coefficients for the Dirichlet problem, θ_j is

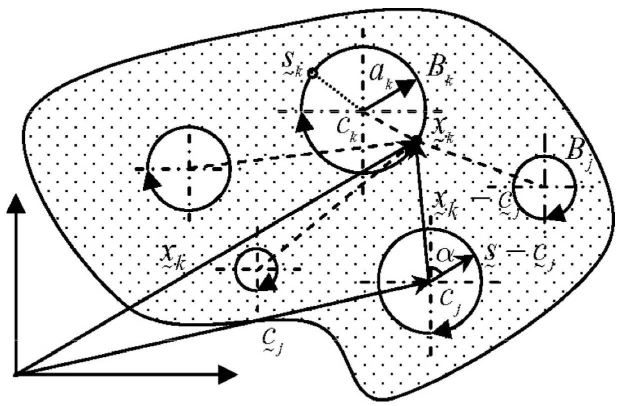


Figure 1. Boundary integral in the null-field BIE.

the polar angle centered at c_j. Based on the boundary integral formulation of the domain point for potential problems [9], we have

$$2\pi u(\underset{\sim}{x}) = \int_B T(\underset{\sim}{s}, \underset{\sim}{x}) u(\underset{\sim}{s}) dB(\underset{\sim}{s}) - \int_B U(\underset{\sim}{s}, \underset{\sim}{x}) t(\underset{\sim}{s}) dB(\underset{\sim}{s}), \underset{\sim}{x} \in D, \tag{3}$$

where $\underset{\sim}{s}$ and $\underset{\sim}{x}$ are the source and field points, respectively, D is the domain of interest, B is the boundary and $U(\underset{\sim}{s}, \underset{\sim}{x}) = \ln r$ is the fundamental solution which satisfies

$$\nabla^2 U(\underset{\sim}{x}, \underset{\sim}{s}) = \delta(\underset{\sim}{x} - \underset{\sim}{s}), \tag{4}$$

in which, $\delta(\underset{\sim}{x} - \underset{\sim}{s})$ denotes the Dirac-delta function. $T(\underset{\sim}{s}, \underset{\sim}{x})$ is defined by

$$T(\underset{\sim}{s}, \underset{\sim}{x}) = \frac{\partial U(\underset{\sim}{s}, \underset{\sim}{x})}{\partial n_{\underset{\sim}{s}}} \tag{5}$$

where $n_{\underset{\sim}{s}}$ denotes the outward normal vector at the source point s. By collocating $\underset{\sim}{x}$ outside the domain ($\underset{\sim}{x} \in D^e$), we obtain the null-field integral equation as shown below

$$0 = \int_B T(\underset{\sim}{s}, \underset{\sim}{x}) u(\underset{\sim}{s}) dB(\underset{\sim}{s}) - \int_B U(\underset{\sim}{s}, \underset{\sim}{x}) t(\underset{\sim}{s}) dB(\underset{\sim}{s}), \underset{\sim}{x} \in D^{\mathrm{e}}, \tag{6}$$

Based on the separable property, the U kernel function can be expanded into degenerate form as shown below

$$U(\underset{\sim}{s}, \underset{\sim}{x}) = \begin{cases} U^{\mathrm{i}} = \ln|\underset{\sim}{s} - \underset{\sim}{c}_j| - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{|\underset{\sim}{x} - \underset{\sim}{c}_j|}{|\underset{\sim}{s} - \underset{\sim}{c}_j|} \right)^m \cos m\alpha, \ |\underset{\sim}{s} - \underset{\sim}{c}_j| > |\underset{\sim}{x} - \underset{\sim}{c}_j| \\ U^{\mathrm{e}} = \ln|\underset{\sim}{x} - \underset{\sim}{c}_j| - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{|\underset{\sim}{s} - \underset{\sim}{c}_j|}{|\underset{\sim}{x} - \underset{\sim}{c}_{\mathrm{j}}|} \right)^m \cos m\alpha, \ |\underset{\sim}{x} - \underset{\sim}{c}_j| > |\underset{\sim}{s} - \underset{\sim}{c}_j| \end{cases} \tag{7}$$

where α is the angle between $\underset{\sim}{s} - \underset{\sim}{c}_j$ and $\underset{\sim}{x} - \underset{\sim}{c}_j$, the superscripts i and e denote the interior and exterior cases, respectively. After taking the normal derivative, the $T(\underset{\sim}{s}, \underset{\sim}{x})$ kernel can be derived as

$$T(\underset{\sim}{s}, \underset{\sim}{x}) = \begin{cases} T^{\mathrm{i}} = \frac{1}{|\underset{\sim}{s} - \underset{\sim}{c}_j|} + \sum_{m=1}^{\infty} \left(\frac{|\underset{\sim}{x} - \underset{\sim}{c}_j|^m}{|\underset{\sim}{s} - \underset{\sim}{c}_j|^{m+1}} \right) \cos m\alpha, \ |\underset{\sim}{s} - \underset{\sim}{c}_j| > |\underset{\sim}{x} - \underset{\sim}{c}_j| \\ T^e = -\sum_{m=1}^{\infty} \left(\frac{|\underset{\sim}{s} - \underset{\sim}{c}_j|^{m-1}}{|\underset{\sim}{x} - \underset{\sim}{c}_j|^m} \right) \cos m\alpha, \ |\underset{\sim}{x} - \underset{\sim}{c}_j| > |\underset{\sim}{s} - \underset{\sim}{c}_j| \end{cases} \tag{8}$$

