Introduction of BEM/BIEM and their mathematical issues

J. T. Chen

Department of Harbor and River Engineering

National Taiwan Ocean University, Keelung, Taiwan

Aug. 14, 2006

Presentation for Summer School of Academi Sinica (Institute of Mathematics)

(AademiaSinica206talk.ppt)

哲人日已遠 典型在宿昔

(1909-1993)

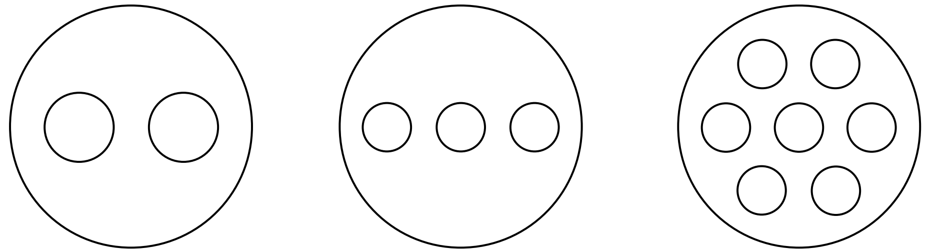
省立中興大學第一任校長

林致平校長
(民國五十年~民國五十二年)

林致平所長(中研院數學所)

林致平院士(中研院)

Torsional rigidity (Ling's problem)



Caulk (First-order Approximate) [14]	0.8739	0.8741	0.7261
Caulk (BIE formulation) [14]	0.8713	0.8732	0.7261
Ling's results	**0.8809**	**0.8093**	**0.7305**
Present method (L=10)	**0.8712**	**0.8732**	**0.7244**

Because there is no apparent reason for the unusually large difference in the second example, Ling's rather lengthy calculations are probably in error here.

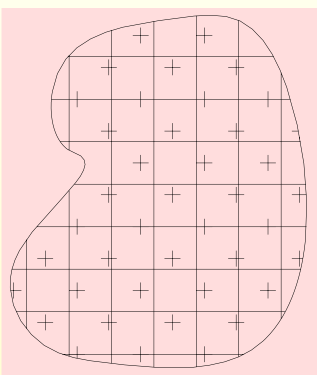
--ASME JAM

Outlines

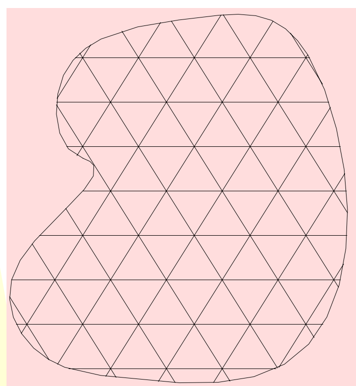
- Overview of BIE and BEM
- Mathematical tools
  - Hypersingular BIE
  - Degenerate kernel
  - Circulants
  - SVD updating term
  - SVD updating document
  - Fredholm alternative theorem
- Nonuniqueness and its treatments
  - Degenerate scale
  - Degenerate boundary
  - True and spurious eigensolution (interior prob.)
  - Fictitious frequency (exterior acoustics)
  - Corner
- Conclusions and further research

Overview of numerical methods

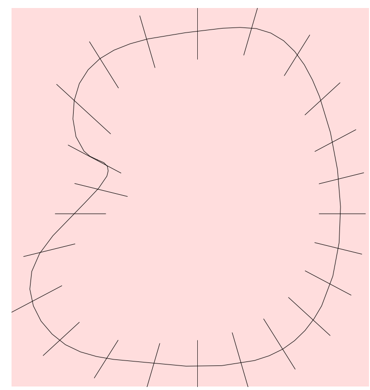
- Numerical Methods
  - Mesh Methods
    - Finite Difference Method
    - Finite Element Method
    - Boundary Element Method
  - Meshless Methods



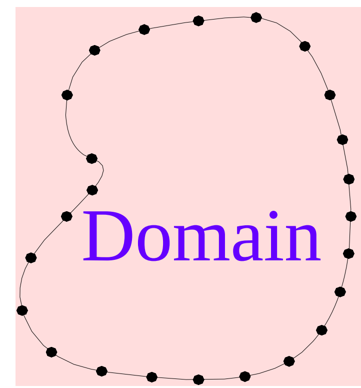
DE



PDE-variational



IE



Boundary

MFS
Trefftz method
MLS

Numeber of Papers of FEM, BEM and FDM

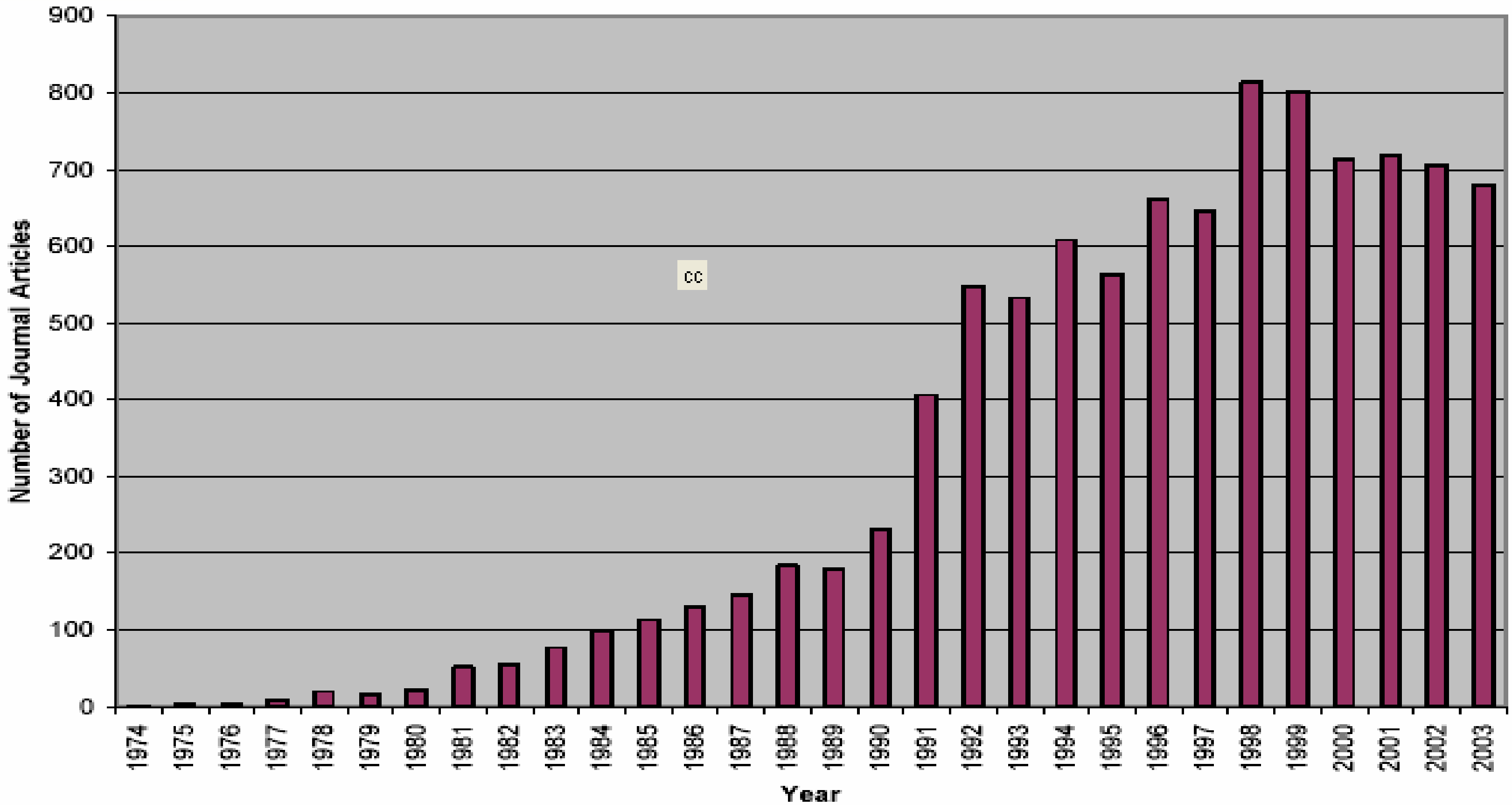
Table 1
Bibliographic database search based on the Web of Science

Numerical method	Search phrase in topic field	No. of entries	
FEM	'Finite element' or 'finite elements'	66,237	6
FDM	'Finite difference' or 'finite differences'	19,531	2
BEM	'Boundary element' or 'boundary elements' or 'boundary integral'	10,126	1
FVM	'Finite volume method' or 'finite volume methods'	1695	
CM	'Collocation method' or 'collocation methods'	1615	

Refer to Appendix A for search criteria. (Search date: May 3, 2004).

(Data form Prof. Cheng A. H. D.)

Growth of BEM/BIEM papers



(data from Prof. Cheng A.H.D.)

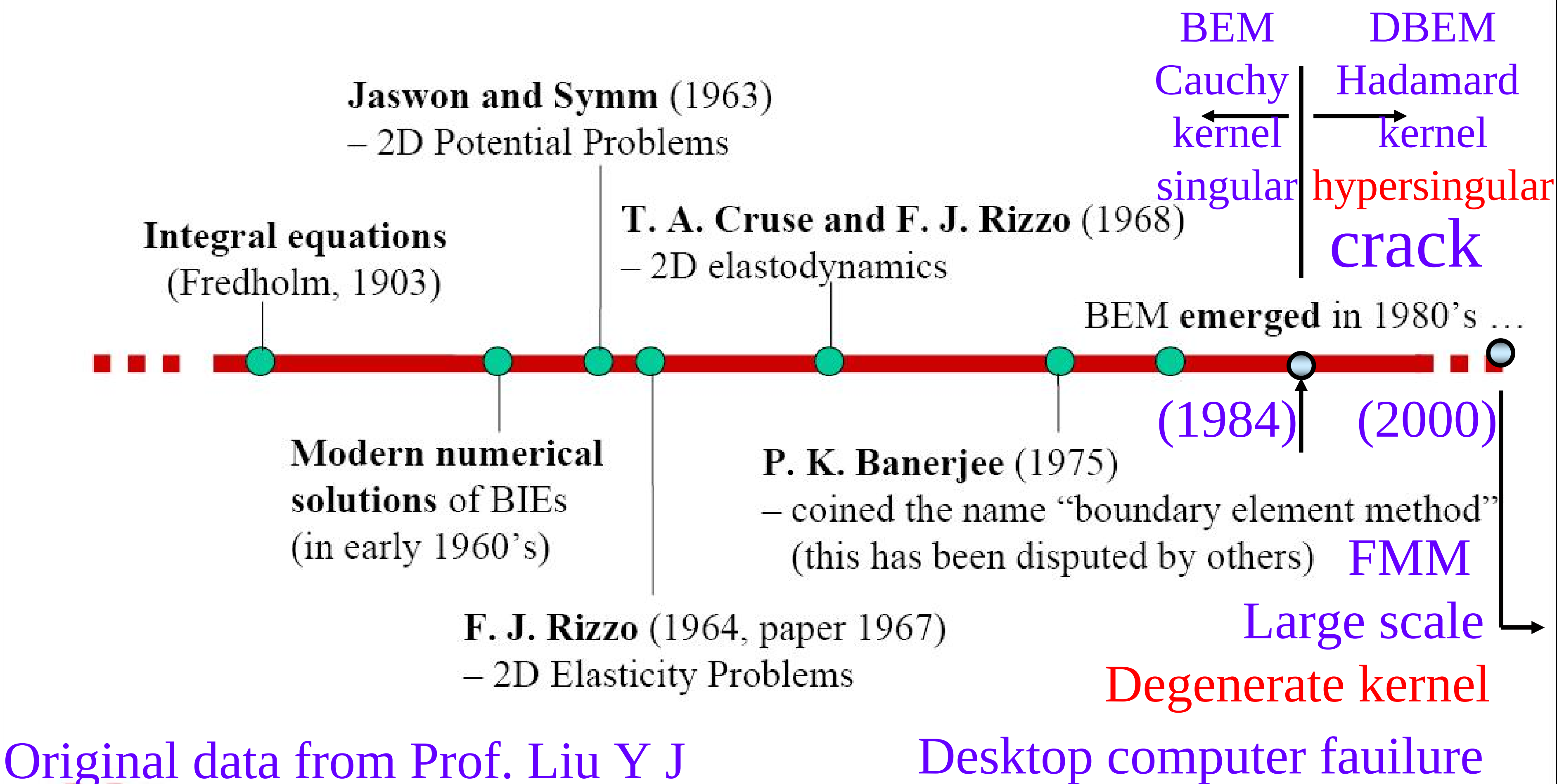
Advantages of BEM

- Discretization dimension reduction
- Infinite domain (half plane)
- Interaction problem
- Local concentration

Disadvantages of BEM

- Integral equations with singularity 北京清華
- Full matrix (nonsymmetric)

A Brief History of the BEM



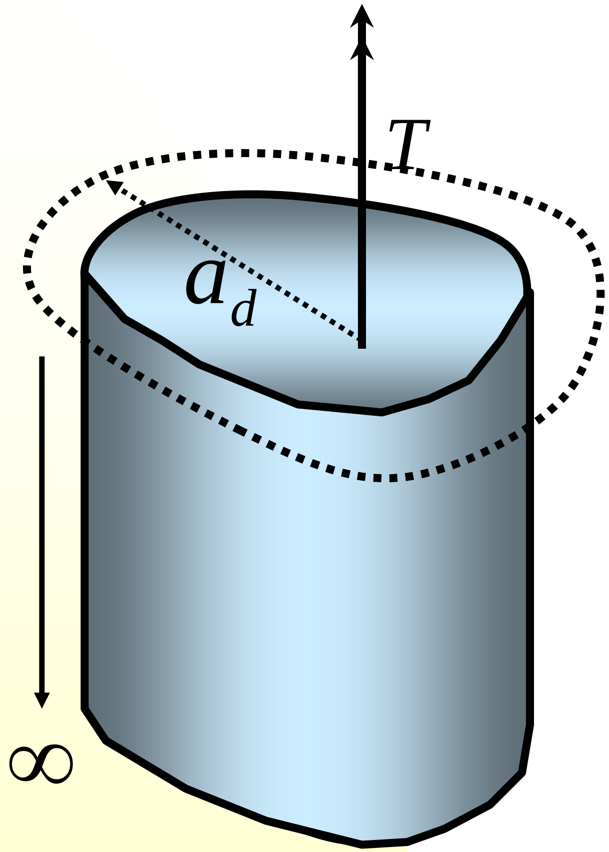
Why engineers should learn mathematics ?

- Well-posed ?
- Existence ?
- Unique ?

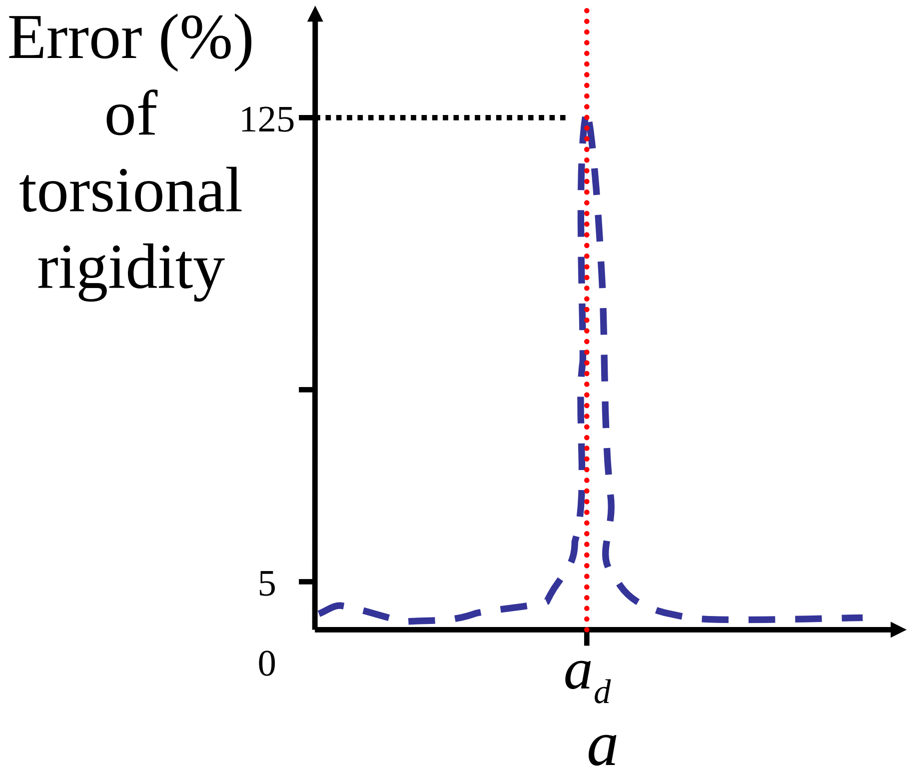
- Mathematics versus Computation

- Some examples

Numerical phenomena
(Degenerate scale)



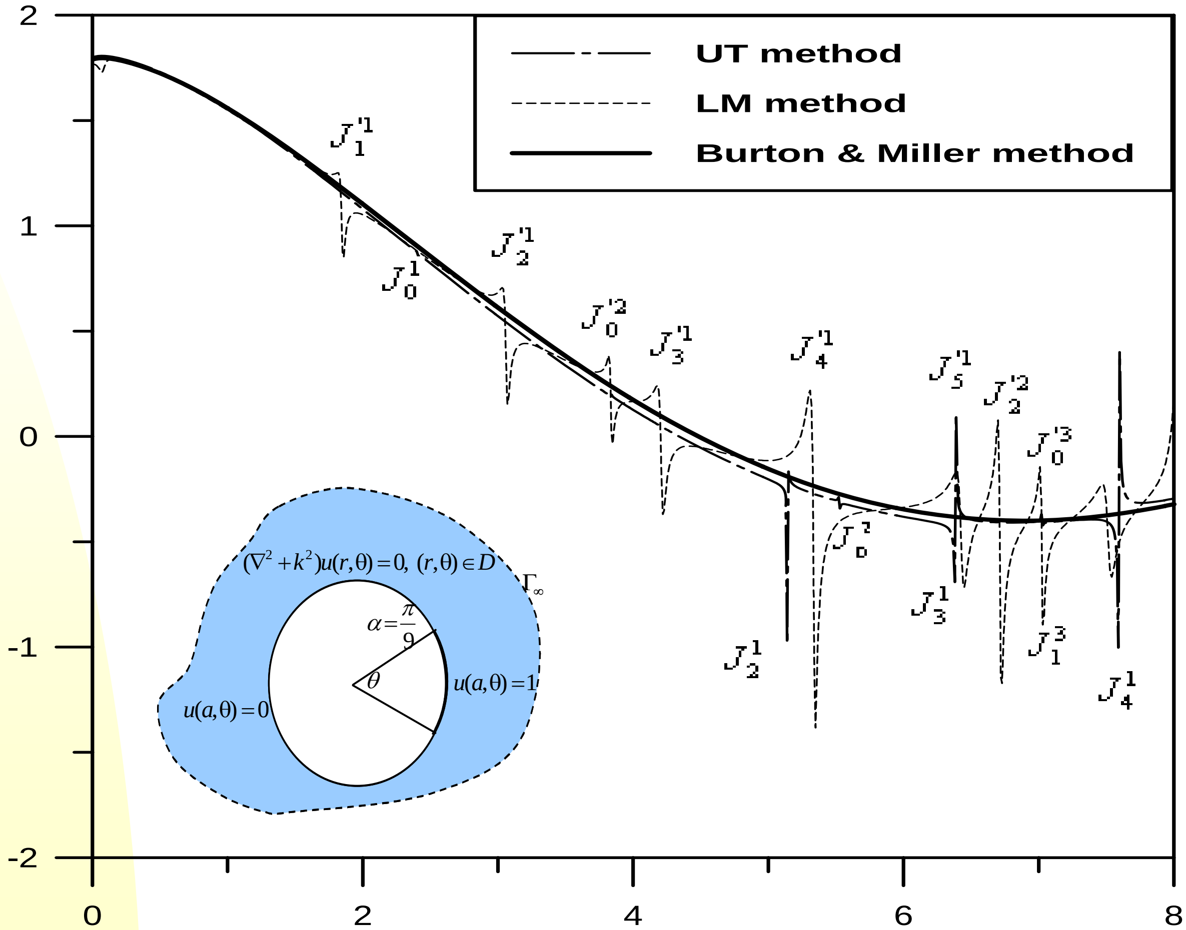
Commercial ode output ?



Previous approach : Try and error on *a*
Present approach : Only one trial

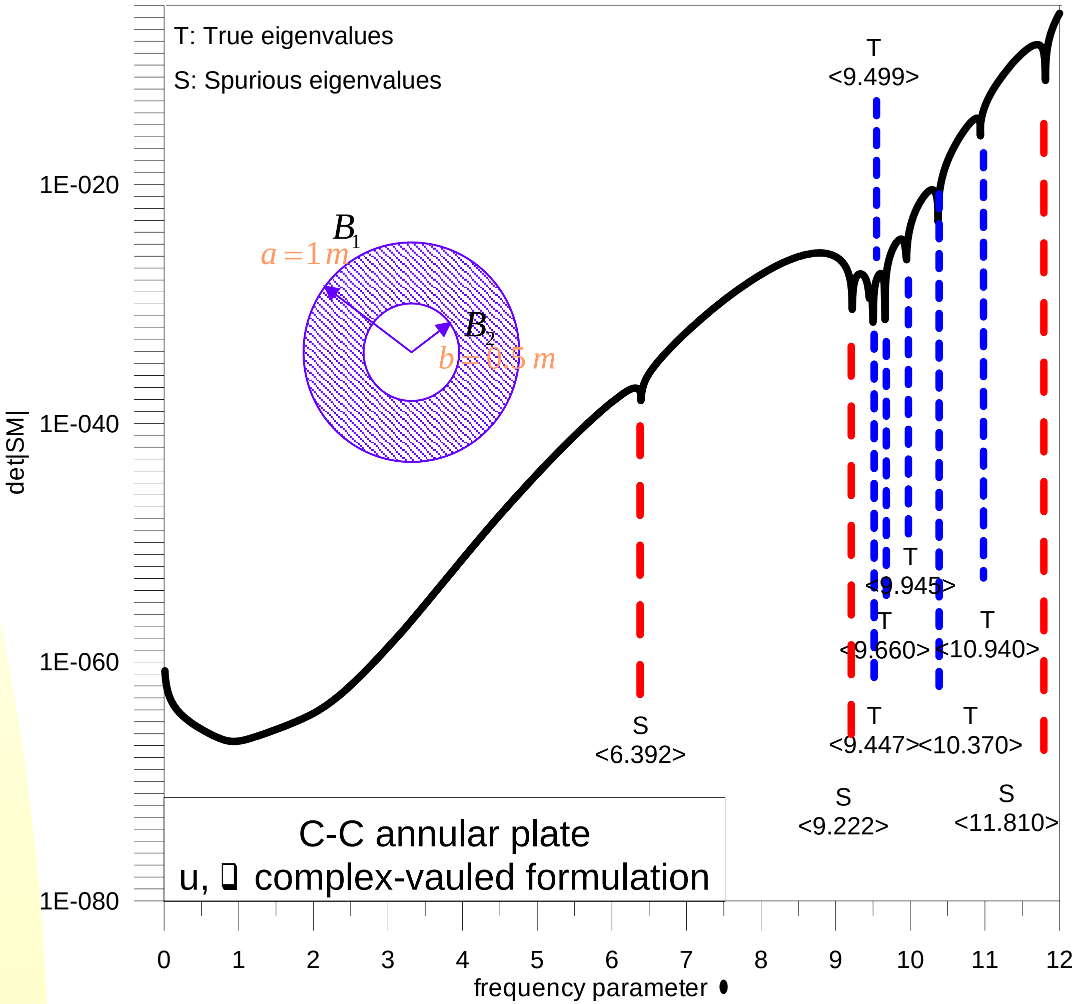
Numerical phenomena
(Fictitious frequency)

$t(a,0)$



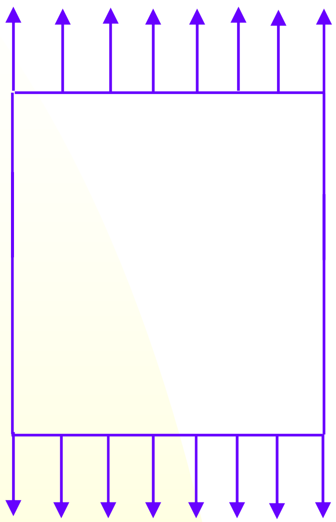
A story of NTU Ph.D. students

Numerical phenomena
(Spurious eigensolution)

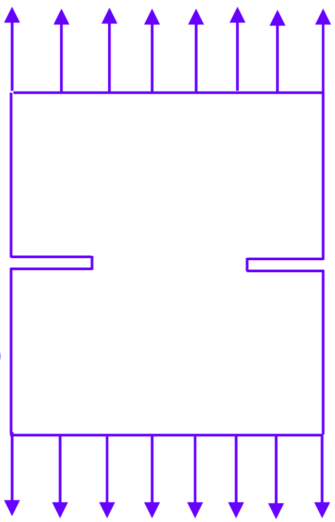


Numerical phenomena
(Degenerate boundary)

normal boundary



1967-1984-2006



degenerate boundary

Singular integral equation → Hypersingular integral equation

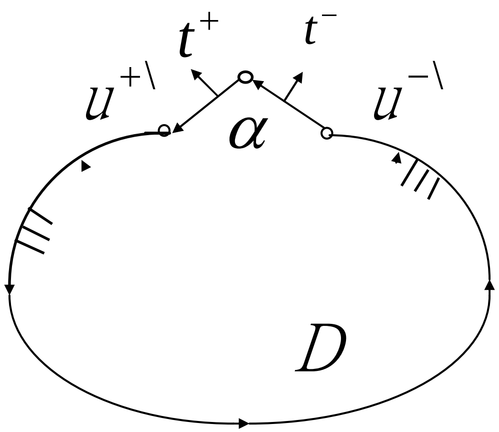
Cauchy principal value → Hadamard principal value

Boundary element method → Dual boundary element method

Numerical phenomena
(Corner)

$$\alpha u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)t(s)dB(s), \quad x \in B$$

$$\alpha t^-(x) + \sin(\alpha)t^+(x) = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)t(s)dB(s), \quad x \in B$$



Boundary

Motivation

Five pitfalls in BEM

- **Numerical instability occurs in BEM ?**

  (1) degenerate scale

  (2) degenerate boundary

  (3) fictitious frequency

  (4) corner

- **Spurious eigenvalues appear ?**

  (5) true and spurious eigenvalues

Mathematical essence—rank deficiency ?
(How to deal with ?) nonuniqueness ?