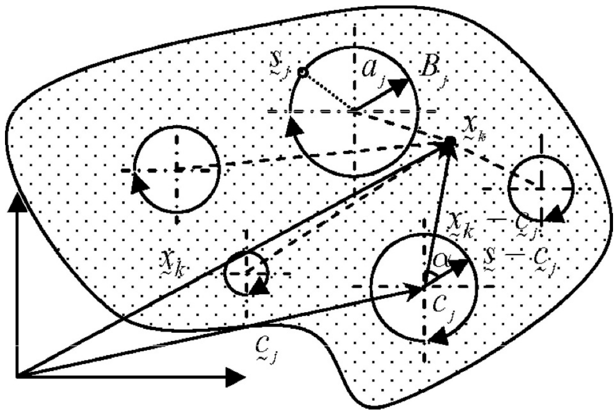


Figure 2. Boundary integral for the BIE of domain point.

By collocating the null-field point $|\underset{\sim}{x}_k - \underset{\sim}{c}_k| = a_k^-$ on the k^{th} circular boundary for Equation (6), we have

$$0 = \sum_{j=1}^{N} \int_{B_j} T(\underset{\sim}{s}, \underset{\sim}{x}_k) u(\underset{\sim}{s}) dB(\underset{\sim}{s}) - \sum_{j=1}^{N} \int_{B_j} U(\underset{\sim}{s}, \underset{\sim}{x}_{\mathrm{k}}) t(\underset{\sim}{s}) dB(\underset{\sim}{s}), \ \underset{\sim}{x} \in D^{\mathrm{e}}, \qquad (9)$$

For the B_j integral of the circular boundary, the kernels of $U(\underset{\sim}{s}, \underset{\sim}{x})$ and $T(\underset{\sim}{s}, \underset{\sim}{x})$ are respectively expressed in terms of degenerate kernels of Equations (7) and (8), and $u(\underset{\sim}{s})$ and $t(\underset{\sim}{s})$ are respectively substituted by using Fourier series of Equations (1) and (2), respectively. It is worth noting that $|\underset{\sim}{x}_k - \underset{\sim}{c}_k| = a_k^-$ must be transformed to the new coordinate of the coordinate system of origin c_j in the B_j integration as shown in Figure 1. We obtain a linear algebraic system

$$[A]\{x\} = [B]\{y\}, \qquad (10)$$

where $[A]$ and $[B]$ are the influence matrices, $\{x\}$ and $\{y\}$ denote the unknown and specified vectors of Fourier coefficient, respectively. After obtaining the unknown Fourier coefficients, we also need to transform the new coordinate as shown in Figure 2 to obtain the interior potential by employing Equation (3).

3. NUMERICAL EXAMPLE

In this section, we consider an exterior Dirichlet problem containing two circular holes in an infinite plane as shown in Figure 3[10]. As shown in Figure 4,

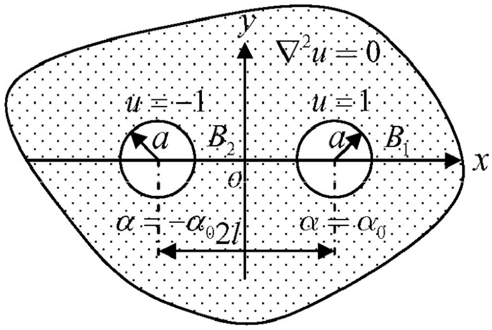
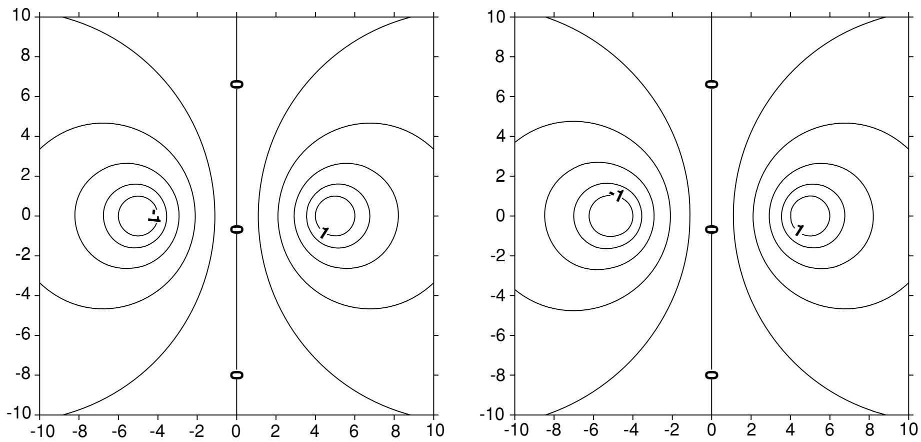


Figure 3. Problem statement (Lebedev et al., 1979).

numerical results show good agreement after comparing the exact solution by Lebedev et al.

4. CONCLUSIONS

A novel method by using degenerate kernels, null-field integral equation and Fourier series was proposed to solve infinite-plane problems with multiple circular holes. The method shows great generality and versatility for the problems of multiple circular holes with arbitrary sizes and positions in an infinite plane. The Fourier series give very accurate representations of boundary densities and numerical errors come from truncation. Numerical results agree very well with the exact solution of Lebedev et al.



(a) Exact solution (b) Present method

Figure 4. Contour of potential for the infinite plane with two circular holes.

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