Mathematical tools

Hypersingular BIE

Degenerate kernel

Circulants

SVD updating term

SVD updating document

Fredholm alternative theorem

Mathematical tools

Hypersingular BIE

(potential theory)

Degenerate kernel

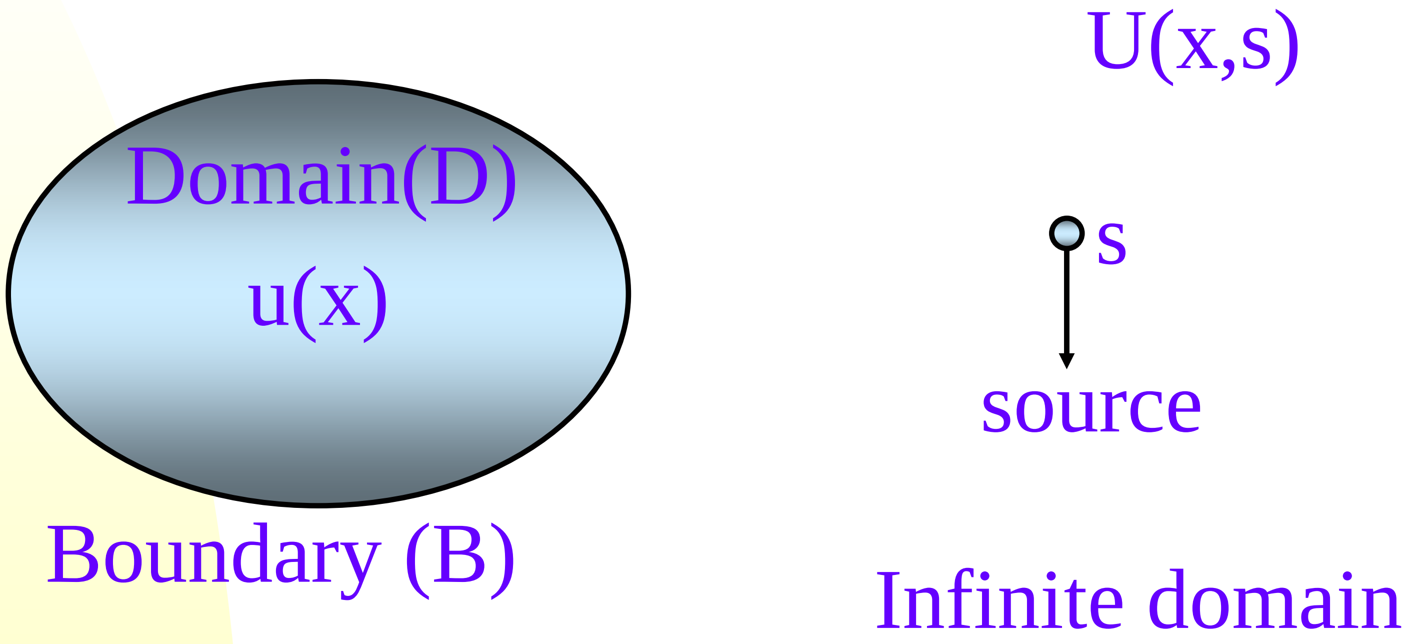
Circulants

SVD updating term

SVD updating document

Fredholm alternative theorem

Two systems u and U



Dual integral equations for a domain point (Green's third identity for two systems, u and U)

Primary field

$$2\pi \quad u(x) = \int_B T(s,x) \quad u(s) \quad dB(s) \quad - \int_B U(s,x) \quad t(s) \quad dB(s), \quad x \in D$$

Secondary field

$$2\pi \quad t(x) = \int_B M(s,x) \quad u(s) \quad dB(s) \quad - \int_B L(s,x) \quad t(s) \quad dB(s), \quad x \in D$$

where $U(s,x)=\ln(r)$ **is the fundamental solution.**

$$T(s,x) \equiv \frac{\partial U}{\partial n_s} \qquad L(s,x) \equiv \frac{\partial U}{\partial n_x} \qquad M(s,x) \equiv \frac{\partial^2 U}{\partial n_s \partial n_x}$$

$$t = \frac{\partial u}{\partial n}$$

Dual integral equations for a boundary point (x push to boundary)

Singular integral equation

$$\pi u(x) = C.P.V.\int_B T(s,x)u(s)\,dB(s) - R.P.V.\int_B U(s,x)t(s)\,dB(s), \ x \in B$$

Hypersingular integral equation

$$\pi t(x) = H.P.V.\int_B M(s,x)u(s)\,dB(s) - C.P.V.\int_B L(s,x)t(s)\,dB(s), \ x \in B$$

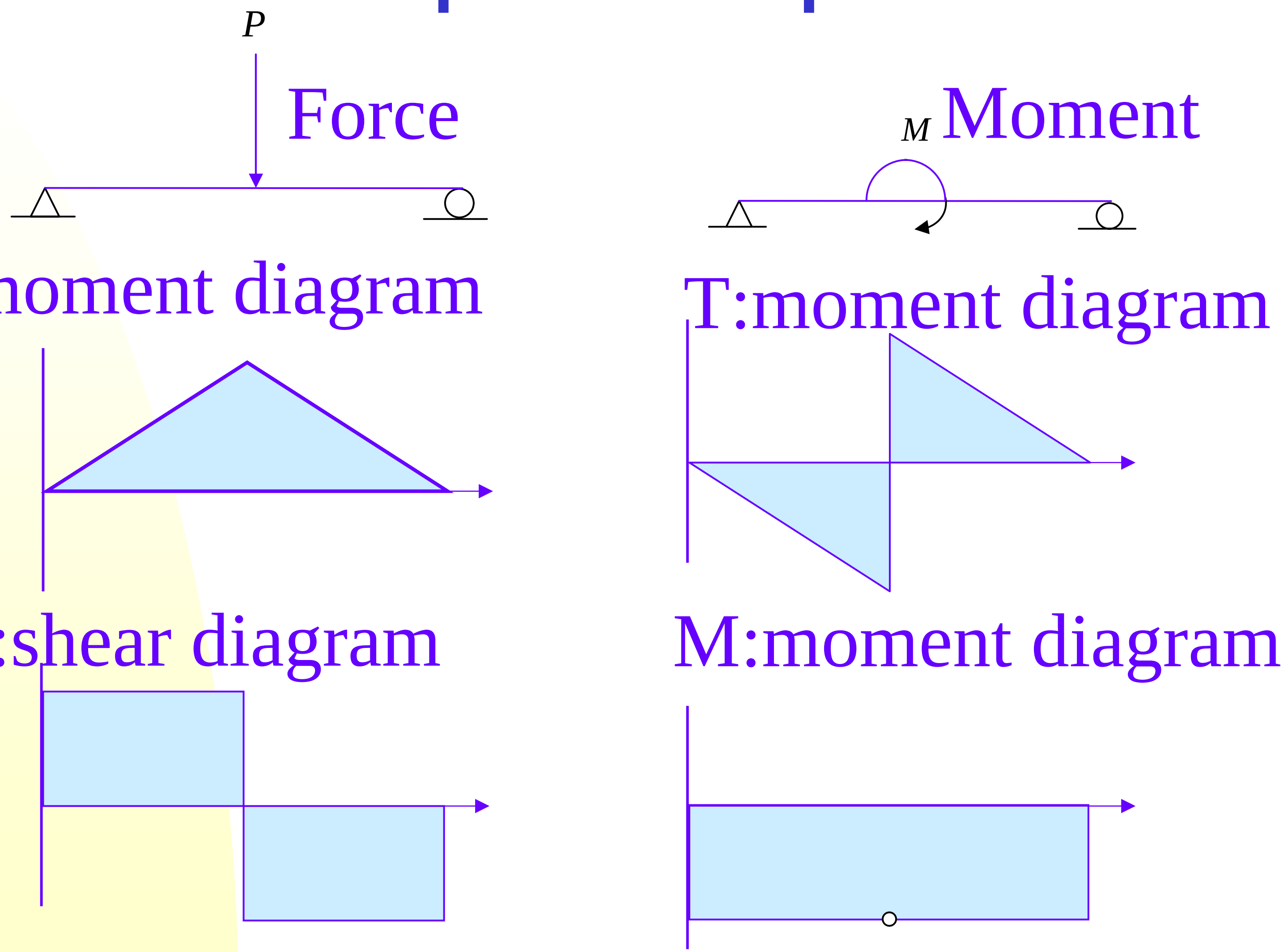
where $U(s,x)$ is the fundamental solution.

$$T(s,x) \equiv \frac{\partial U}{\partial n_s} \qquad L(s,x) \equiv \frac{\partial U}{\partial n_x} \qquad M(s,x) \equiv \frac{\partial^2 U}{\partial n_s \partial n_x}$$

Potential theory

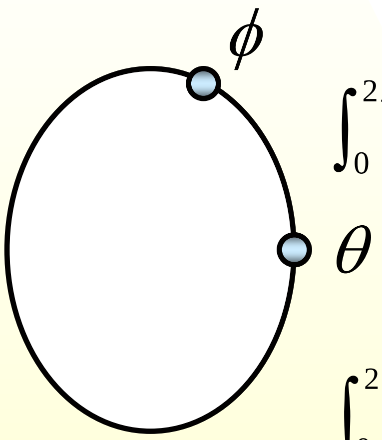
- Single layer potential (U)
- Double layer potential (T)
- Normal derivative of single layer potential (L)
- Normal derivative of double layer potential (M)

Physical examples for potentials



Order of pseudo-differential operator

- Single layer potential (U) --- (-1)
- Double layer potential (T) --- (0)



$$\int_0^{2\pi} T(\phi,\theta)u(\theta)d\theta = \pi u(\phi) + CPV\int_0^{2\pi} T(\phi,\theta)u(\theta)d\theta$$

- Normal derivative of single layer potential (L) --- (0)

$$\int_0^{2\pi} L(\phi,\theta)t(\theta)d\theta = -\pi\ t(\phi) + CPV\int_0^{2\pi} L(\phi,\theta)t(\theta)d\theta$$

- Normal derivative of double layer potential (M) --- (1)

$$\int_0^{2\pi} M(\phi,\theta)u(\theta)d\theta = \mathrm{M}(u)$$

$\mathrm{M}(\mathrm{M}(u)) = -u''$ Pseudo differential operator

$D(D(u)) = u''$ Real differential operator

Calderon projector

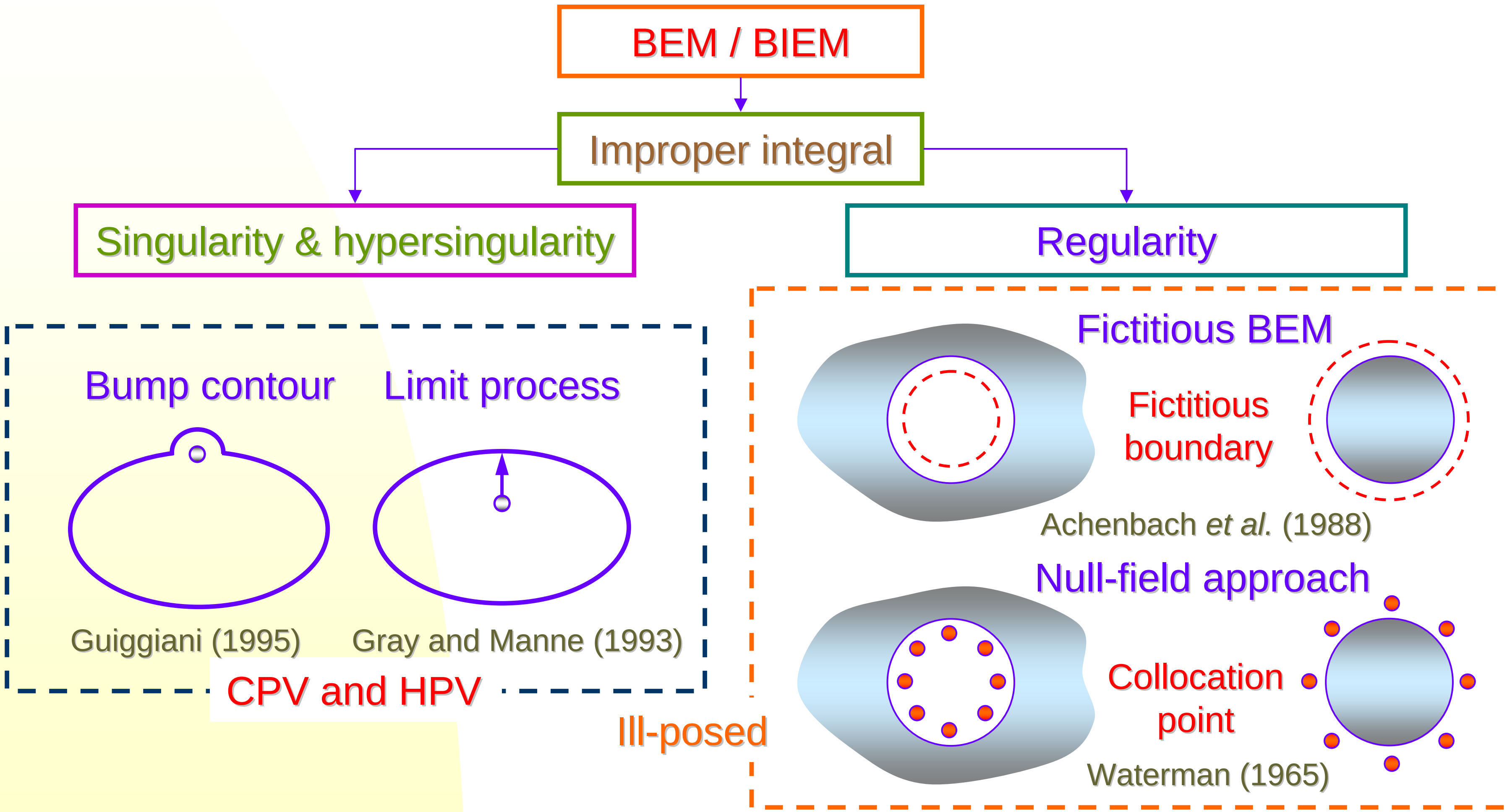
$$-\frac{1}{4}[I]+[T]^2-[U][M]=[0]$$

$$[U][L]=[T][M]$$

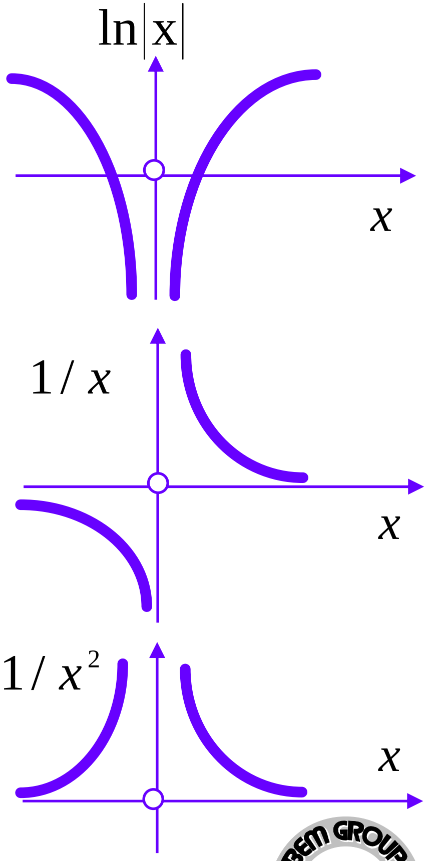
$$[M][T]=[L][M]$$

$$-\frac{1}{4}[I]+[L]^2-[M][U]=[0]$$

How engineers avoid singularity



Definitions of R.P.V., C.P.V. and H.P.V. using bump approach



- **R.P.V. (Riemann principal value)**

$$R.P.V.\int_{-1}^{1} \ln|x| \; dx = (x\ln|x| - x) \Big|_{x=-1}^{x=1} = -2$$

- **C.P.V.(Cauchy principal value)**

$$C.P.V.\int_{-1}^{1} \frac{1}{x} \; dx = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{1}{x} \; dx = 0$$

- **H.P.V.(Hadamard principal value)**

$$H.P.V.\int_{-1}^{1} \frac{1}{x^2} \; dx = \lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1} \frac{1}{x^2} \; dx - \frac{2}{\varepsilon} = -2$$

Principal value in who's sense

- Common sense
- Riemann sense
- Lebesgue sense
- Cauchy sense
- Hadamard sense (elasticity)
- Mangler sense (aerodynamics)
- Liggett and Liu's sense

  The singularity that occur when the base point and field point coincide are not integrable. (1983)

Two approaches to understand HPV

$$H.P.V.\int_{-1}^{1}\frac{1}{x^2}\,dx=\lim\left[\int_{-1}^{-\varepsilon}+\int_{\varepsilon}^{1}\frac{1}{x^2}\,dx-\frac{2}{\varepsilon}\right]=-2$$

Differential first and then trace operator

$$\lim_{y\to 0}\int_{-1}^{1}\frac{1}{x^2+y^2}\,dx=-2$$

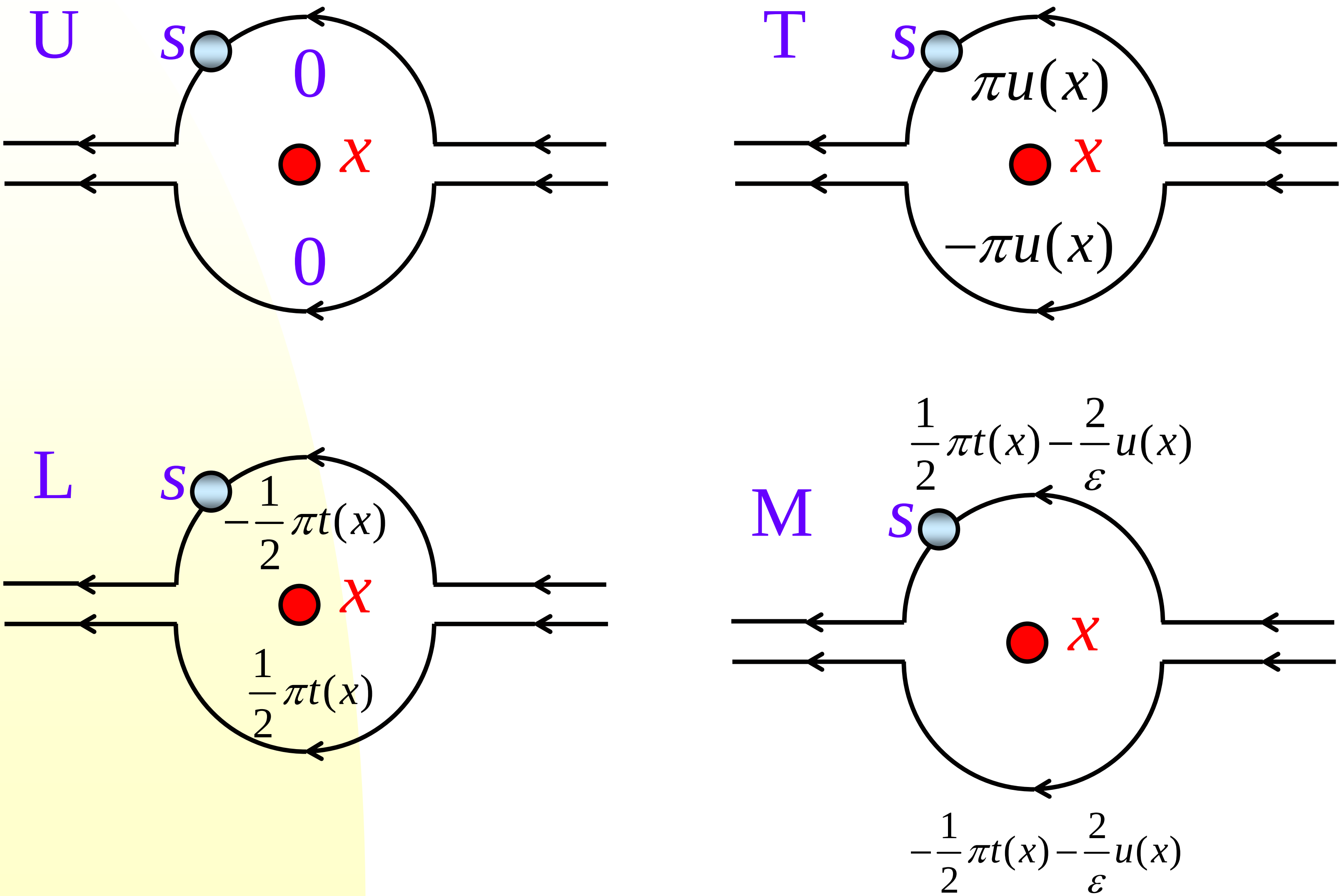
(Limit and integral operator can not be commuted)

Trace first and then differential operator

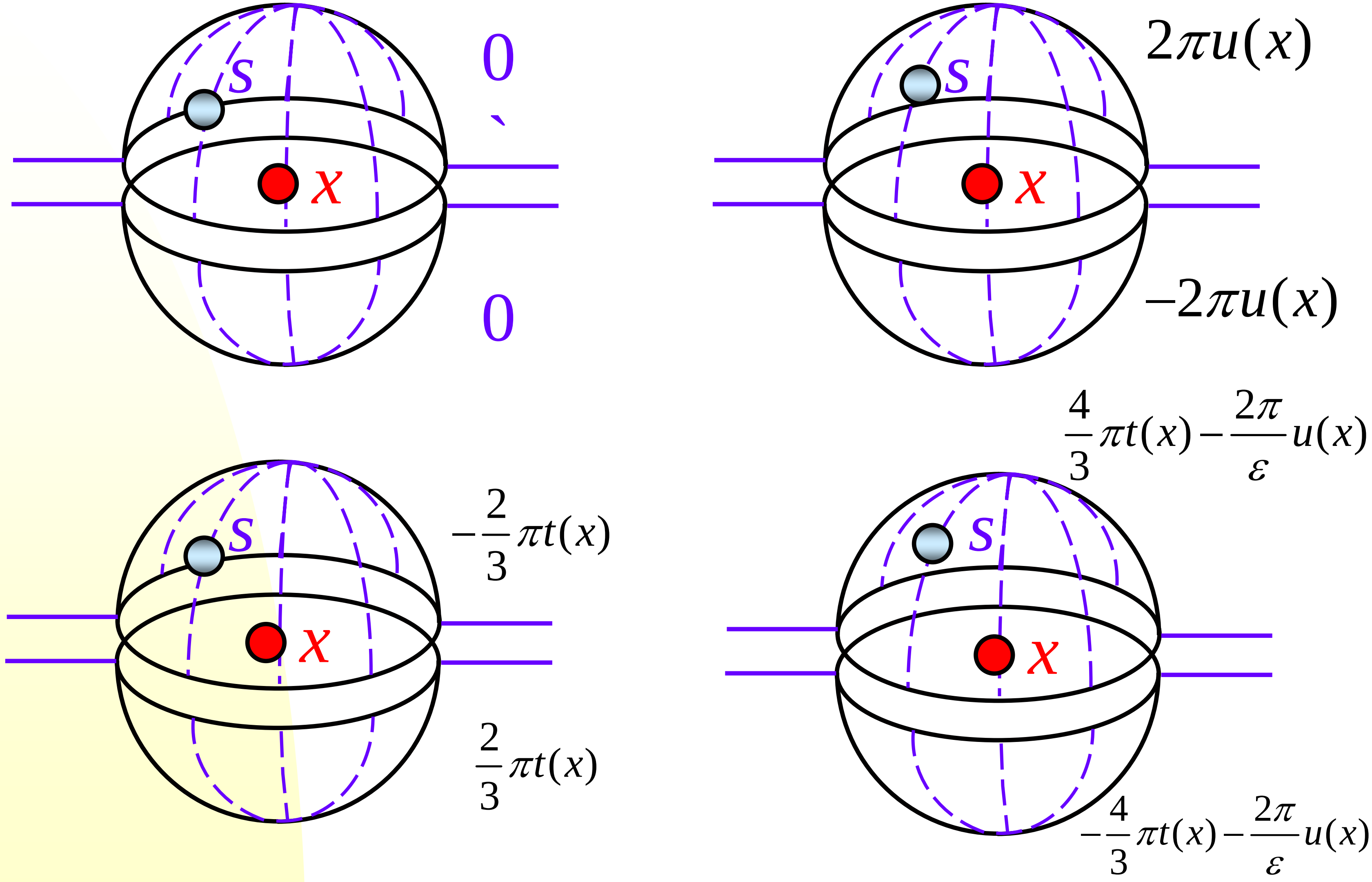
$$\frac{d}{dt}\left\{CPV\int_{-1}^{1}\frac{-1}{x-t}dx\right\}\Big|_{t=0}=-2$$

(Leibnitz rule should be considered)

Bump contribution (2-D)



Bump contribution (3-D)



Hypersingular BIE

Degenerate kernel

Circulants

SVD updating term

SVD updating document

Fredholm alternative theorem

Fundamental solution

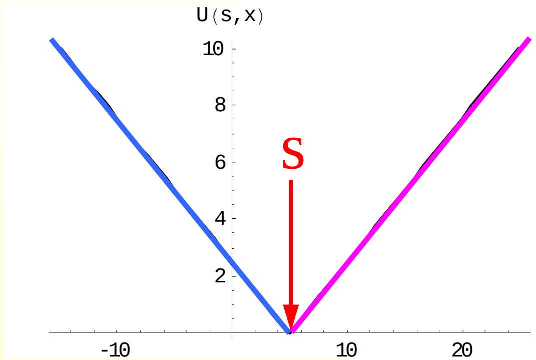
- Field response due to source (space)
- Green's function
- Casual effect (time)

$$K(x,s;t,\tau)$$

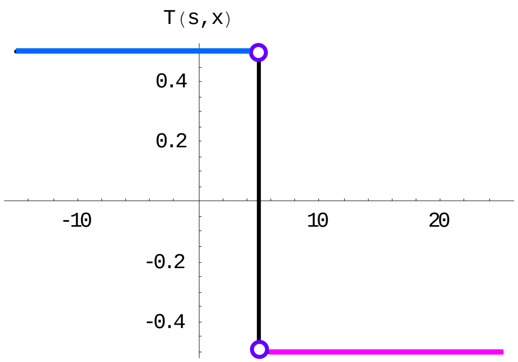
Separable form of fundamental solution (1D)

Separable property

$$U(s,x)=\begin{cases}\sum_{i=1}^{2} a_i(x)b_i(s), & s\geq x\\ \sum_{i=1}^{2} a_i(s)b_i(x), & x>s\end{cases}$$



continuous



discontinuous

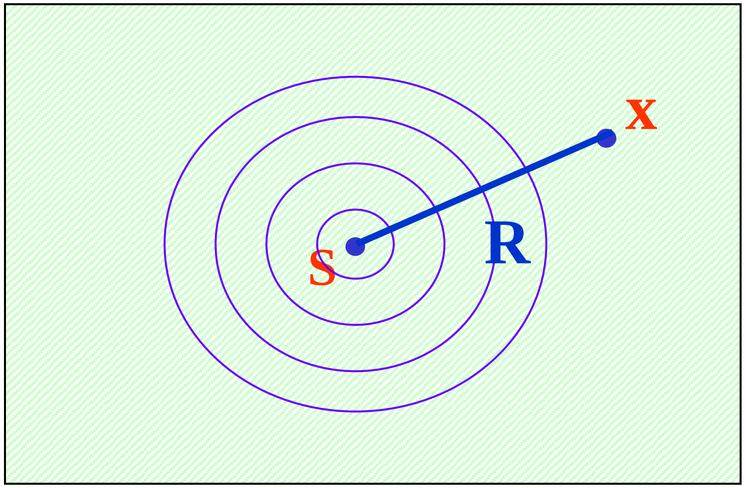
$$U(s,x)=\frac{1}{2}r=\begin{cases}\frac{1}{2}(s-x), & s\geq x\\ \frac{1}{2}(x-s), & x>s\end{cases}$$

$$T(s,x)=\begin{cases}\frac{1}{2}, & s>x\\ \frac{-1}{2}, & x>s\end{cases}$$

Degenerate kernel (step1)

Step 1

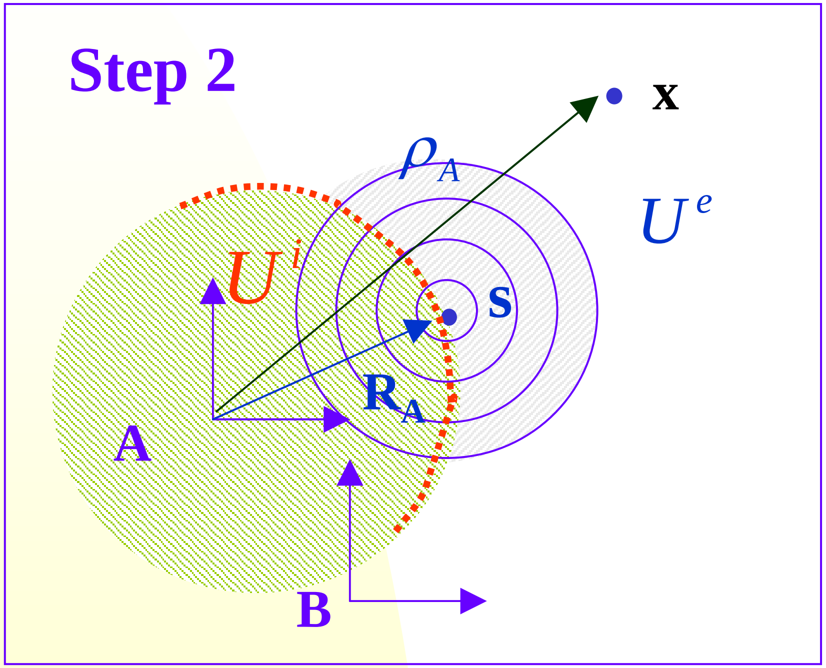
$$U(s,x) = \ln(R) = \ln\left|\underline{s} - \underline{x}\right|$$

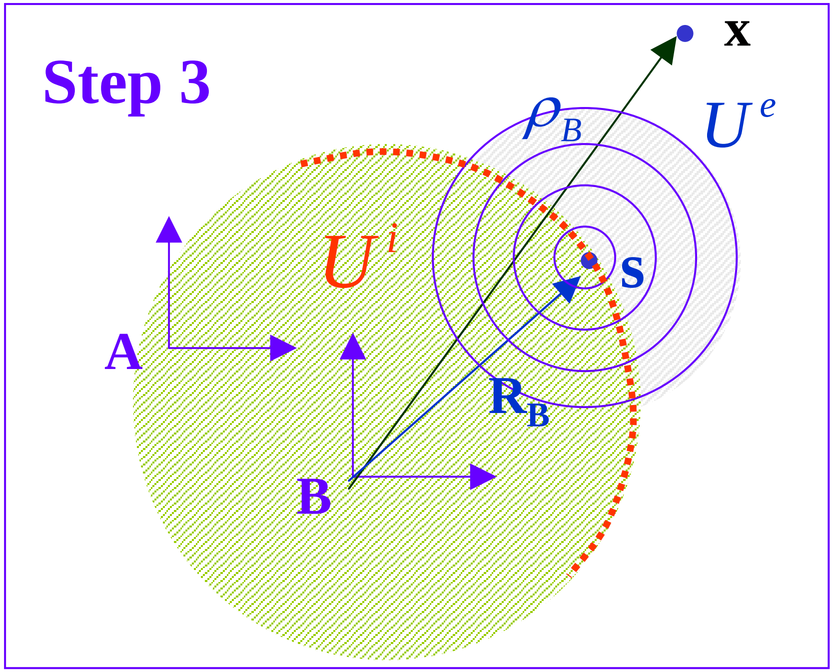


x: variable

s: fixed

Degenerate kernel (step2, step3)





$$U^e(R,\theta,\rho,\phi) = \ln(\rho) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{R}{\rho}\right)^m \cos(m(\theta-\phi)), \quad R > \rho$$

$$U^i(R,\theta,\rho,\phi) = \ln(R) - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{\rho}{R}\right)^m \cos(m(\theta-\phi)), \quad R < \rho$$

Mathematical tools

Hypersingular BIE

Degenerate kernel

Circulants

SVD updating term

SVD updating document

Fredholm alternative theorem

Circulant

$$[U]=\begin{bmatrix} z_0 & z_1 & z_2 & \cdots & z_{2N-1} \\ z_{2N-1} & z_0 & z_1 & \cdots & z_{2N-2} \\ z_{2N-2} & z_{2N-1} & z_0 & \cdots & z_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_1 & z_2 & z_3 & z_{2N-1} & z_0 \end{bmatrix}_{2N\times 2N}$$

$$z_m=\int_{(m-\frac{1}{2})\Delta\bar{\phi}}^{(m+\frac{1}{2})\Delta\bar{\phi}}[-U(a,\bar{\phi},a,\phi)]a\,d\bar{\phi}\approx -U(a,\bar{\phi}_m,a,\phi)a\,\Delta\bar{\phi},$$

$$m=0,1,2,\ldots,2N-1$$

Mathematical tools

- Hypersingular BIE
- Degenerate kernel
- Circulants
- SVD updating term
- SVD updating document
- Fredholm alternative theorem

SVD $A = \Phi \Sigma \Psi^H$

Diagonal matrix

$$[\Sigma] = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n}$$

Unitary matrix $[\Phi]_{m \times m}$, $[\Psi]_{n \times n}$

SVD updating terms

Direct method for Dirichlet B. C. :

Singular equation (UT method)

$$[T^E]\,\underset{\sim}{u} = [U^E]\,\underset{\sim}{t} = 0,$$

Hypersingular equation (LM method)

$$[M^E]\,\underset{\sim}{u} = [L^E]\,\underset{\sim}{t} = 0.$$

$$\underset{\sim}{t} = \{\psi_j\}$$

$$\begin{bmatrix} U^E \\ L^E \end{bmatrix} \{\psi_j\} = 0$$

↓

SVD updating terms

Mathematical tools

Hypersingular BIE

Degenerate kernel

Circulants

SVD updating term

SVD updating document

Fredholm alternative theorem

SVD updating documents

For double-layer potential approach:

$$\underbrace{\begin{matrix} u(x) \\ t(x) \end{matrix}}_{b} \begin{matrix} = \\ = \end{matrix} \underbrace{\begin{bmatrix} [T(s,x)] \\ [M(s,x)] \end{bmatrix}}_{A} \underbrace{\begin{matrix} \{\psi\} \\ \{\psi\} \end{matrix}}_{x}$$

b, A, x

$$A^T\{\phi\} = 0 \quad \text{or} \quad \{\phi\}^T A = 0$$

$$\begin{bmatrix} [T]^T \\ [M]^T \end{bmatrix} \{\phi\} = 0 \quad \text{or} \quad \{\phi\}^T \left[[T] \quad [M] \right] = 0$$

Mathematical tools

Hypersingular BIE

Degenerate kernel

Circulants

SVD updating term

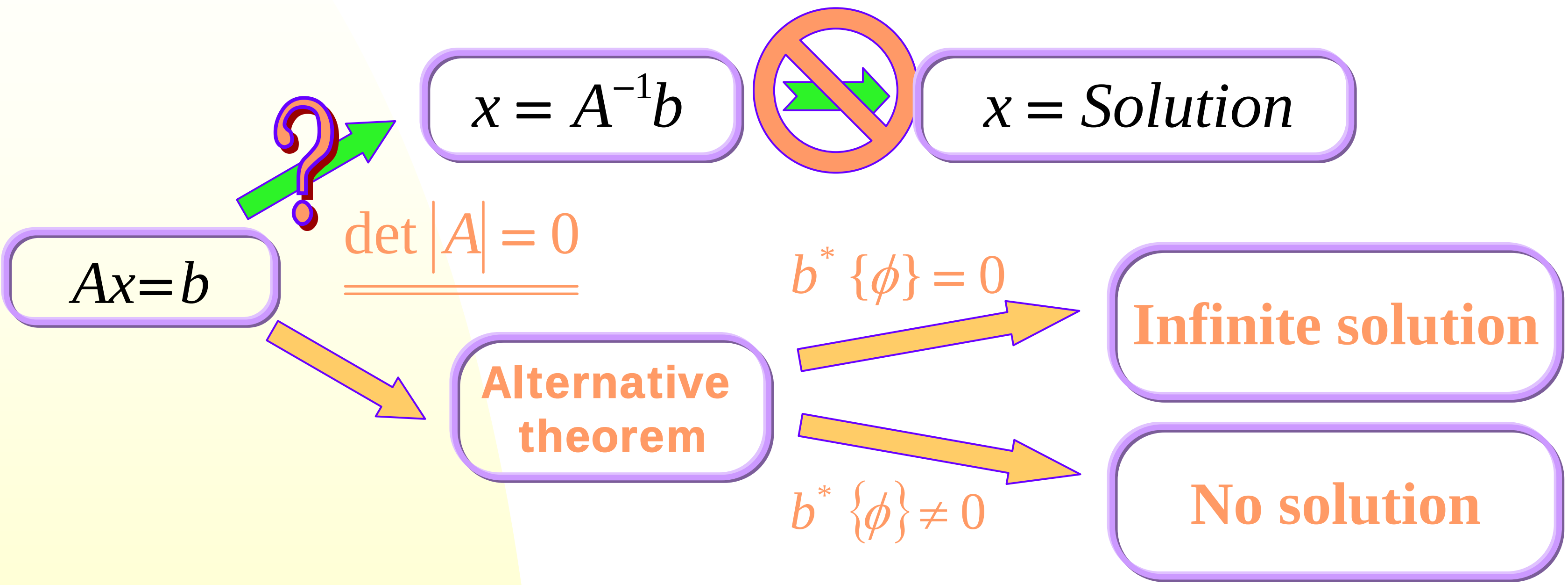
SVD updating document

Fredholm alternative theorem

Fredholm alternative theorem

Fredholm's alternative theorem:

For solving an algebraic system: $Ax = b$



A^* : the transpose conjugate matrix of A

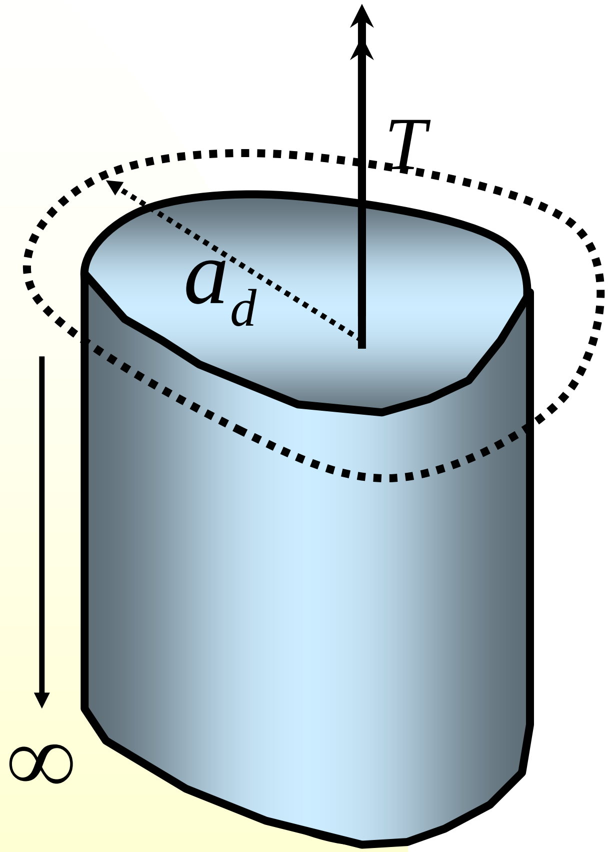
$A^* = A^T$ if A is real

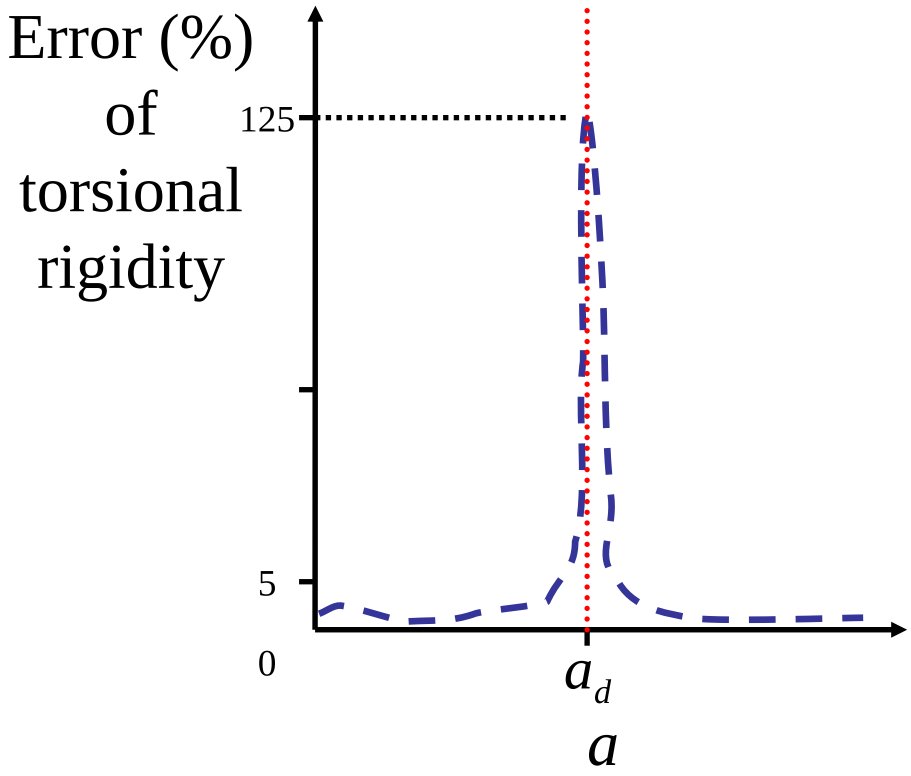
where ♐ satisfies $A^* \{\phi\} = 0$

Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner

The degenerate scale for torsion bar using BEM





Previous approach : Try and error on *a*

Present approach : Only one trial

Mathematical terminology

- Critical value
- Logarithmatic capacity
- Transfinite radius
- Transfinite boundary
- Degenerate scale
- Γ Contour

Determination of the degenerate scale by trial and error

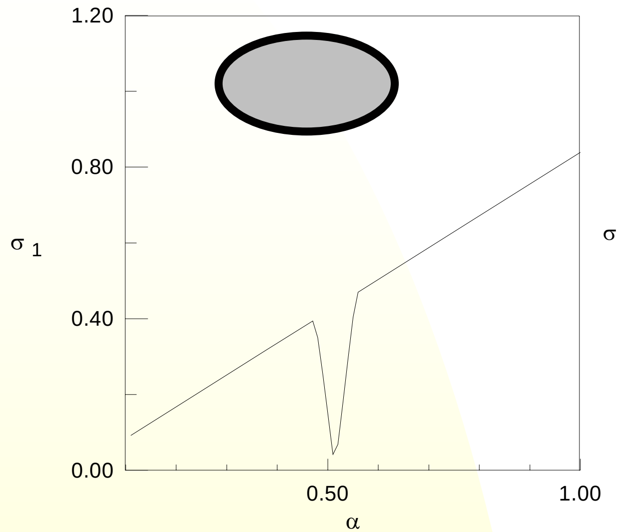


Fig.2-13 The minimum singular value σ_1 of [U] versus semiaxes α for the interior potential problem with an elliptic domain.

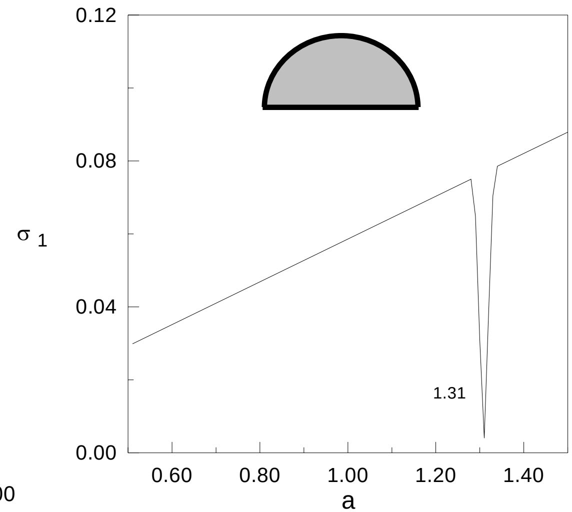


Fig.2-19 The minimum singular value σ_1 of [U] versus radius a for the interior potential problem with a semicircular domain.

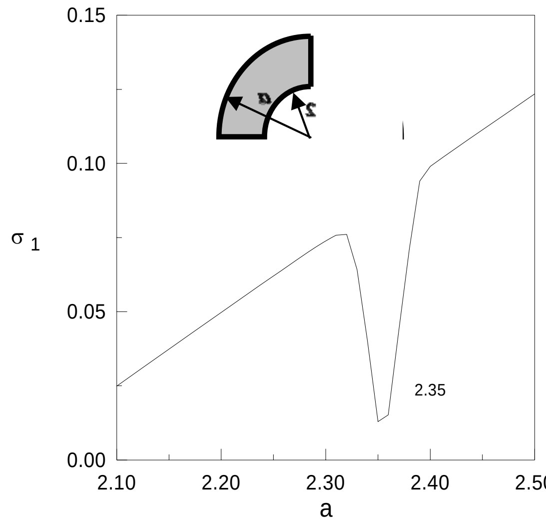
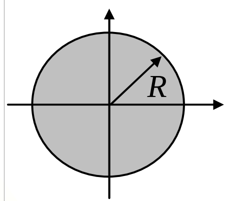
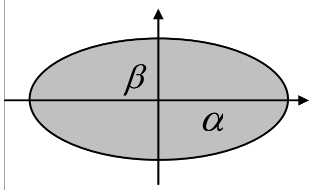
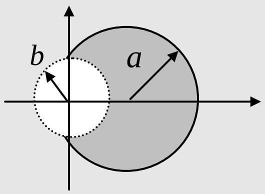
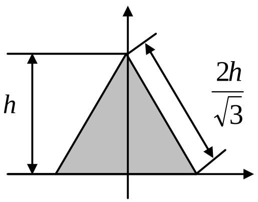
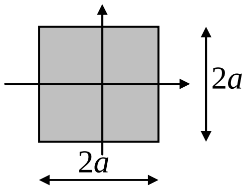


Fig.2-20 The minimum singular value σ_1 of [U] versus parameter a for the interior potential problem with a sector domain.

Direct searching for the degenerate scale

Trial and error---detecting zero singular value by using SVD [Lin (2000) and Lee (2001)]

Determination of the degenerate scale for the two-dimensional Laplace problems

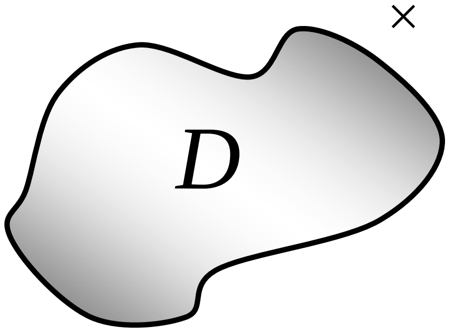
Cross Section					
Normal scale	$R = 2.0$	$\alpha = 3.0, \quad \beta = 1.0$	$a = 2.0, \quad b = \frac{2}{3}a$	$h = 3.0$	$a = 1.0$
Torsional rigidity	$G\frac{\pi}{2}R^4$	$G\frac{\pi\alpha^3\beta^3}{\alpha^2+\beta^2}$	$2Ga^4k_2$	$G\frac{\sqrt{3}}{45}h^4$	Ga^4k_1
Reference equation	$u(x) = \int_B U(s,x)\psi_1(s)dB(s)$, where , x on B, . $[U]\{\psi\} = \{1\}$				
$\Gamma = \int_B \psi_1(s)dB(s)$	1.4480 $(\frac{1}{\ln(2)})$	1.4509 $(\frac{1}{\ln(2)})$	1.5539 (N.A.)	2.6972 (N.A.)	6.1530 (6.1538)
Expansion ratio $d = e^{-\frac{1}{\Gamma}}$	0.5020 (0.5)	0.5019 (0.5)	0.5254 (N.A.)	0.6902 (N.A.)	0.8499 (0.85)
Degenerate scale	R=1.0040 (1.0)	$\alpha + \beta$ = 2.0058 (2.0)	a=1.0508 (N.A.)	h=2.0700 (N.A.)	a=0.8499 (0.85)

Note: Data in parentheses are exact solutions.

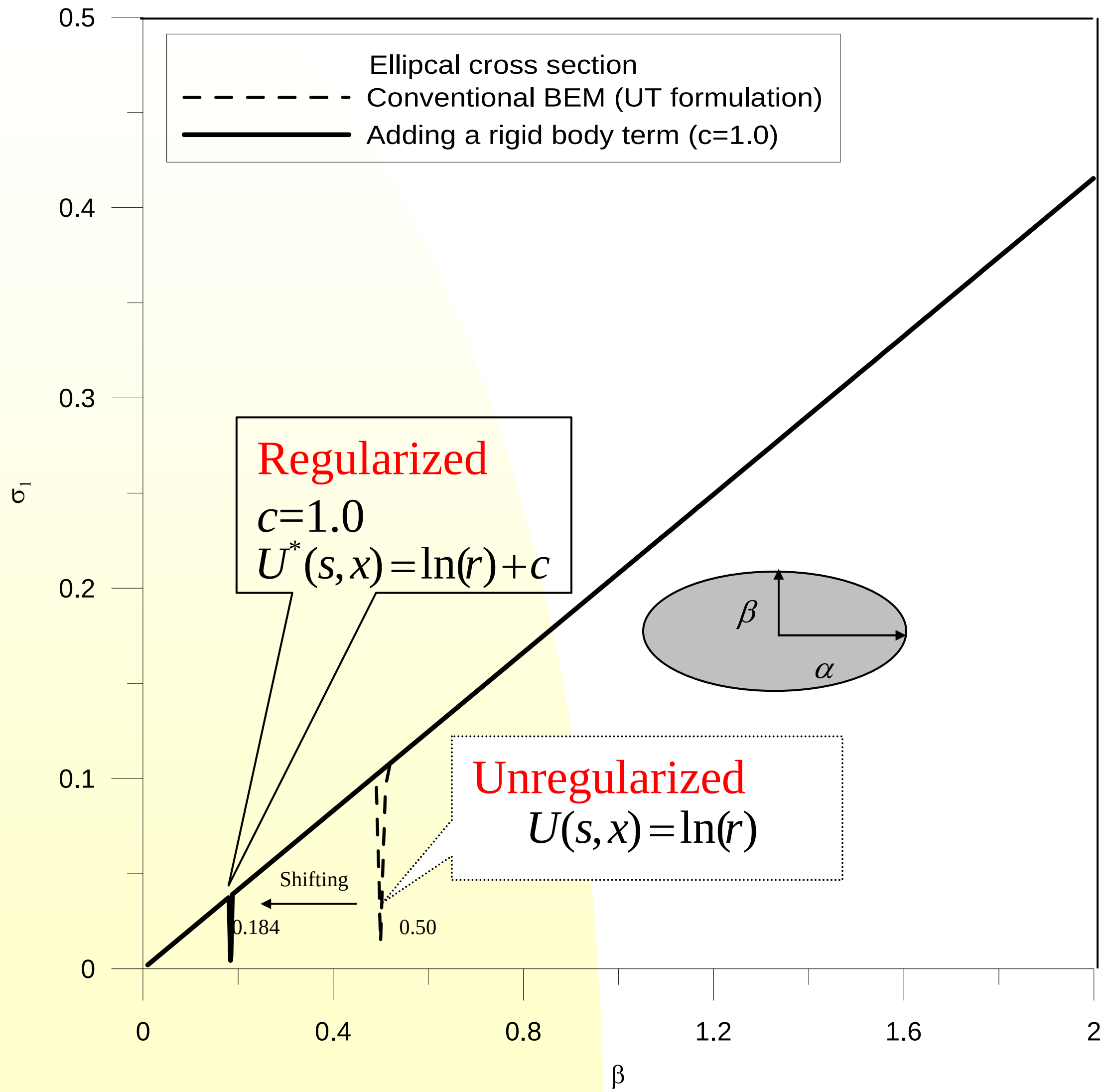
Data marked in the shadow area are derived by using the polar coordinate.

Three regularization techniques to deal with degenerate scale problems in BEM

- Hypersingular formulation (*LM* equation)
- Adding a rigid body term **($U^*(s,x)=U(s,x)+c$)**
- CHEEF concept



Degenerate scale for torsion bar problems with arbitrary cross sections



Normal scale

$\alpha = 3.0,\ \beta = 1.0\ (\alpha = 3\beta)$

$U(s,x) = \ln(r)$ $\quad d = e^{-\frac{1}{\Gamma}}$

↓

Original degenerate scale

$\alpha = 1.5,\ \beta = 0.5$

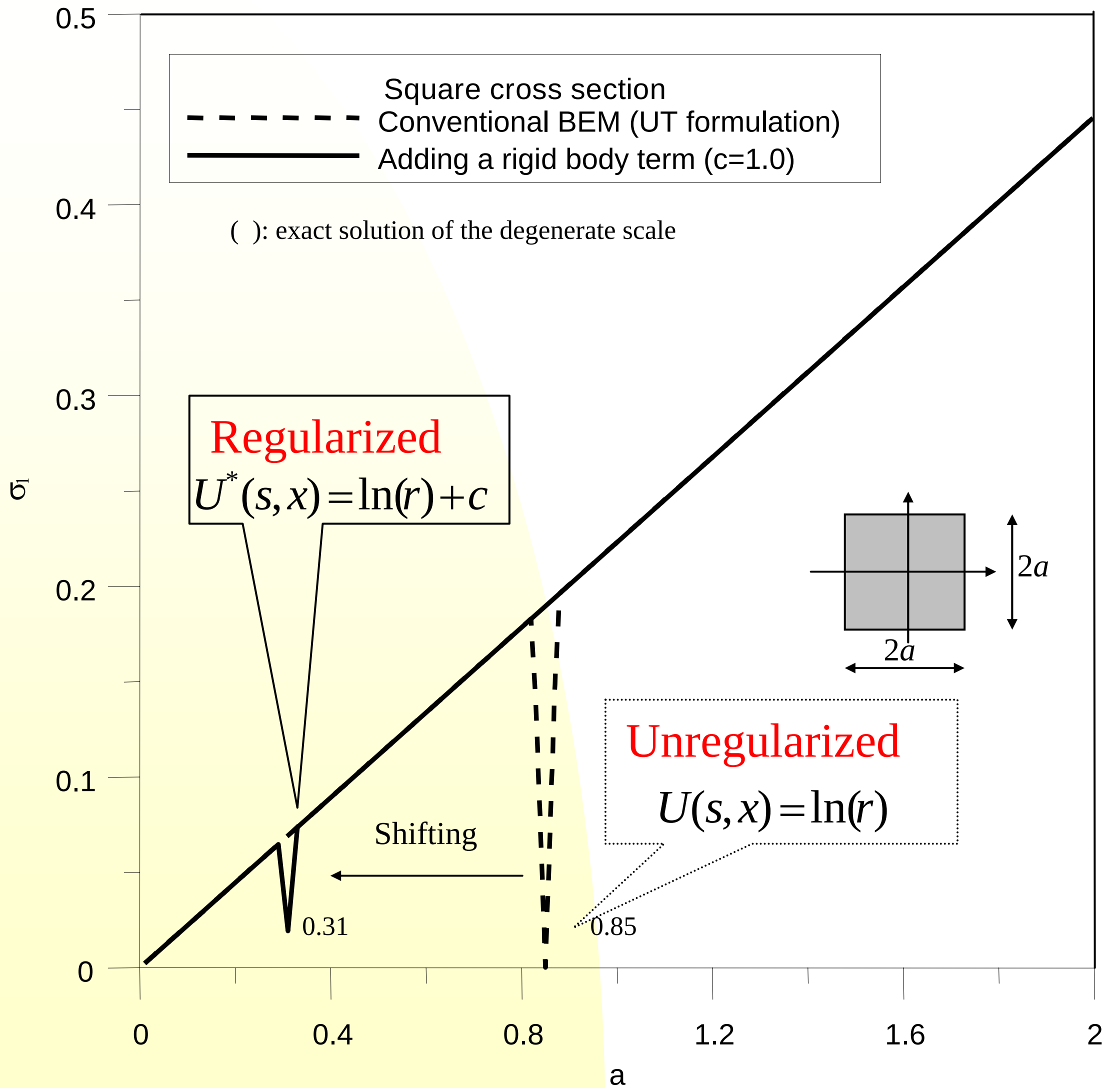
$U^*(s,x) = \ln(r) + c$ $\quad d^* = e^{-c}$

↓

New degenerate scale

$\alpha = 0.552,\ \beta = 0.184$

Degenerate scale for torsion bar problems with arbitrary cross sections



Normal scale
$a = 1.0$

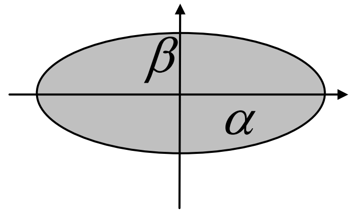
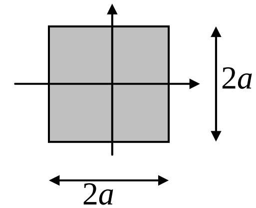
$U(s,x) = \ln(r)$ $\quad d = e^{-\frac{1}{\Gamma}}$

Original degenerate scale
$a = 0.85$

$U^*(s,x) = \ln(r) + c$ $\quad d^* = e^{-c}$

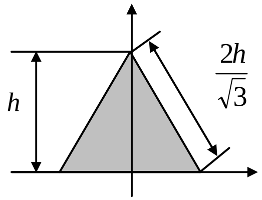
New degenerate scale
$a = 0.31$

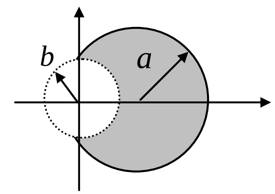
Numerical results

Torsion rigidity / cross section / method	Ellipse 		Square 	
	Normal scale (α =3.0, β=1.0)	Degenerate scale (α =1.5, β=0.5)	Normal scale (**a**=1.0)	Degenerate scale (**a**=0.85)
Analytical solution	$G\dfrac{\pi\alpha^3\beta^3}{\alpha^2+\beta^2}$ 8.4823	$G\dfrac{\pi\alpha^3\beta^3}{\alpha^2+\beta^2}$ 0.5301	$16k_1Ga^4$ 2.249	$16k_1Ga^4$ 1.174
U T Conventional BEM	8.7623 (3.30%)	-0.8911 (268.10%)	2.266 (0.76%)	2.0570 (75.21%)
L M formulation	**Regularization techniques are not necessary.**	0.4812 (9.22%)	**Regularization techniques are not necessary.**	1.1472 (2.31%)
Add a rigid body term, *c*=1.0		0.5181 (2.26%)		1.1721(0.19%)
Add a rigid body term, *c*=2.0		0.5176 (2.36%)		1.1723 (0.17%)
CHEEF concept		0.5647 (6.53%) CHEEF POINT (2.0, 2.0)		1.1722 (0.18%) CHEEF POINT (5.0, 5.0)

Note: data in parentheses denote error.

Numerical results

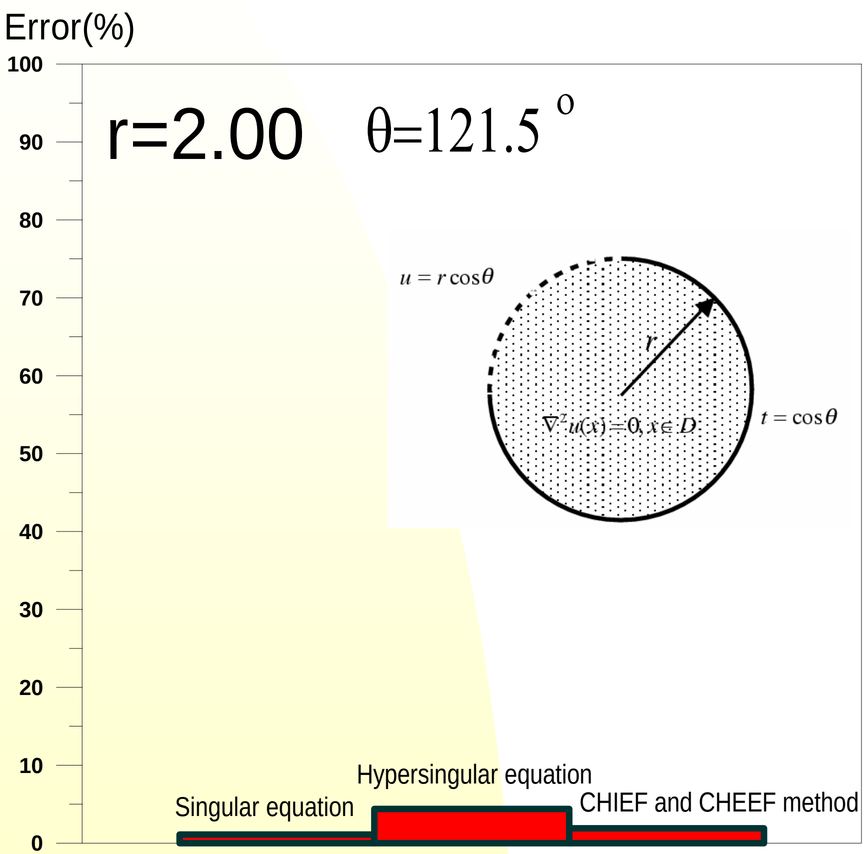


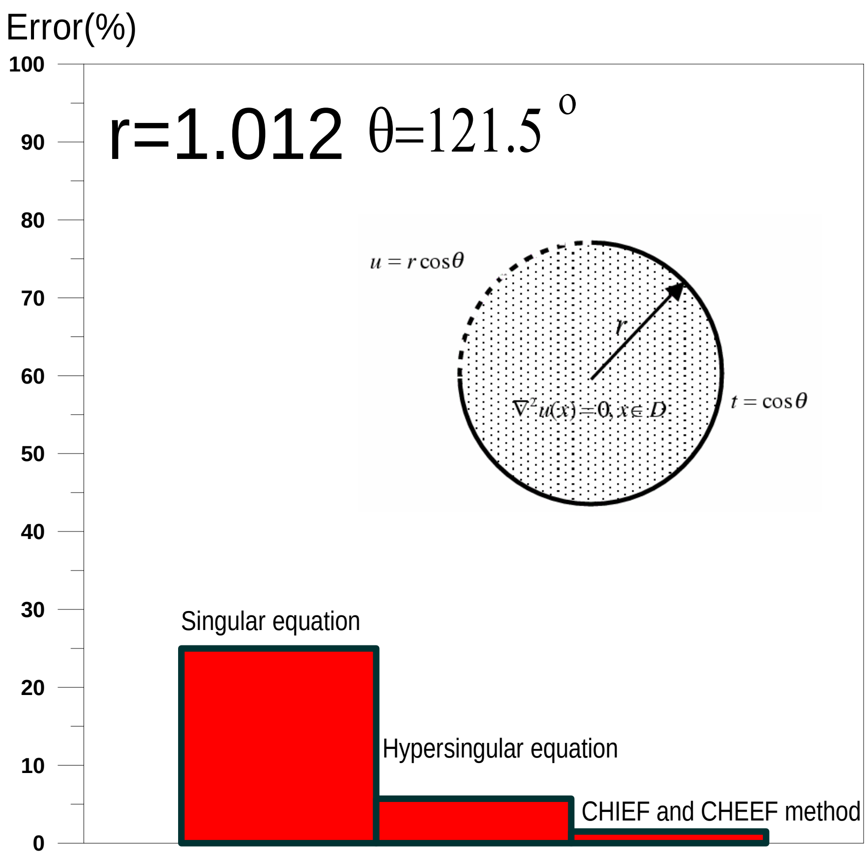


Torsion rigidity / cross section / method	**Triangle** Normal scale h=3.0	**Triangle** Degenerate scale h=2.07	**Keyway** Normal scale (a=2.0)	**Keyway** Degenerate scale (a=1.05)
Analytical solution	3.1177 $G\frac{\sqrt{3}}{45}h^4$	0.7067 $G\frac{\sqrt{3}}{45}h^4$	12.6488 $2Ga^4k_2$	0.9609 $2Ga^4k_2$
U T Conventional BEM	3.1829 (2.09%)	1.1101 (57.08%)	12.5440 (0.83%)	1.8712 (94.73%)
L M formulation	**Regularization techniques are not necessary.**	0.6837 (3.25%)	**Regularization techniques are not necessary.**	0.9530 (0.82%)
Add a rigid body term, c=1.0		0.7035 (0.45%)		0.9876 (2.78%)
Add a rigid body term, c=2.0		0.7024 (0.61%)		0.9879 (2.84%)
CHEEF concept		0.7453 (5.46%) CHEEF POINT (15.0, 15.0)		0.9272 (3.51%) CHEEF POINT (20.0, 20.0)

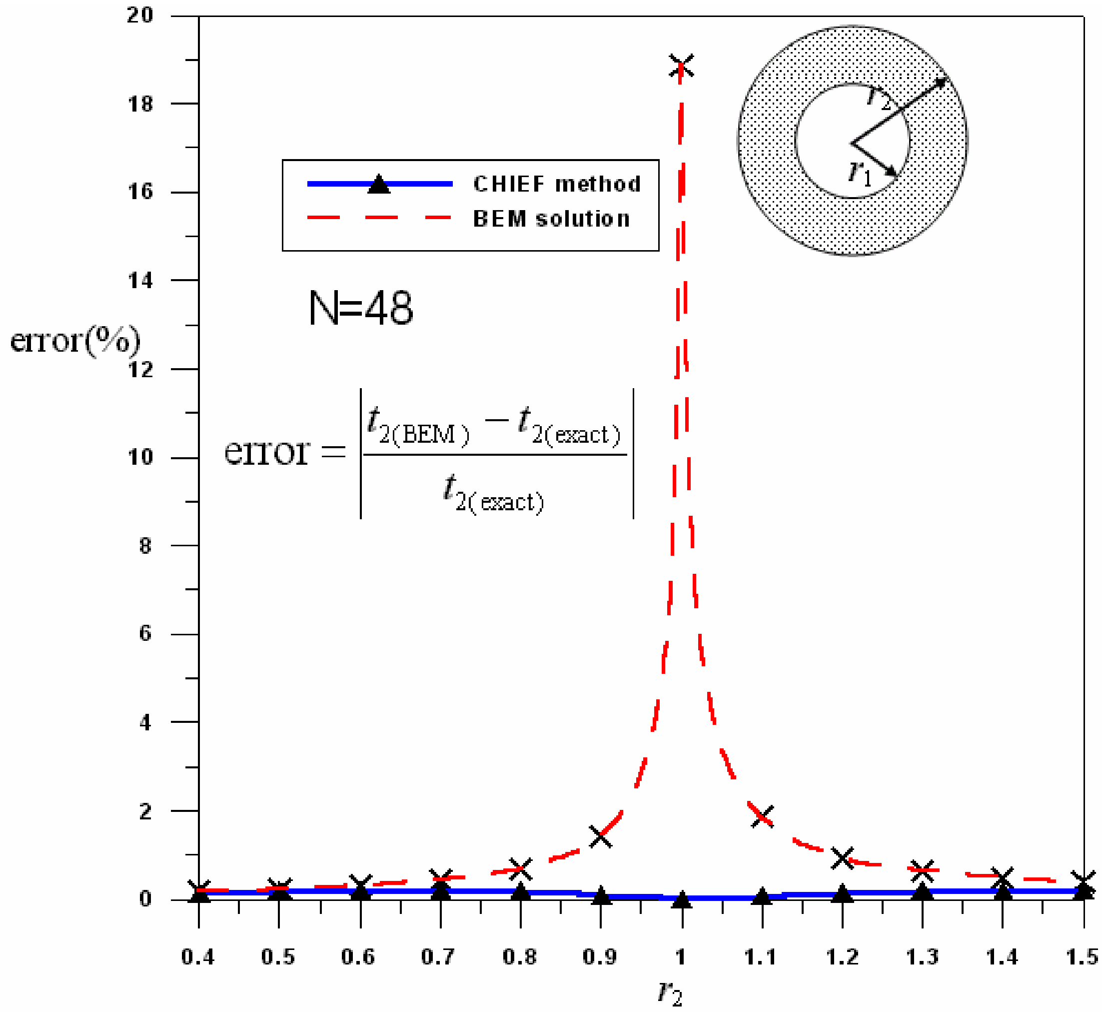
Note: data in parentheses denote error.

Error using three methods



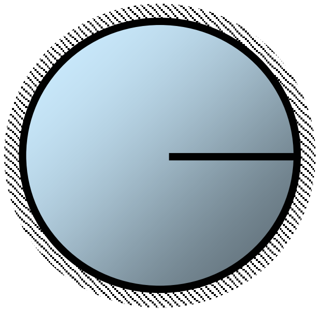


Multiply-connected problem



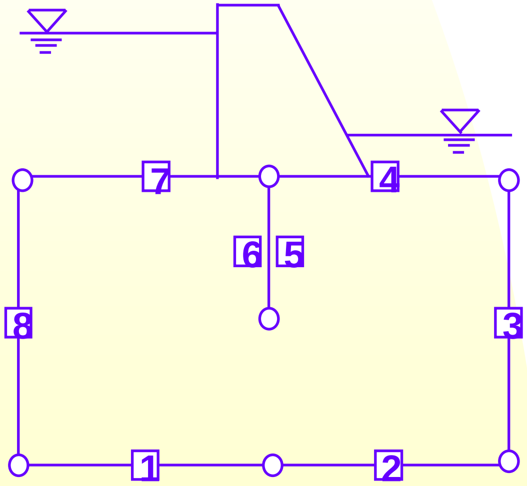
Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner

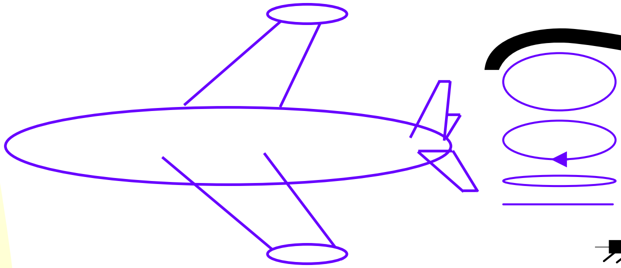


Engineering problems

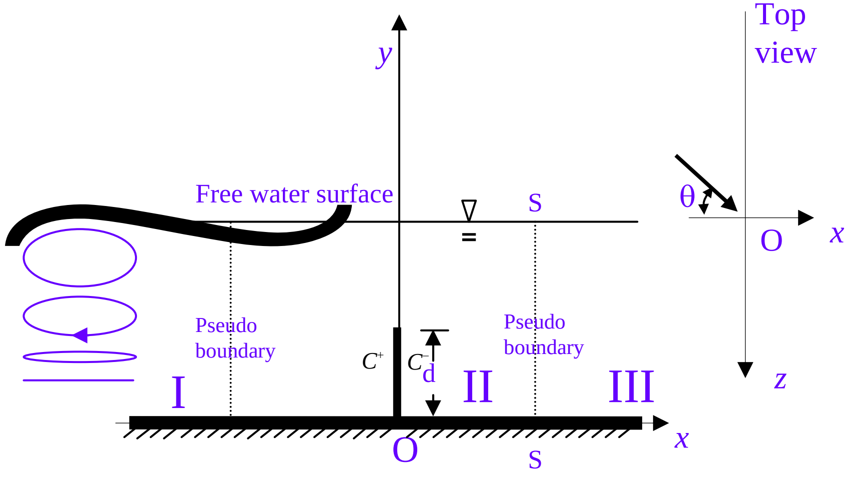
Seepage with sheetpiles



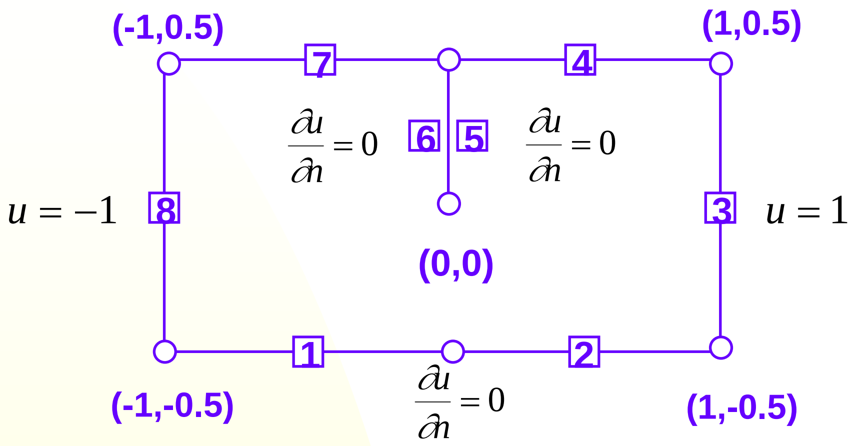
Thin-airfoill Aerodynamics



oblique incident water wave



Degeneracy of the Degenerate Boundary



○ geometry node

N the Nth constant or linear element

$$[U]\{t\} = [T]\{u\}$$

$$[L]\{t\} = [M]\{u\}$$

Columns 5(+), 6(+); rows 5(+), 6(+):

$$[U]=\begin{bmatrix} -1.693 & -0.045 & 0.471 & 0.347 & -0.054 & -0.054 & 0.039 & -0.335 \\ -0.045 & -1.693 & -0.335 & 0.039 & -0.054 & -0.054 & 0.347 & 0.471 \\ 0.445 & -0.335 & -1.693 & -0.335 & 0.019 & 0.019 & 0.445 & 0.703 \\ 0.347 & 0.039 & -0.335 & -1.693 & -0.281 & -0.281 & -0.045 & 0.471 \\ -0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 \\ -0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 \\ 0.039 & 0.347 & 0.471 & -0.045 & -0.281 & -0.281 & -1.693 & -0.334 \\ -0.335 & 0.445 & 0.703 & 0.445 & 0.019 & 0.019 & -0.335 & -1.693 \end{bmatrix}$$

$n(s)$; columns 5(+), 6(-); rows 5(+), 6(+):

$$[T]=\begin{bmatrix} -\pi & 0.000 & 0.588 & 0.519 & -0.321 & 0.321 & 0.927 & 1.107 \\ 0.000 & -\pi & 1.107 & 0.927 & 0.321 & -0.321 & 0.519 & 0.588 \\ 0.219 & 1.107 & -\pi & 1.107 & 0.464 & -0.464 & 0.219 & 0.490 \\ 0.519 & 0.927 & 1.107 & -\pi & 0.785 & -0.785 & 0.000 & 0.588 \\ 0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 \\ 0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 \\ 0.927 & 0.519 & 0.588 & 0.000 & -0.7854 & 0.785 & -\pi & 1.107 \\ 1.107 & 0.219 & 0.490 & 0.219 & -0.464 & 0.464 & 1.107 & -\pi \end{bmatrix}$$

$n(x)$; columns 5(+), 6(+); rows 5(+), 6(-):

$$[L]=\begin{bmatrix} \pi & 0.000 & 0.184 & 0.519 & 0.458 & 0.458 & 0.927 & 0.805 \\ 0.000 & \pi & 0.805 & 0.927 & 0.458 & 0.458 & 0.519 & 0.184 \\ 0.612 & 0.805 & \pi & 0.805 & 0.464 & 0.464 & 0.612 & 0.490 \\ 0.519 & 0.927 & 0.805 & \pi & 0.347 & 0.347 & 0.000 & 0.184 \\ -0.511 & 0.511 & 0.888 & 1.417 & \pi & -\pi & -1.417 & -0.888 \\ 0.511 & -0.511 & -0.888 & -1.417 & -\pi & \pi & 1.417 & 0.888 \\ 0.927 & 0.519 & 0.184 & 0.000 & 0.347 & 0.347 & \pi & 0.805 \\ 0.805 & 0.612 & 0.490 & 0.612 & 0.464 & 0.464 & 0.805 & \pi \end{bmatrix}$$

$n(s)$, $n(x)$; columns 5(+), 6(-); rows 5(+), 6(-):

$$[M]=\begin{bmatrix} 4.000 & -1.333 & -0.205 & -0.061 & 0.600 & -0.600 & -0.800 & -1.600 \\ -1.333 & 4.000 & -1.600 & -0.800 & -0.600 & 0.600 & -0.061 & -0.205 \\ -0.282 & -1.600 & 4.000 & -1.600 & -0.400 & 0.400 & -0.282 & -0.236 \\ -0.061 & -0.800 & -1.600 & 4.000 & -1.000 & 1.000 & -1.333 & -0.205 \\ 0.853 & -0.853 & -0.715 & -3.765 & 8.000 & -8.000 & 3.765 & 0.715 \\ -0.853 & 0.853 & 0.715 & 3.765 & -8.000 & 8.000 & -3.765 & -0.715 \\ -0.800 & -0.062 & -0.205 & -1.333 & 1.000 & -1.000 & 4.000 & -1.600 \\ -1.600 & -0.282 & -0.235 & -0.282 & 0.400 & -0.400 & -1.600 & 4.000 \end{bmatrix}$$

Theory of dual integral equations

$$f(x) = (x-a)^2 Q(x) + px + q$$

$$f(a) = pa + q, \quad when\ x = a$$

The constraint equation is not enough to determine the coefficient *p* and *q*,

Another constraint equation is required

$$f'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x) + p$$

$$f'(a) = p, \quad when\ \ x = a$$

Successful experiences

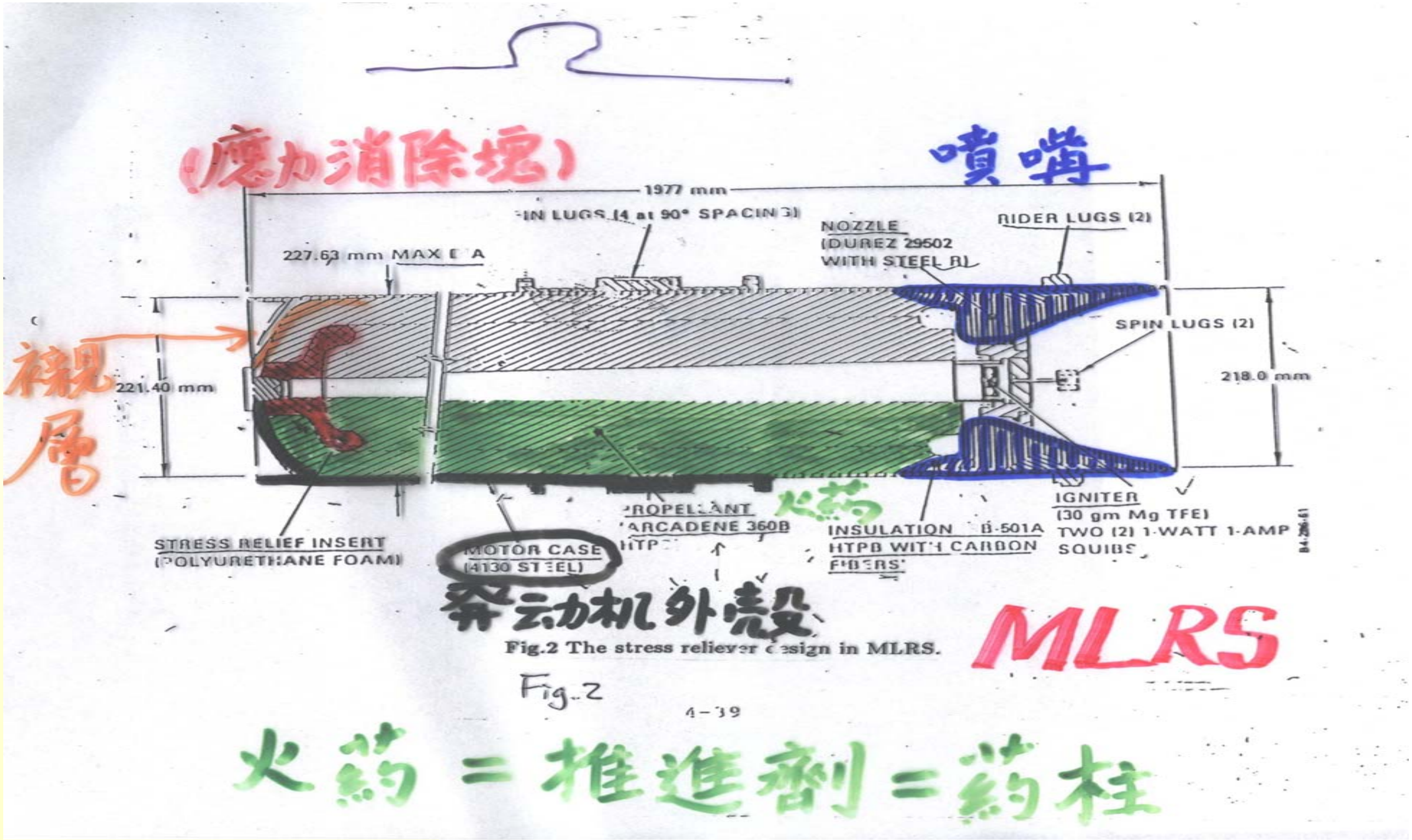


Fig.2 The stress reliever design in MLRS.

Fig. 2

4–39

火药 = 推進劑 = 药柱

X-ray detection

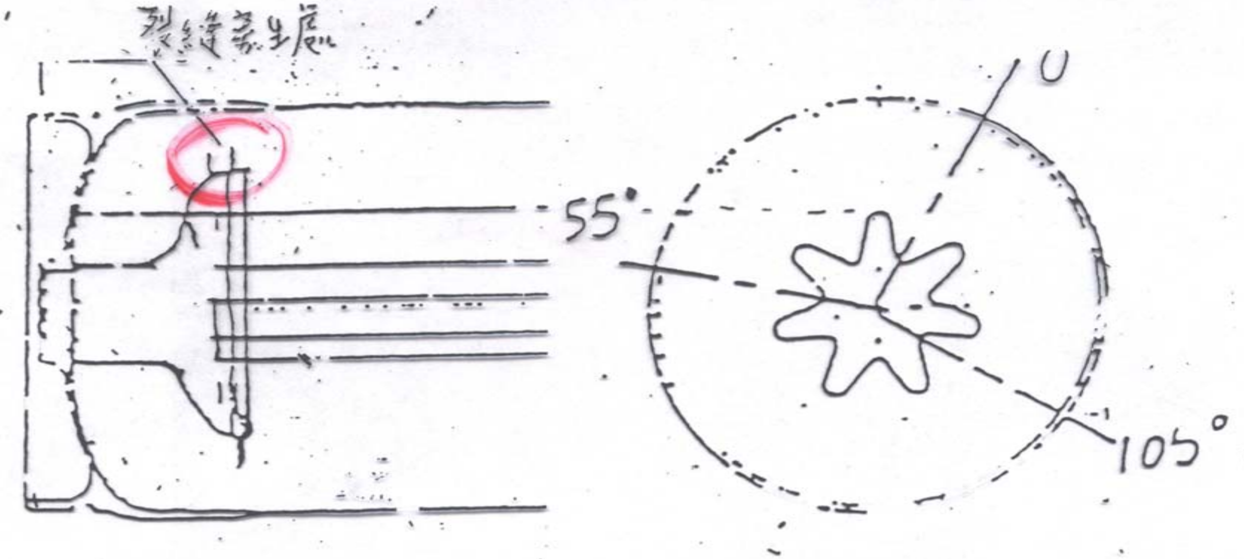
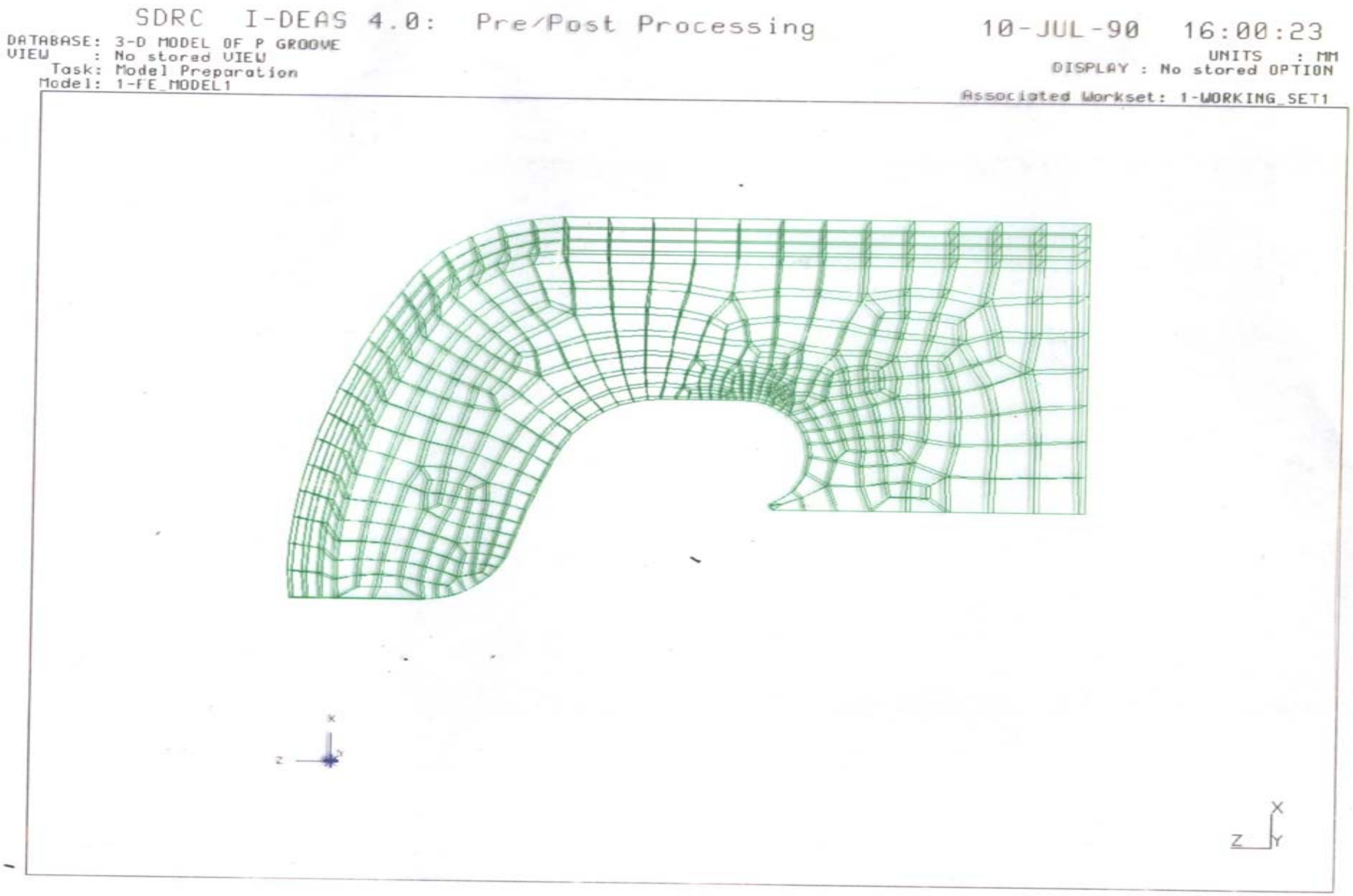


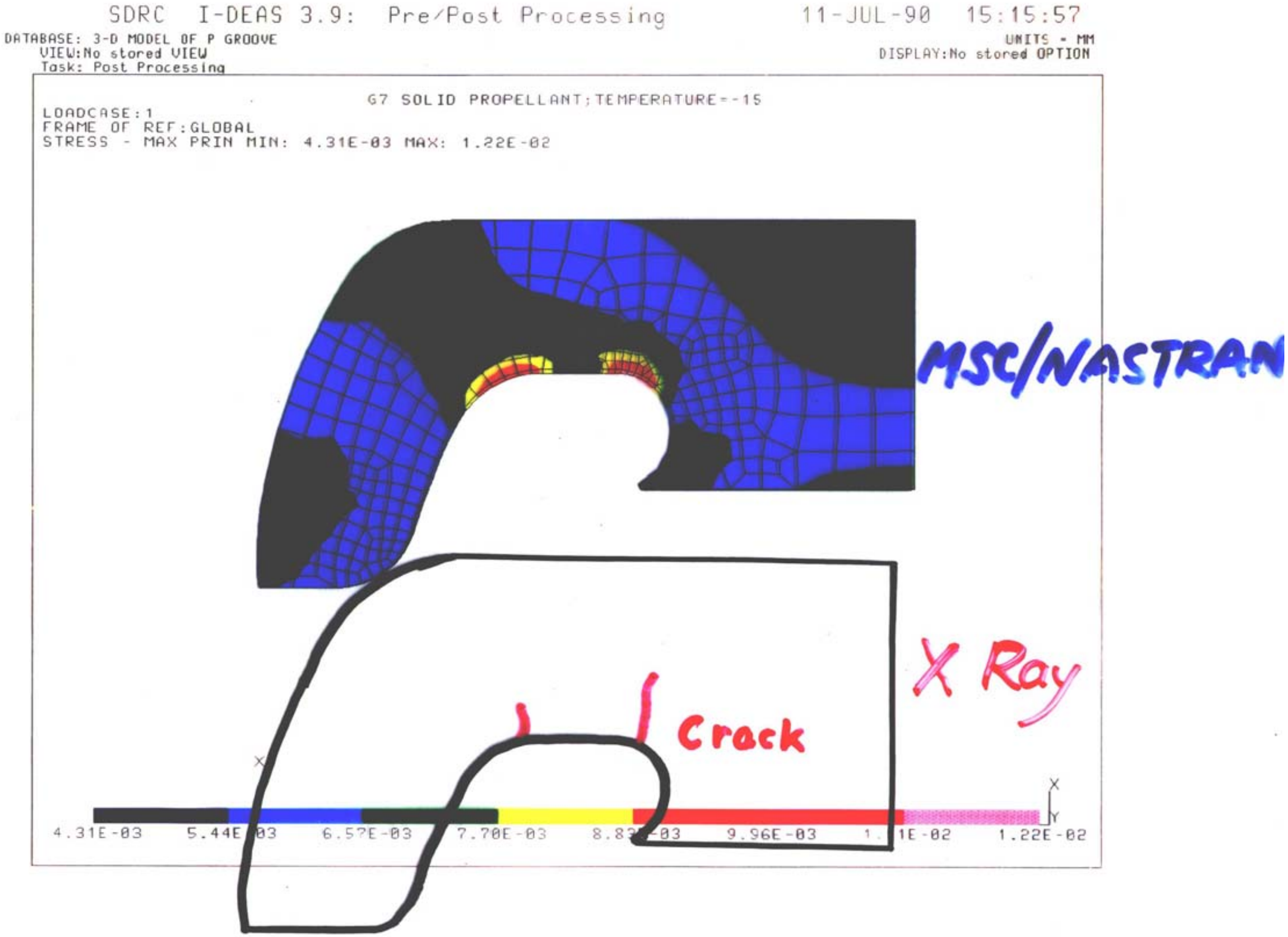
Fig. DT X-ray results.

圖一應力消除塊設計圖 [12,13]

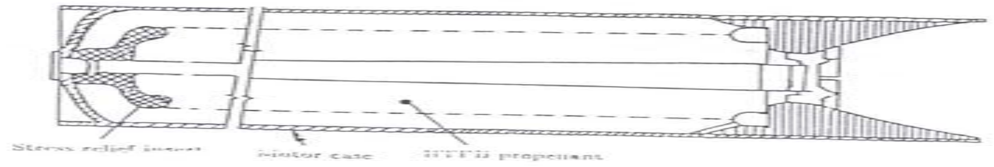
FEM simulation

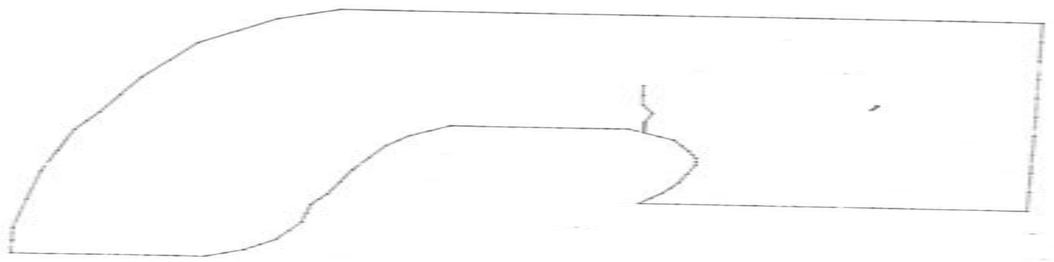


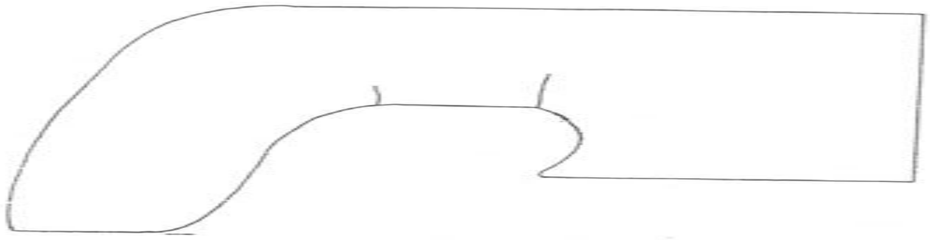
Stress analysis



BEM simulation







V-band structure (Tien-Gen missile)

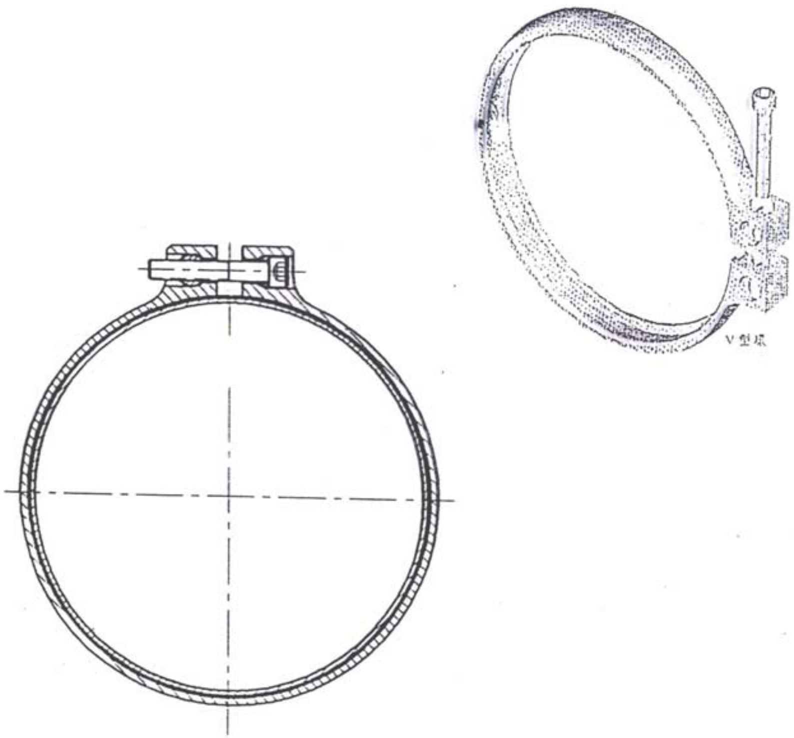


圖 1．V型環的結構示意圖

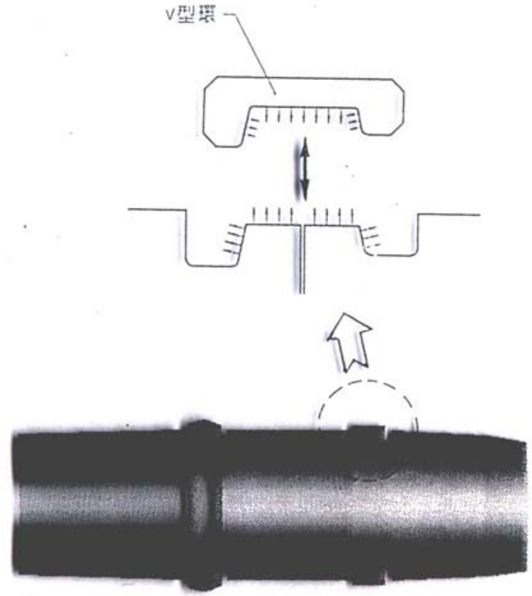
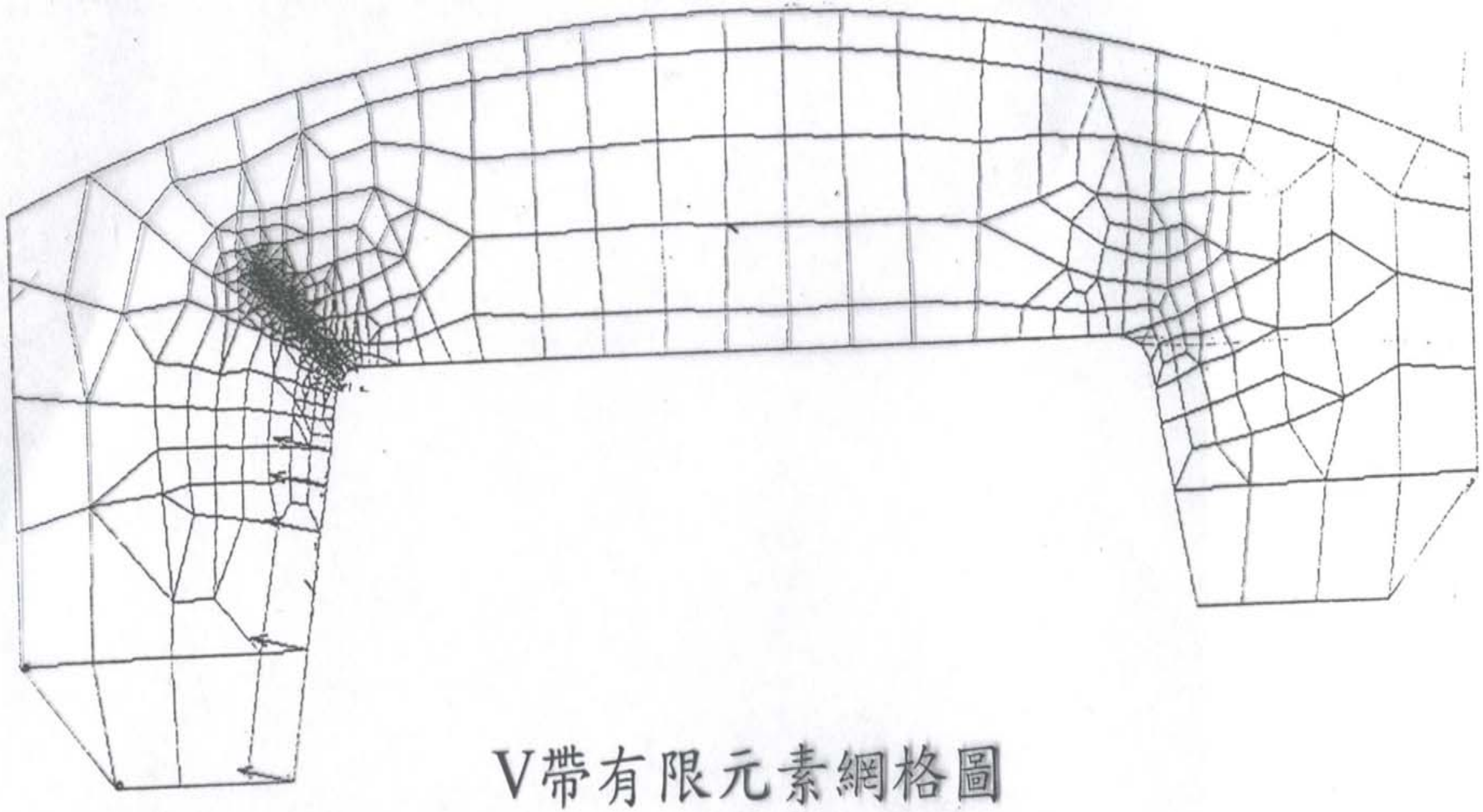


圖 2．V型環的結合功能

V帶結構示意圖

FEM simulation



V帶有限元素網格圖

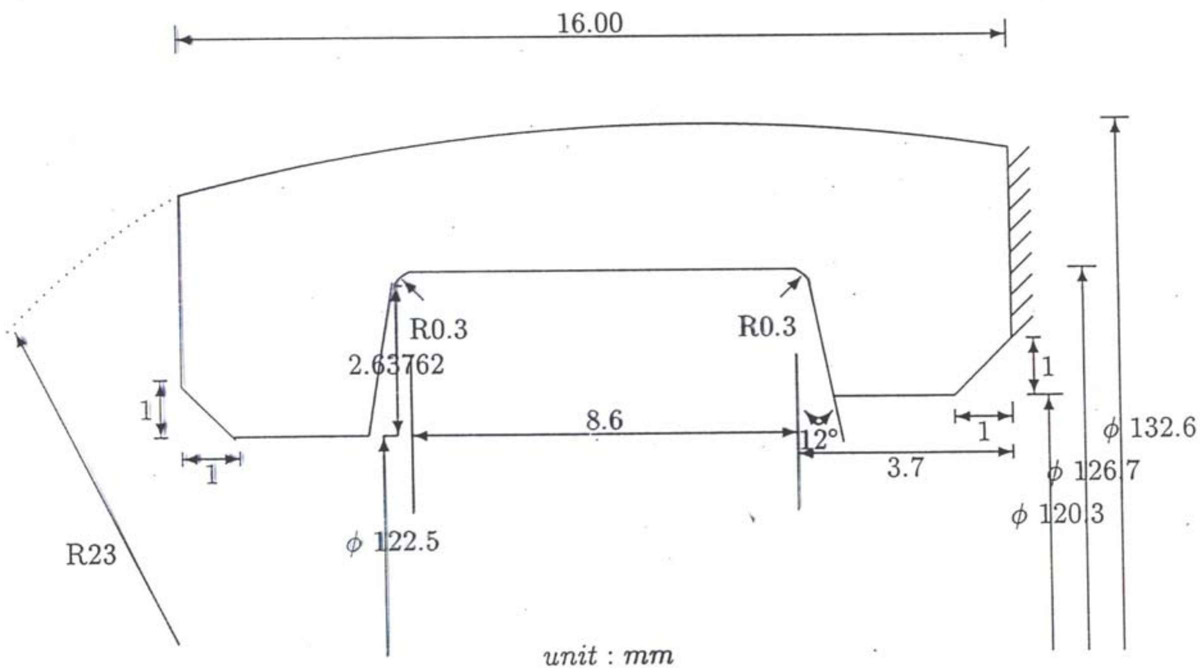
Application to V-band structure:

$E = 19950\ kgf/mm^2$, $\upsilon=0.27$,

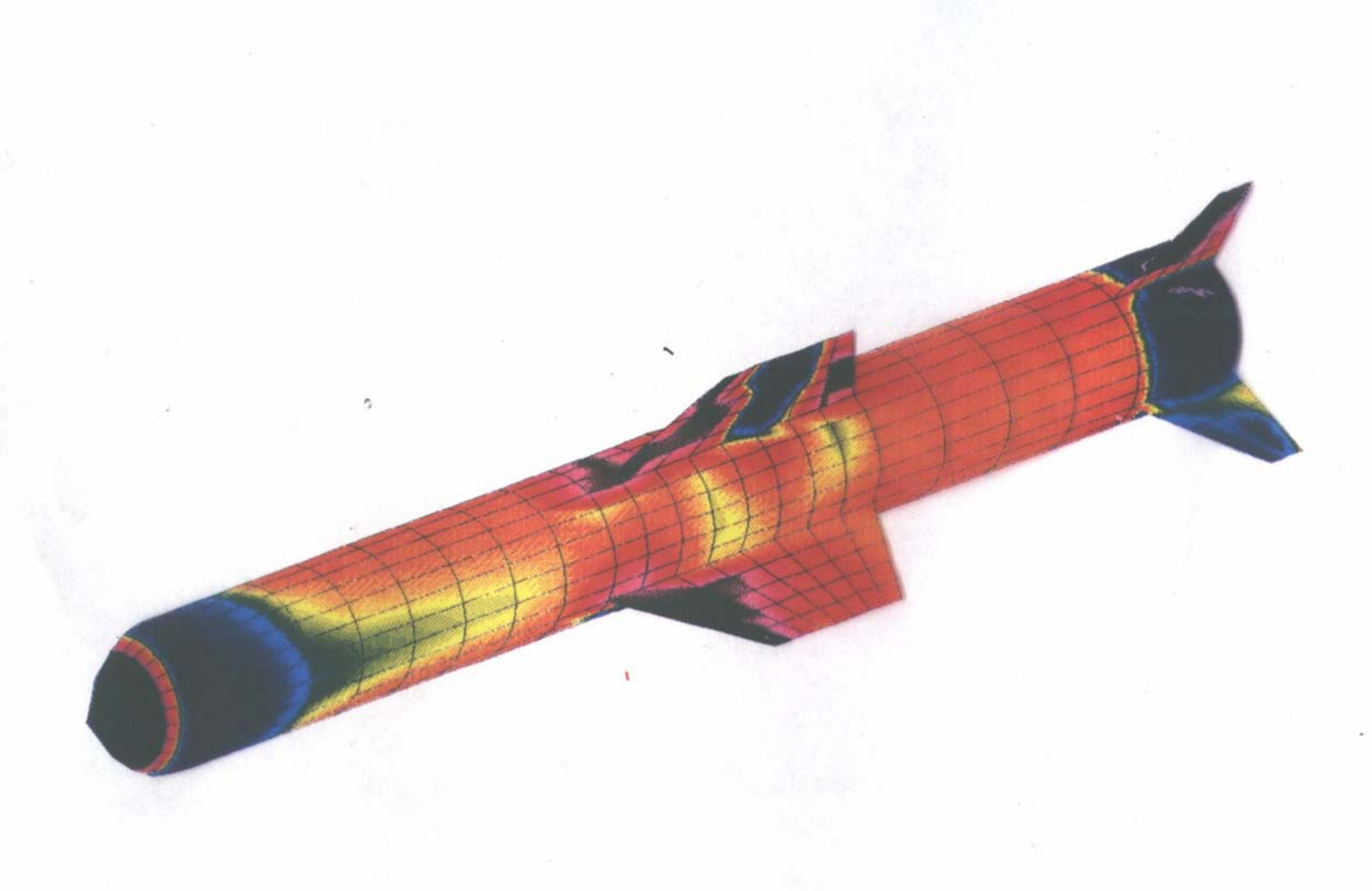
$a=0.125$ $\sigma = 3.63 kgf / mm^2$

Pari's law: $\frac{da}{dN} = C(\Delta K)^m$

$C = 4.624 \times 10^{-12}$, $m=3.3$, $R=\frac{2}{3}$



Shong-Fon II missile



IDF

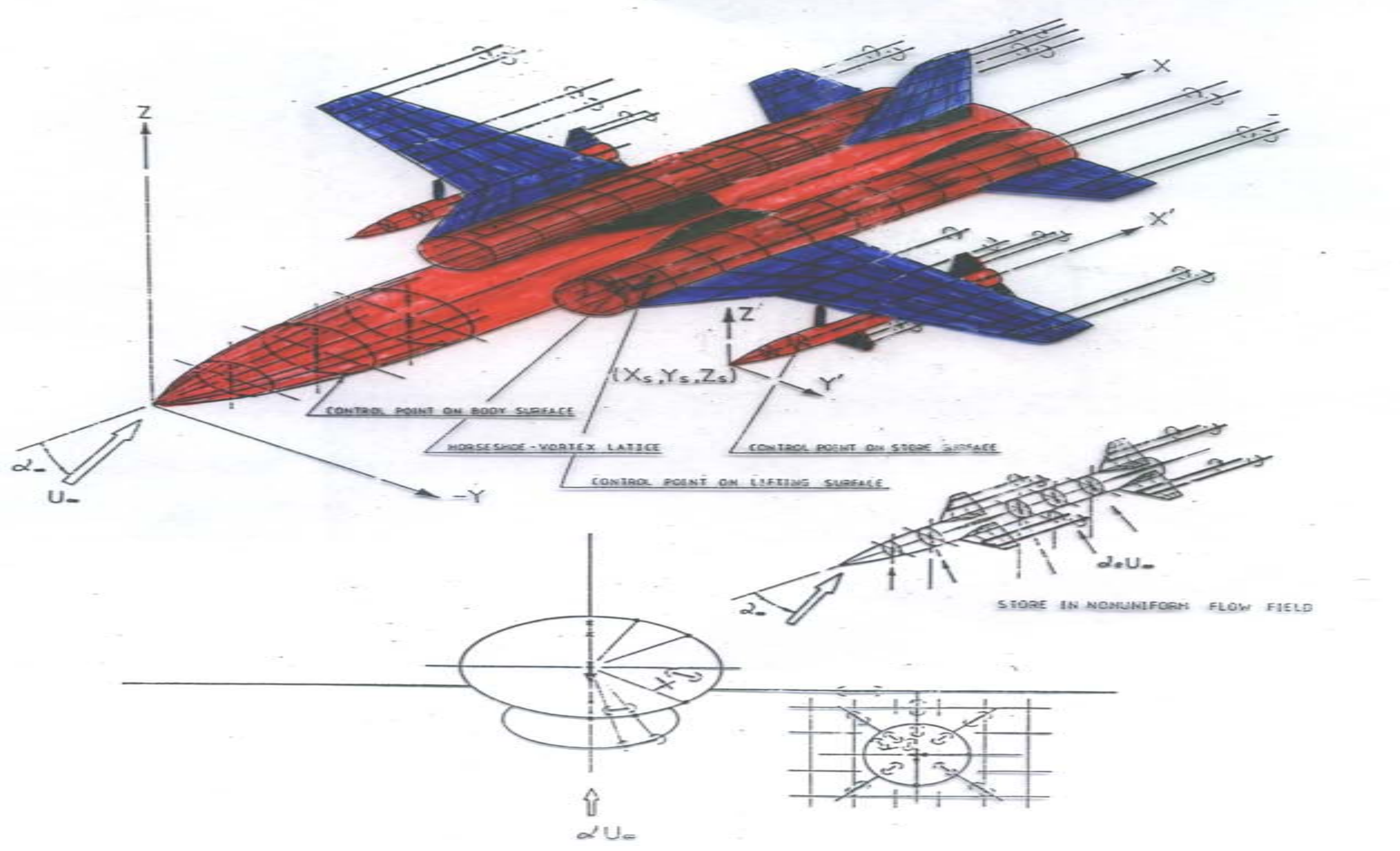


Fig.1 Image system of all the singularities in aircraft/external store configurations.

Flow field

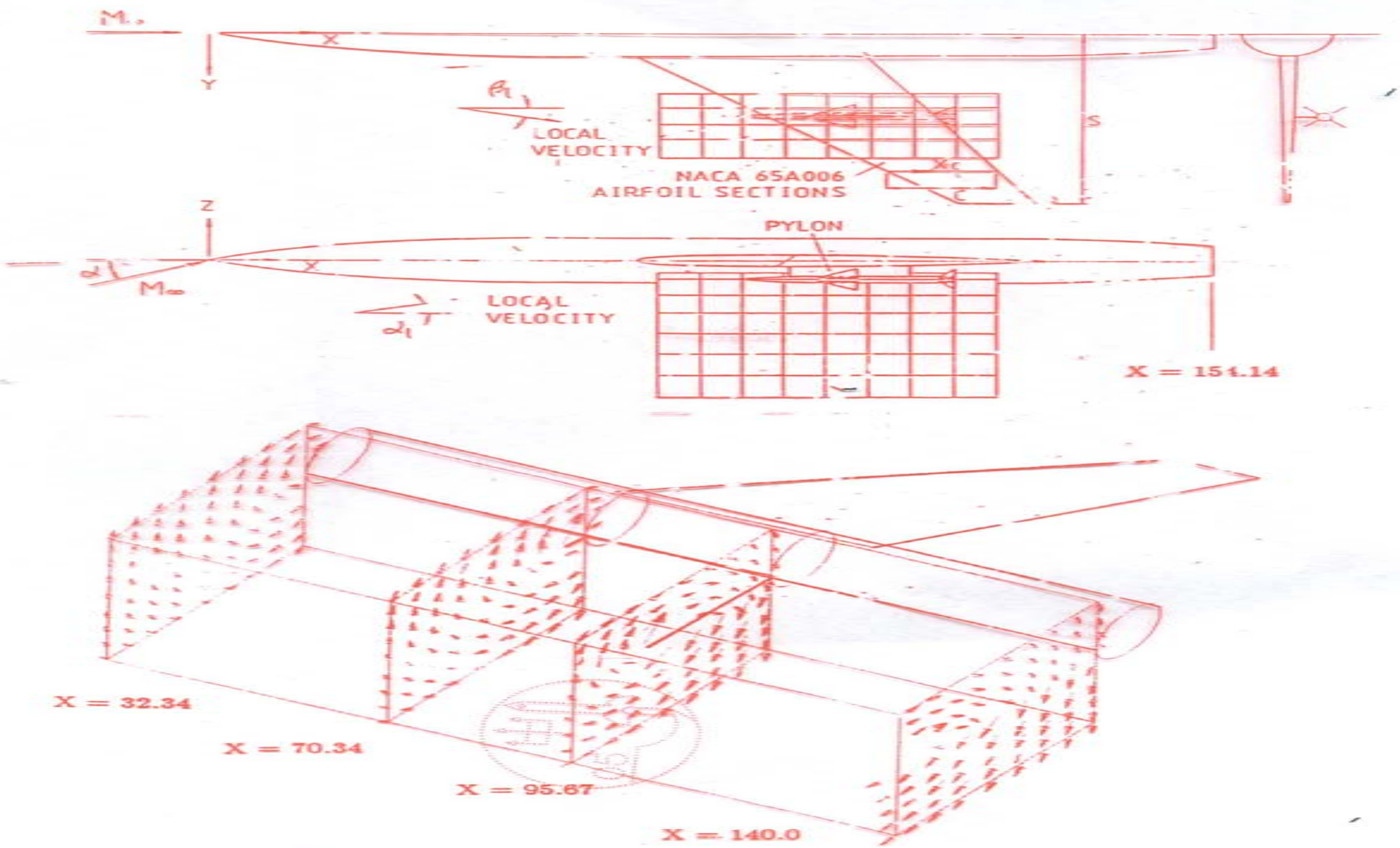


Fig.8 Cross flow velocity field on the reference planes.

Seepage flow

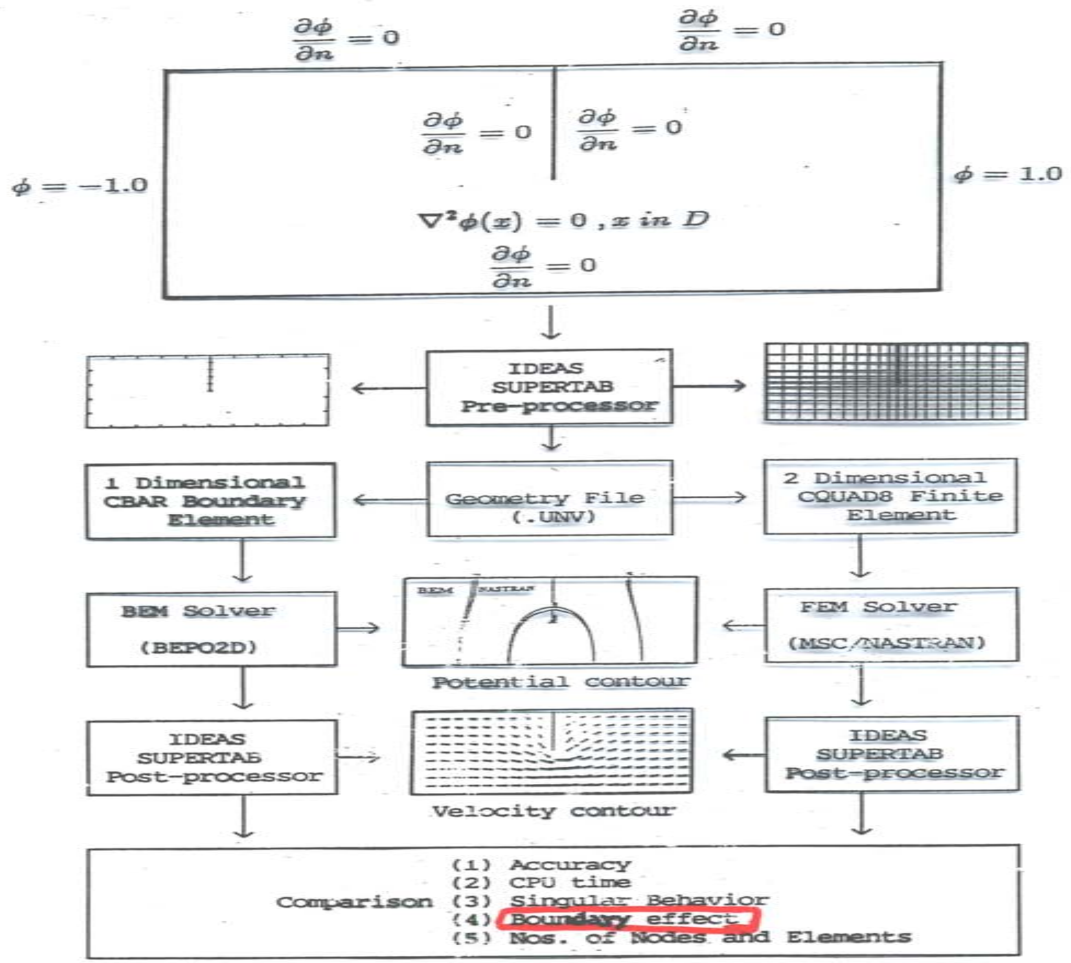
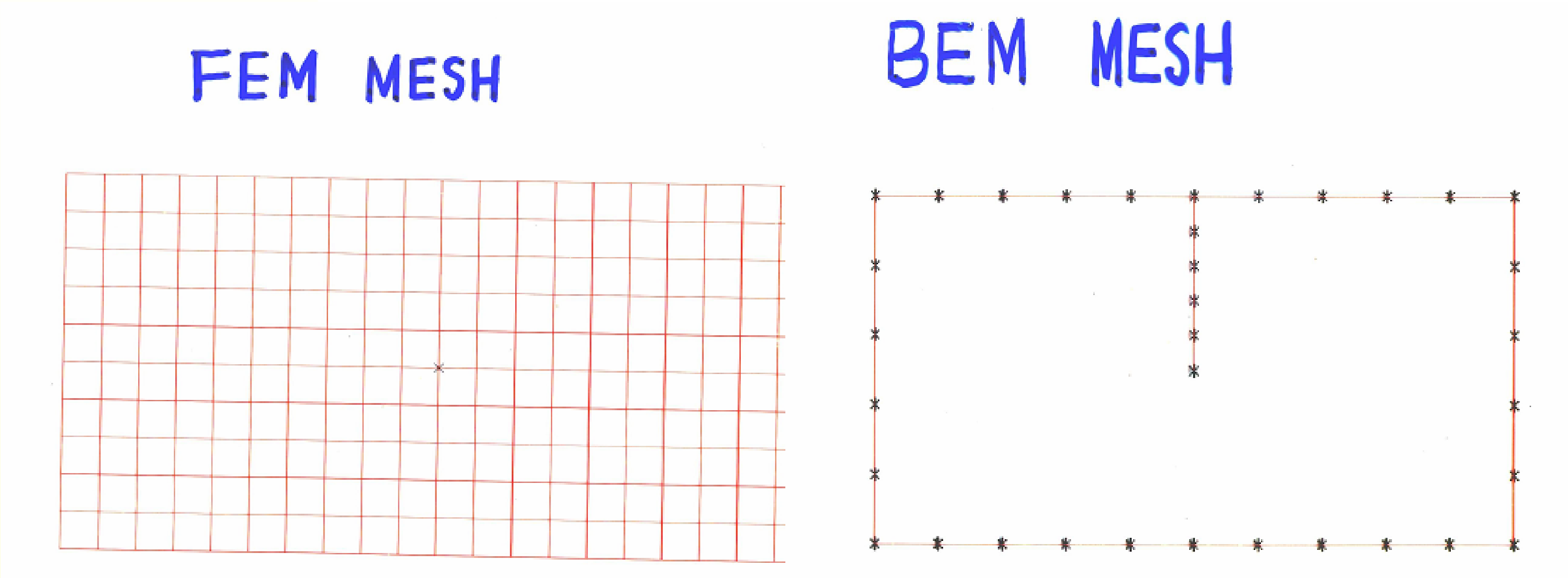
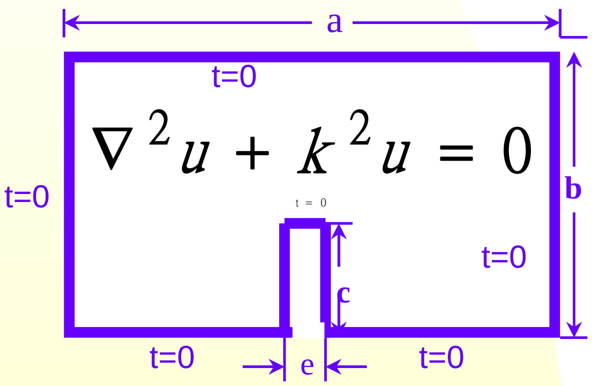


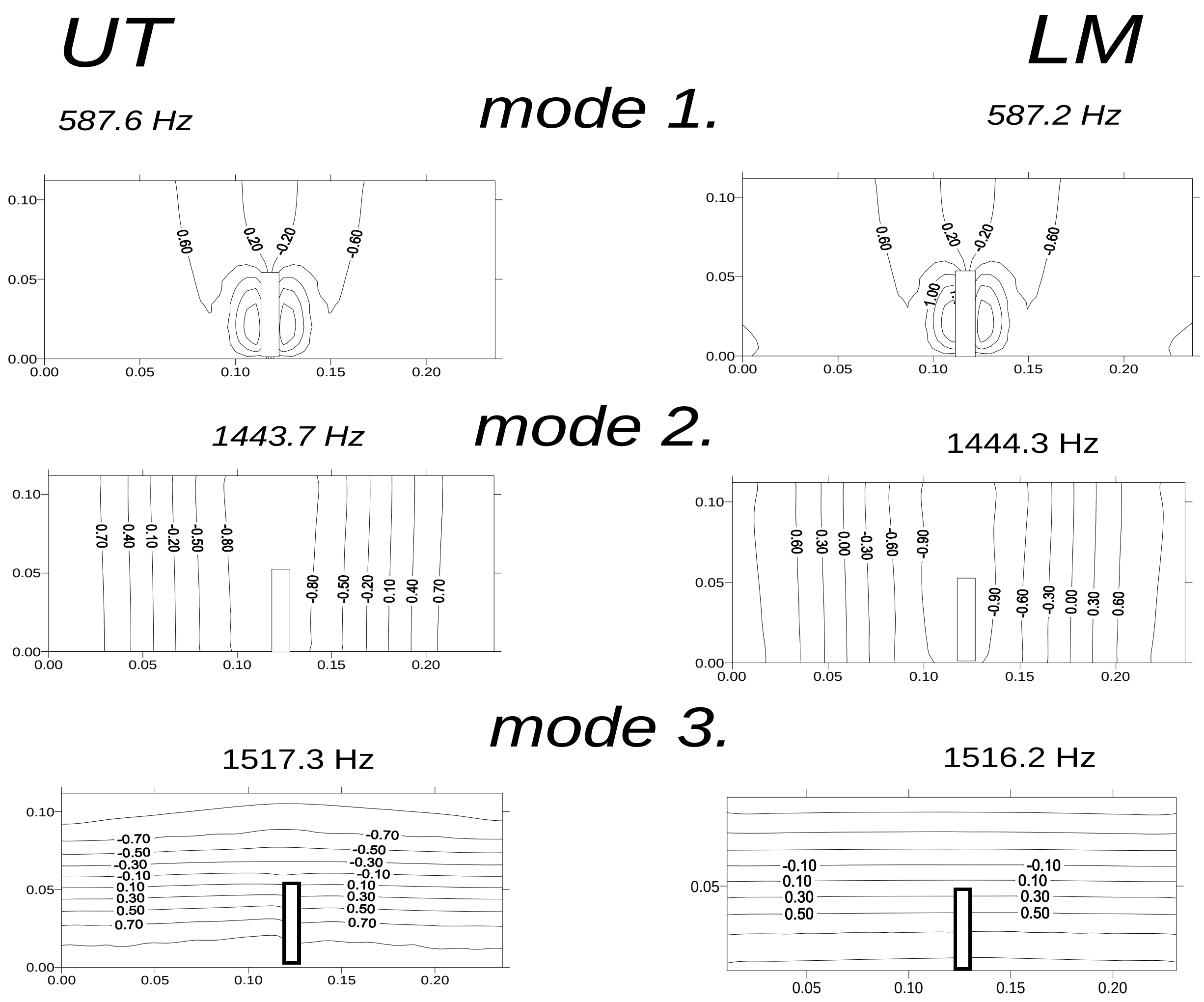
Fig.4 Flowchart of BEM and FEM solver system.

Meshes of FEM and BEM



Screen in acoustics

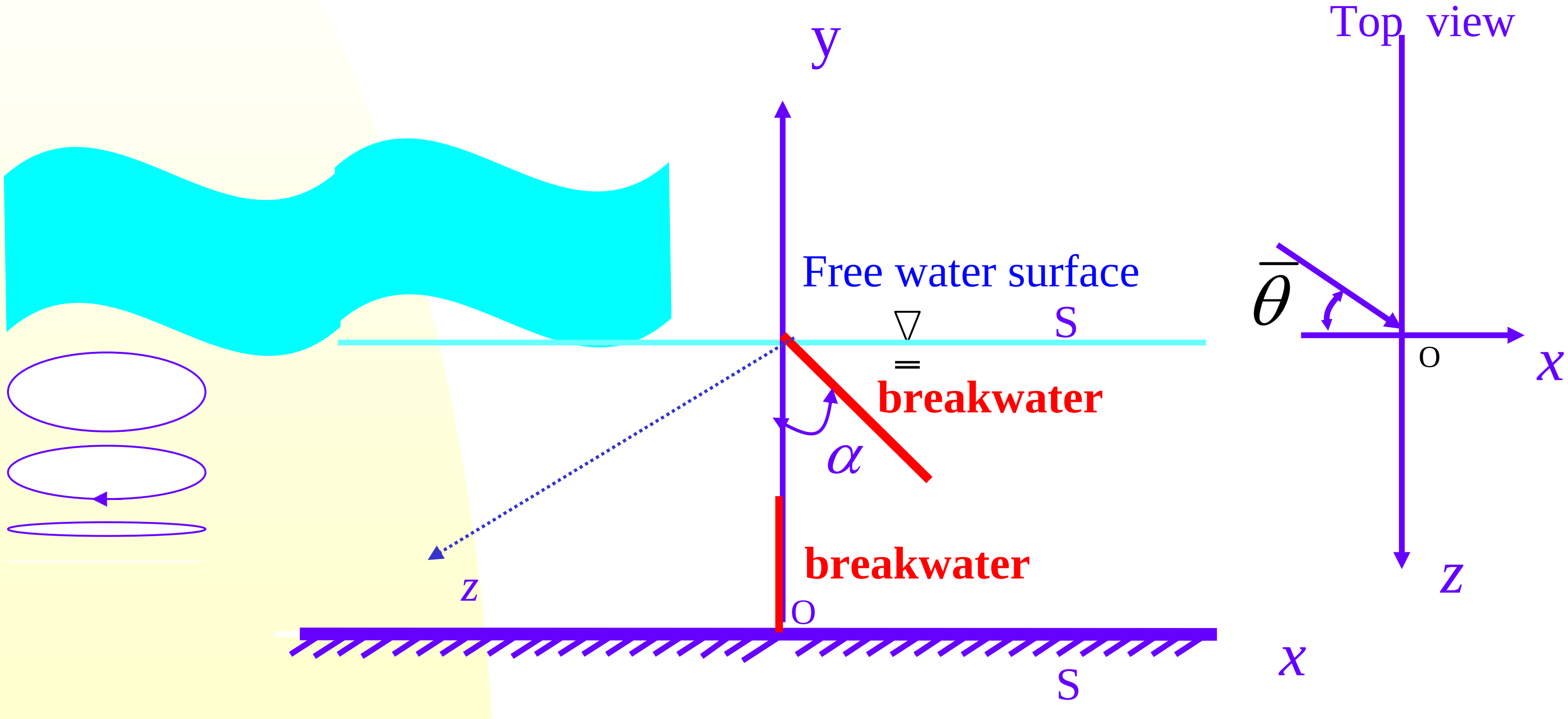




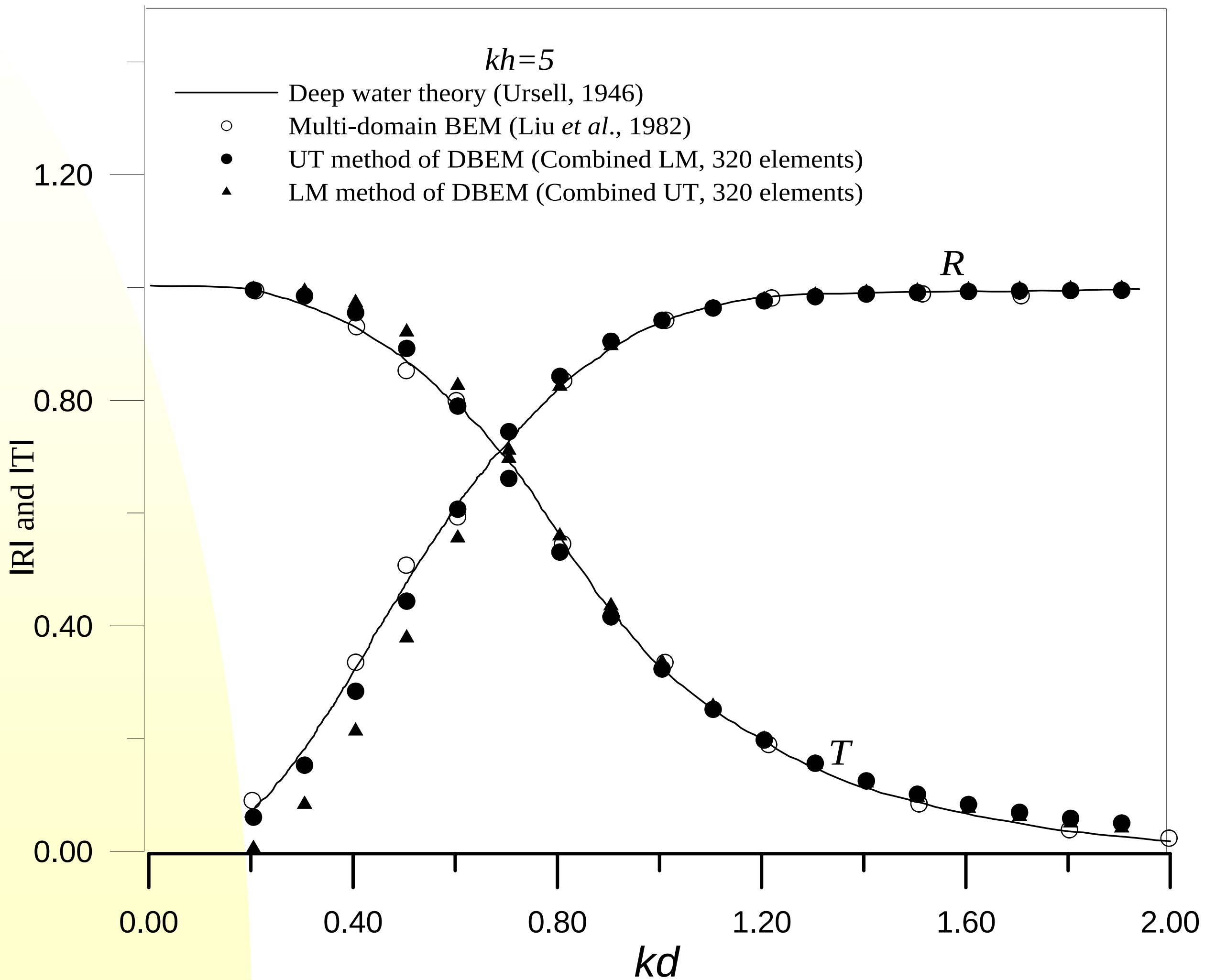
Water wave problem

$$\nabla^2 u(\tilde{x}) - \lambda^2 u(\tilde{x}) = 0$$

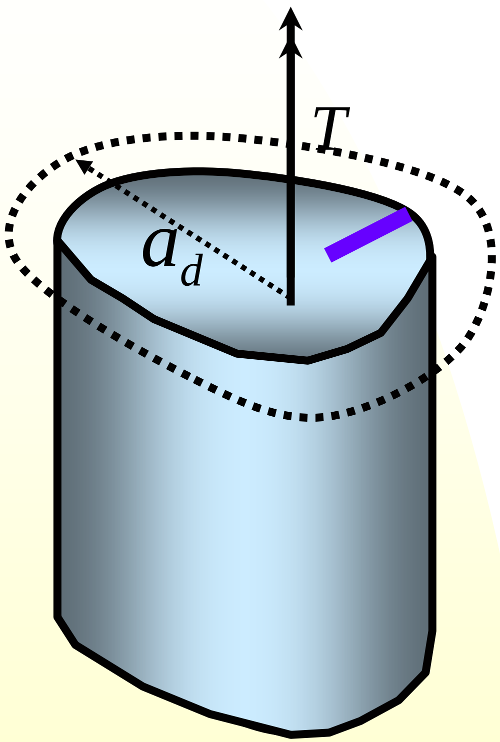
oblique incident water wave

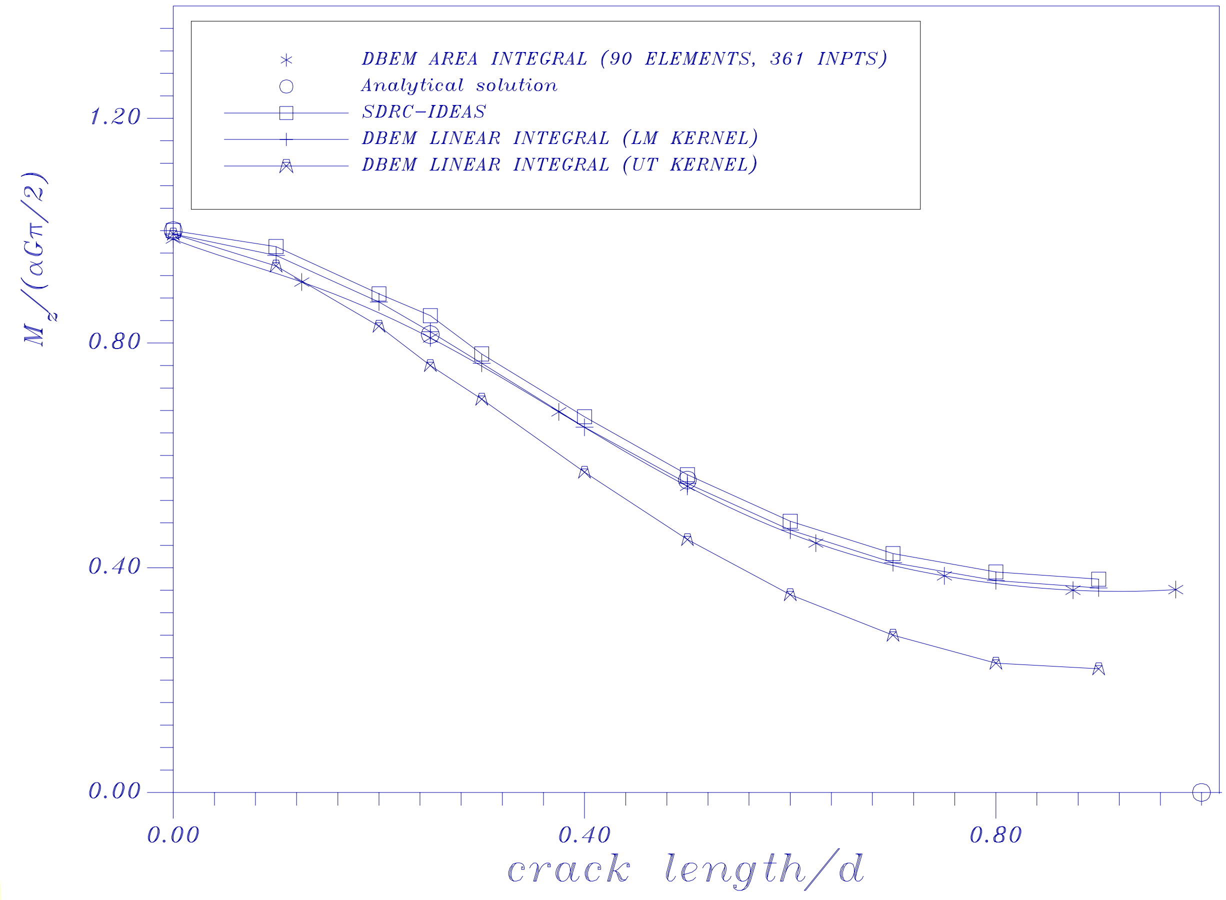


Reflection and Transmission

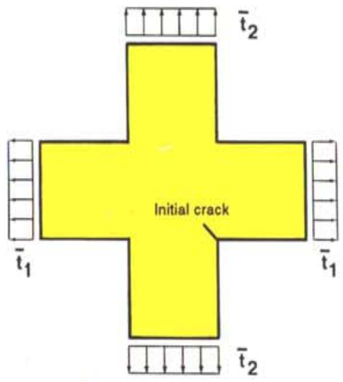


Cracked torsion bar

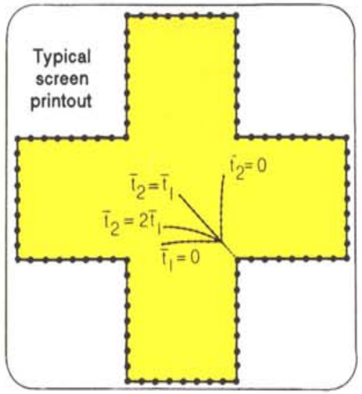




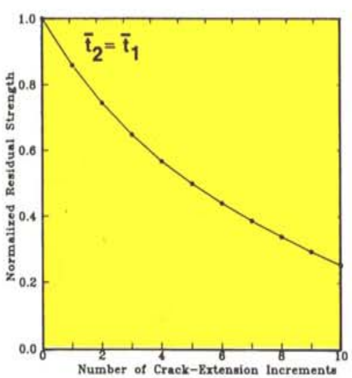
Fatigue life and residual strength calculations



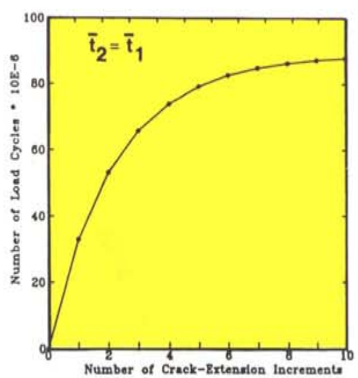
Cruciform cracked plate



Crack paths for the cruciform cracked plate



Residual strength diagram



Fatigue life diagram

COMPUTATIONAL MECHANICS PUBLICATIONS

ORDERS

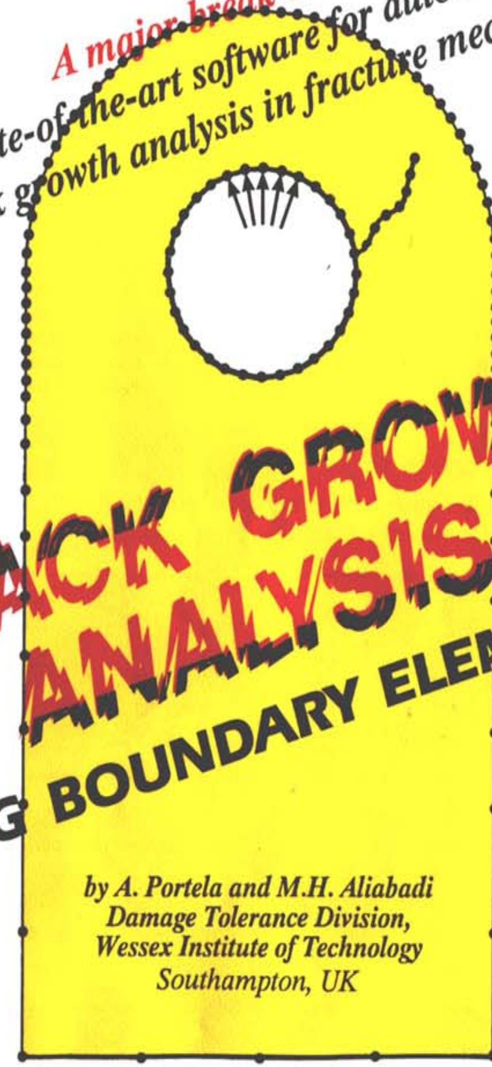
25, Bridge Street,
Billerica MA01821,
U.S.A.
Tel: 508 667 5841
Fax: 508 667 7582

Crack Growth Analysis using Boundary Elements - Software

ISBN: 1 85312 186 X ringbinder/diskette/50 page manual/Topics book. Price: £675/$995

A major break-through

state-of-the-art software for automatic crack growth analysis in fracture mechanics



CRACK GROWTH ANALYSIS USING BOUNDARY ELEMENTS

by A. Portela and M.H. Aliabadi
Damage Tolerance Division,
Wessex Institute of Technology
Southampton, UK

COMPUTATIONAL MECHANICS PUBLICATIONS

CMP

Computational Mechanics Inc.,
25, Bridge Street,
Billerica MA01821,
U.S.A.
Tel: 508 667 5841
Fax: 508 667 7582

Crack Growth Analysis

There are many Finite Element software packages for crack growth analysis currently available. However, they all have a common drawback, which is the requirement for remeshing as the crack propagates. This software utilizes the state-of-the-art development in the boundary element method and for the first time removes the difficult and time consuming task of remeshing. Furthermore, it evaluates accurate stress intensity factors for which the Boundary Element Method is renowned. The software uses the established criterion for crack propagation and evaluates the residual strength as well as fatigue life calculations.

MAIN FEATURES:

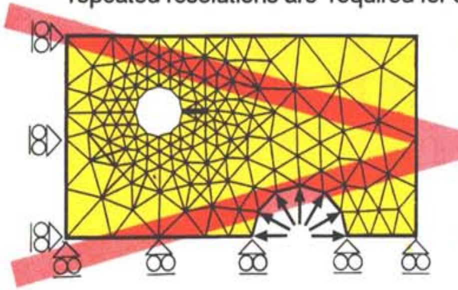
- ☆ Automatic incremental crack propagation
- ☆ Eliminates remeshing for crack growth analysis
- ☆ Accurate evaluation of stress intensity factors
- ☆ Residual strength and fatigue life computations.

MODULES IN THE SOFTWARE:

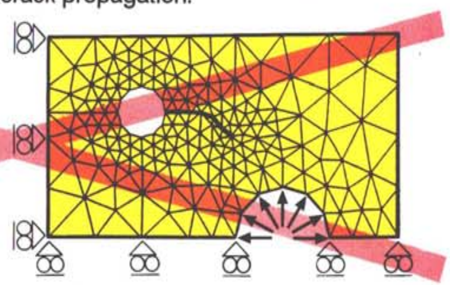
- ☆ Data generation with a minimum of input
- ☆ Plotting of the mesh
- ☆ Automatic fatigue crack growth analysis
- ☆ Plotting of the deformed configuration and principal stresses
- ☆ Plotting of the crack path

The old approach

The Finite Element approach: continuous remeshing and repeated resolutions are required for crack propagation.



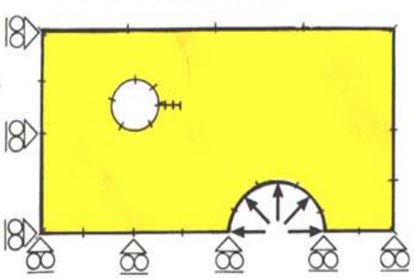
Starting mesh



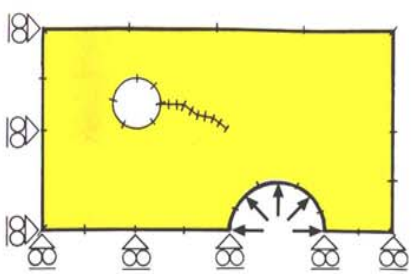
Mesh after three crack propagation increments

The new approach

The Boundary Element approach: No remeshing is required for crack propagation.



Starting Mesh



Mesh after three crack propagation increments

Program Description

The software features include the use of quadratic continuous and discontinuous elements, evaluation of boundary stresses, displacements and tractions, element or point constraint including skew constraints and mixed-mode path independent integrals for the accurate evaluation of stress intensity factors. Automatic crack propagation algorithm is implemented utilizing an incremental crack extension which employs special solver to avoid resolution for each crack extension.

The fracture criterion is based on the maximum principal stress and the fatigue crack growth rates are calculated using established formulae.

The software package is accompanied with a user manual for data generation and the analysis program as well as a book *Boundary Elements in Crack Growth Analysis*- describing the basic theory of the method. The source code in FORTRAN is included along with several example problems to demonstrate the use of the code.

The Boundary Element Method (BEM) is now widely regarded as the most accurate numerical tool for analysis of crack problems in linear elastic fracture mechanics. This software package is based on a new formulation of BEM called Dual Boundary Element Method (DBEM) developed at the Damage Tolerance Division of Wessex Institute of Technology. The Dual Boundary Element Method retains all of the important features of BEM which are: reduced set of equations, simple data preparation, accurate evaluation of stresses, strains and displacements at selected internal points as well as introducing additional improvements which include crack modelling in a single region and accurate stress intensity factors evaluation.

ORDER FORM

Please send me the following software package

Quantity	*Title/Author*	*Price*
	Crack Growth Analysis using Boundary Elements *by A. Portela and M.H. Aliabadi*	*£675**

**$995 for USA, Canada and Mexico - postage & packing UK £4/$7, USA £5/$9.*

Name __________

Organisation __________

Position __________

Address __________

Please indicate method of payment __________

Cheque number __________

From Portela

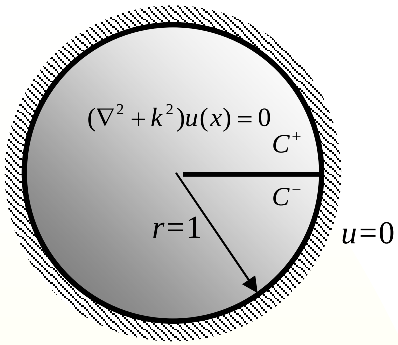
I wish to pay by Credit card ☐ *Expiry date* __________

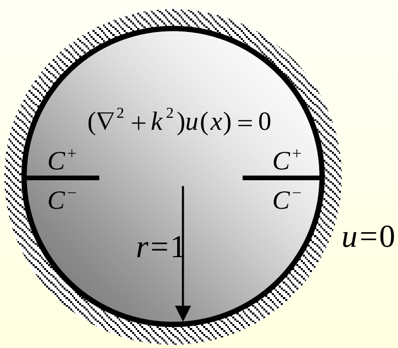
Name __________

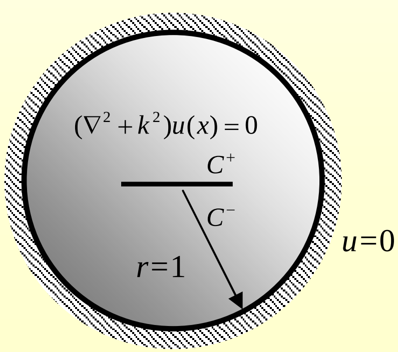
Number __________

Nov. 1993

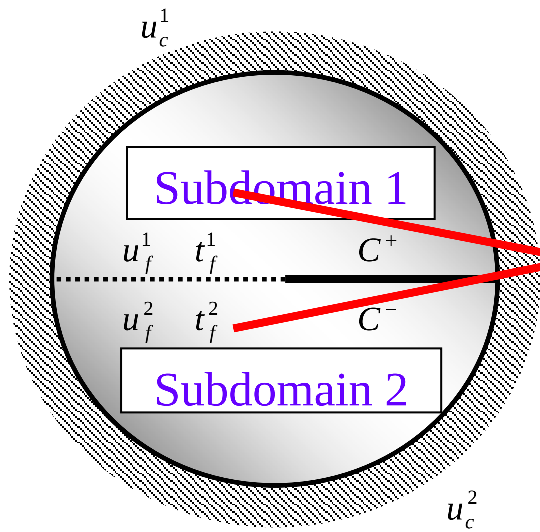
Degenerate boundary problems

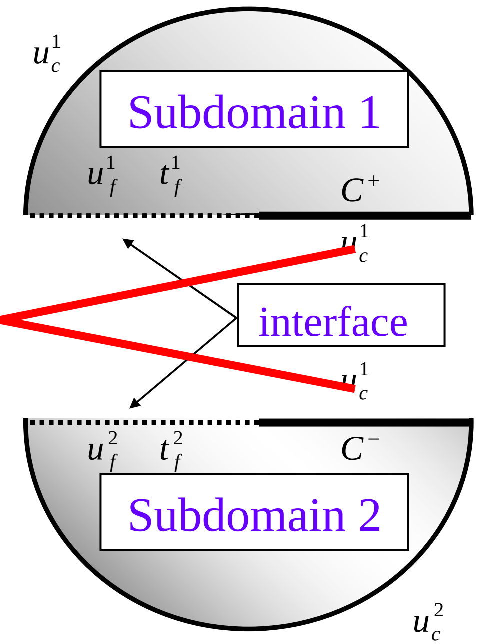






- Multi-domain BEM





- Dual BEM

$$[T]\{u\}=[U]\{t\}$$

$$[M]\{u\}=\{L\}\{t\}$$

Conventional BEM in conjunction with SVD

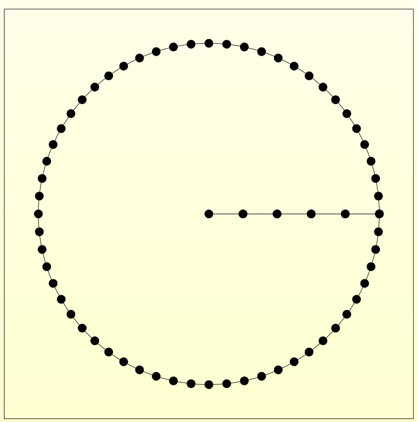
Singular Value Decomposition

$$[U]_{M\times P} = [\Phi]_{M\times M}[\Sigma]_{M\times P}[\Psi]^{H}_{P\times P}$$

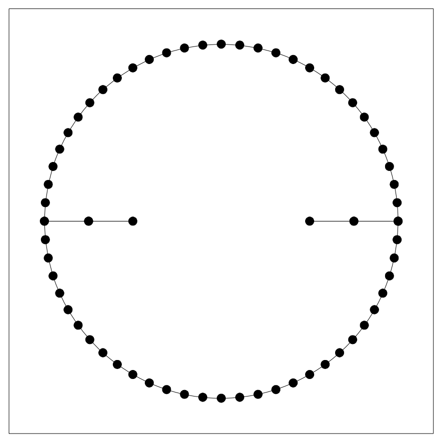
Rank deficiency originates from two sources:

(1). Degenerate boundary

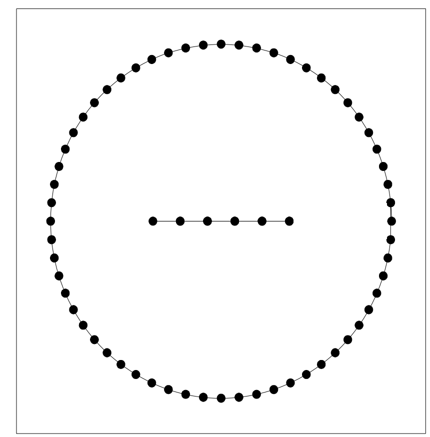
(2). Nontrivial eigensolution



$N_d=5$



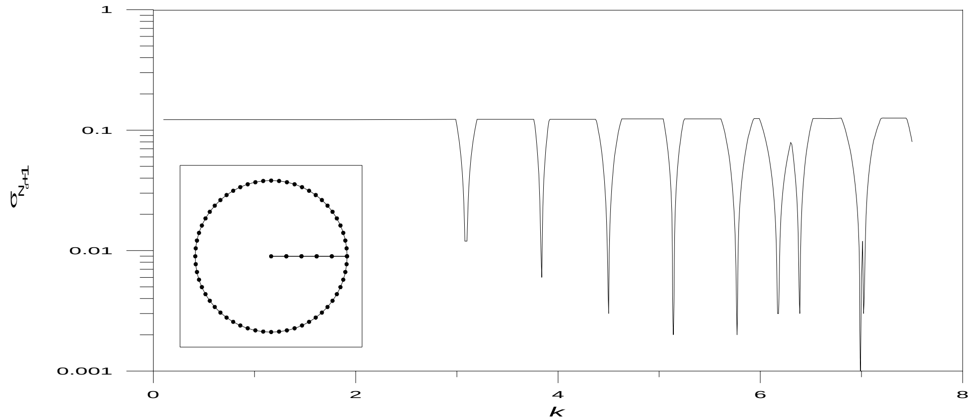
$N_d=4$



$N_d=5$

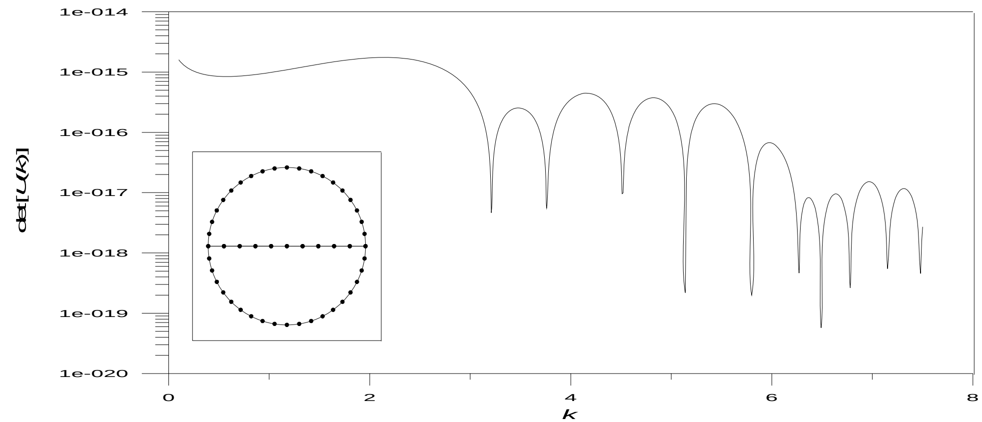
- *UT* BEM + SVD (Present method)

σ_{N_d+1} versus k



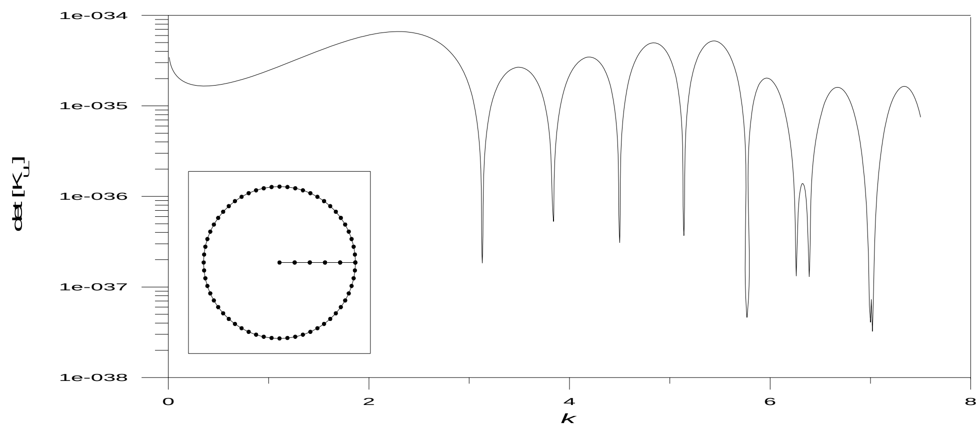
- Multi-domain BEM

Determinant versus k

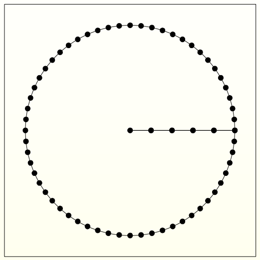


- Dual BEM

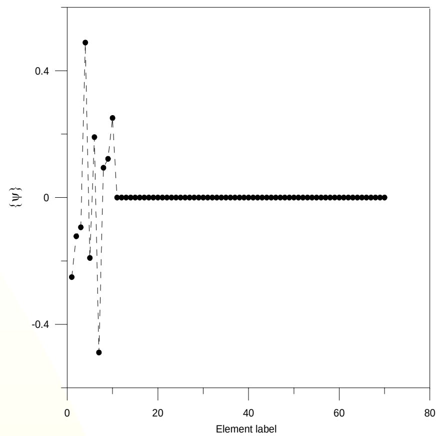
Determinant versus k



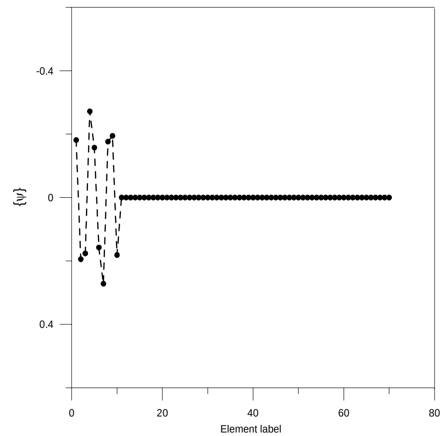
Two sources of rank deficiency (*k*=3.09)



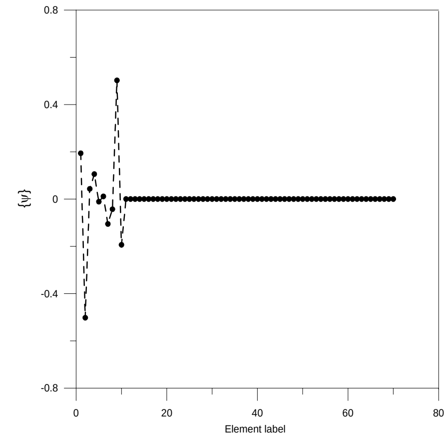
N_d=5



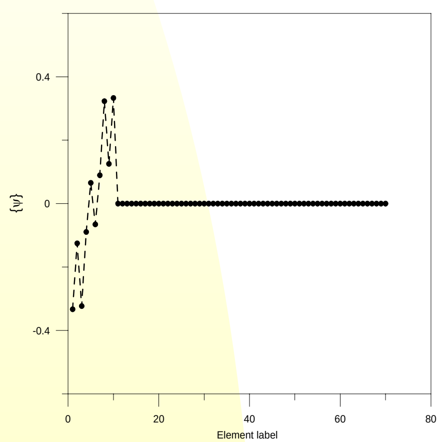
$\{\psi_1\},\ (\sigma_1 = 0)$



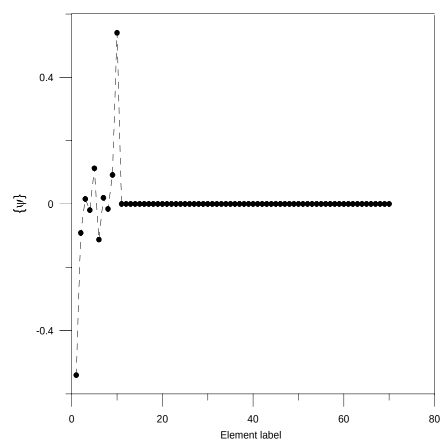
$\{\psi_2\},\ (\sigma_2 = 0)$



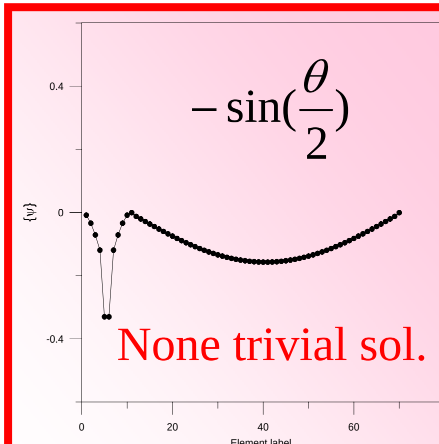
$\{\psi_3\},\ (\sigma_3 = 0)$



$\{\psi_4\},\ (\sigma_4 = 0)$



$\{\psi_5\},\ (\sigma_5 = 0)$

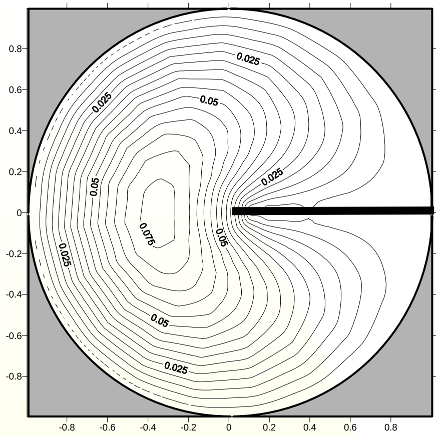


$\{\psi_6\},\ (\sigma_6 = 0)$

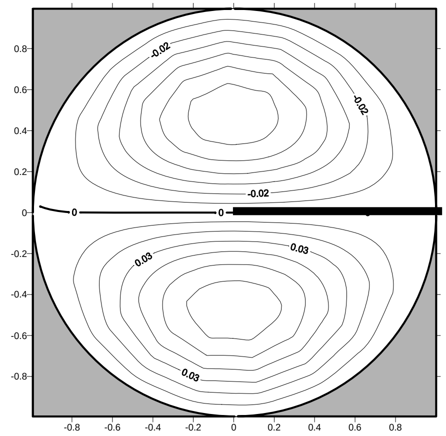
Degenerate boundary

Eigensolution

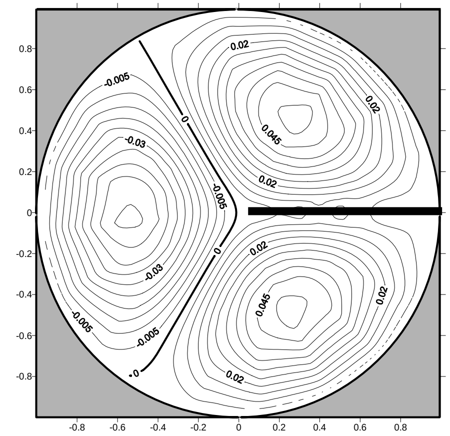
UT BEM+SVD



***k*=3.09**

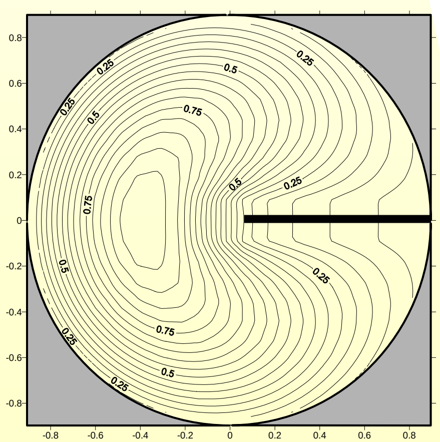


***k*=3.84**

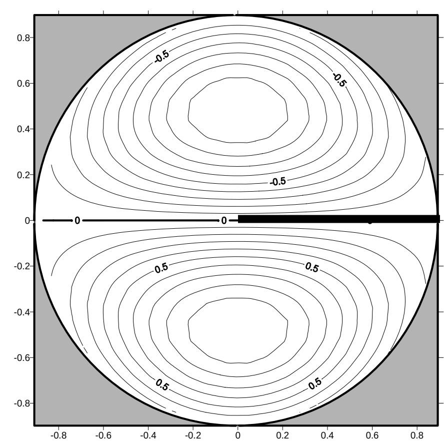


***k*=4.50**

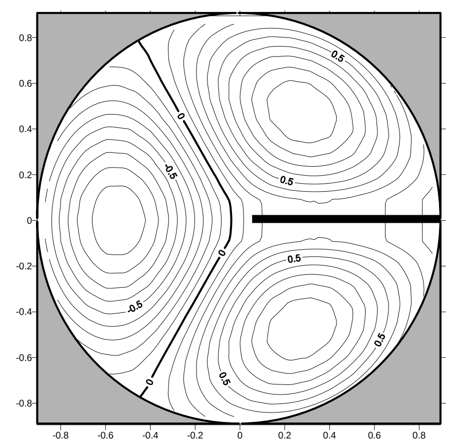
FEM (ABAQUS)



***k*=3.14**



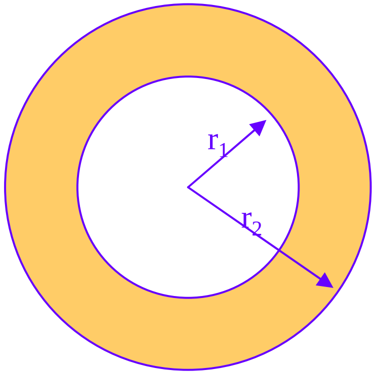
***k*=3.82**



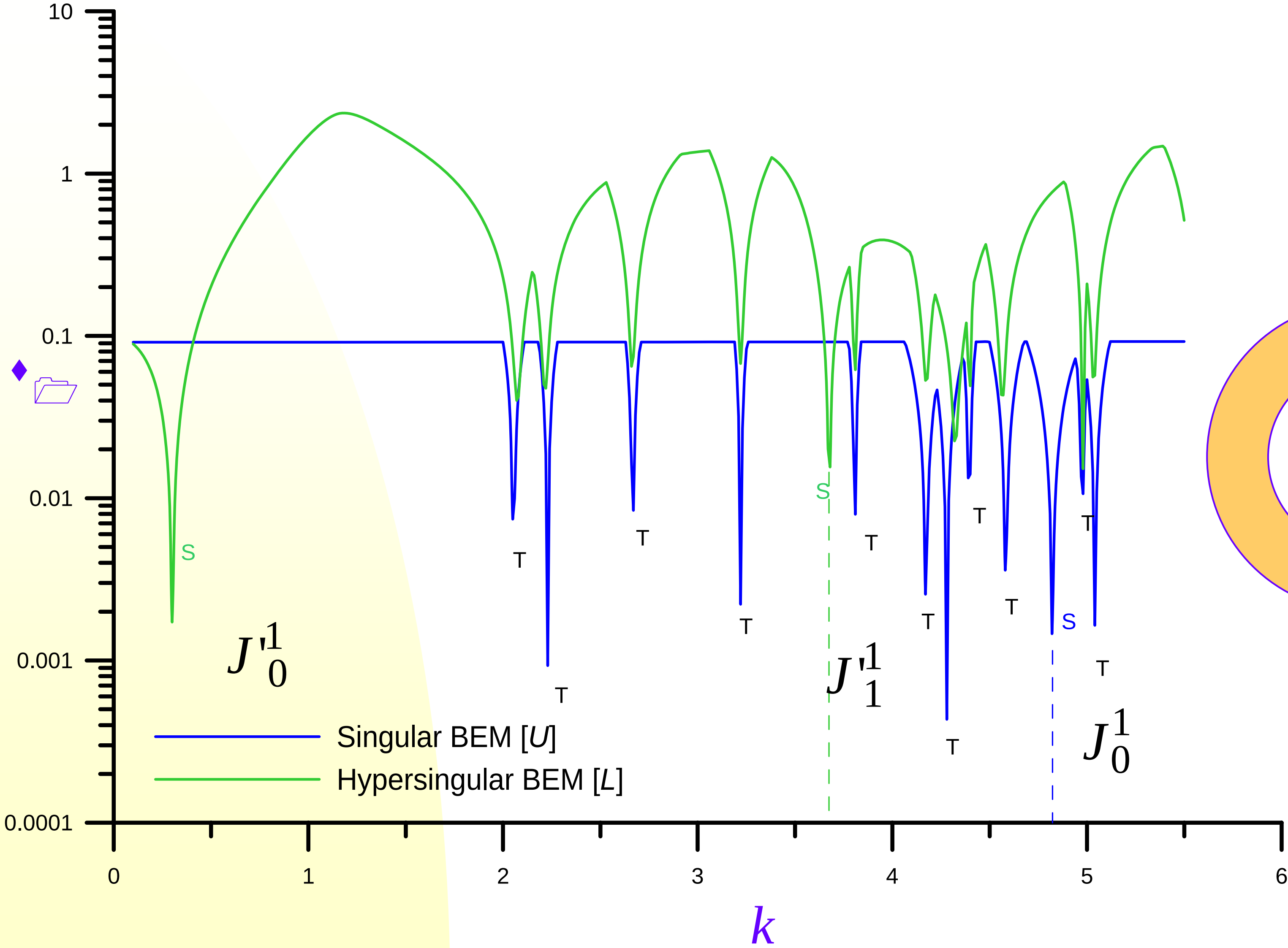
***k*=4.48**

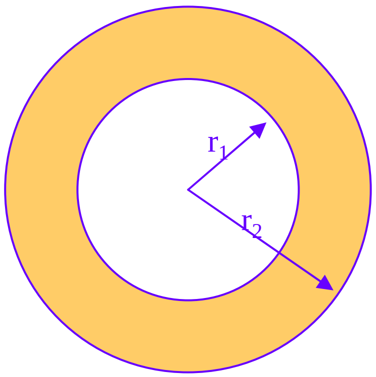
Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner



Spurious eigenvalue of membrane

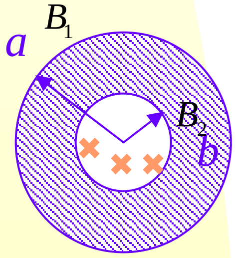




r_1=0.5 m

r_2=2.0 m

Treatments

SVD updating term	$[C] = \begin{bmatrix} U \\ L \end{bmatrix}$
Burton & Miller method	$[U] + i[L]$
CHIEF method 	$[C^*] = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{c1} & U_{c2} \end{bmatrix}_{(4N+N_c)\times 4N}$

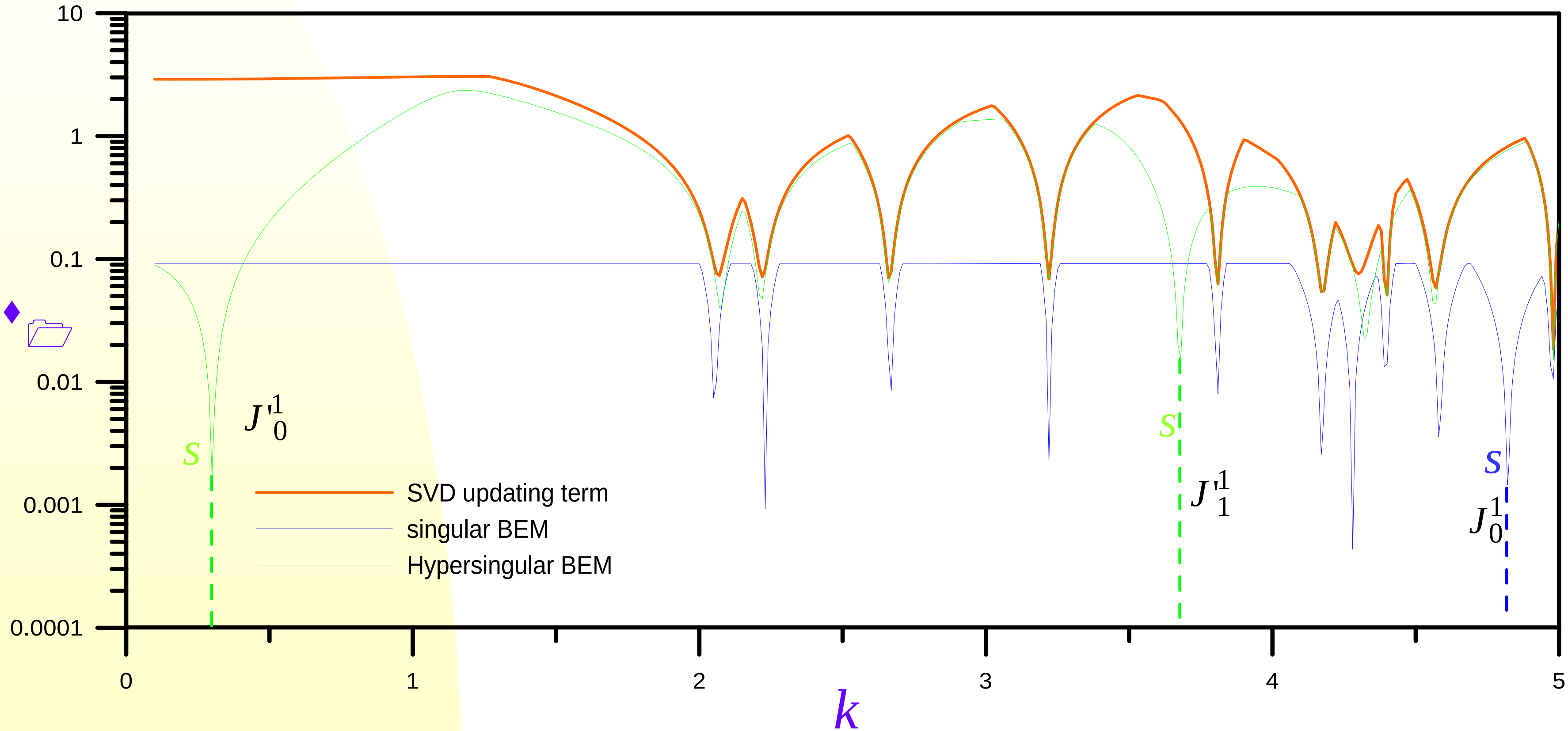
SVD updating term for true eigenvalue

$$[U]\{t\}=\{0\}$$
$$[L]\{t\}=\{0\}$$

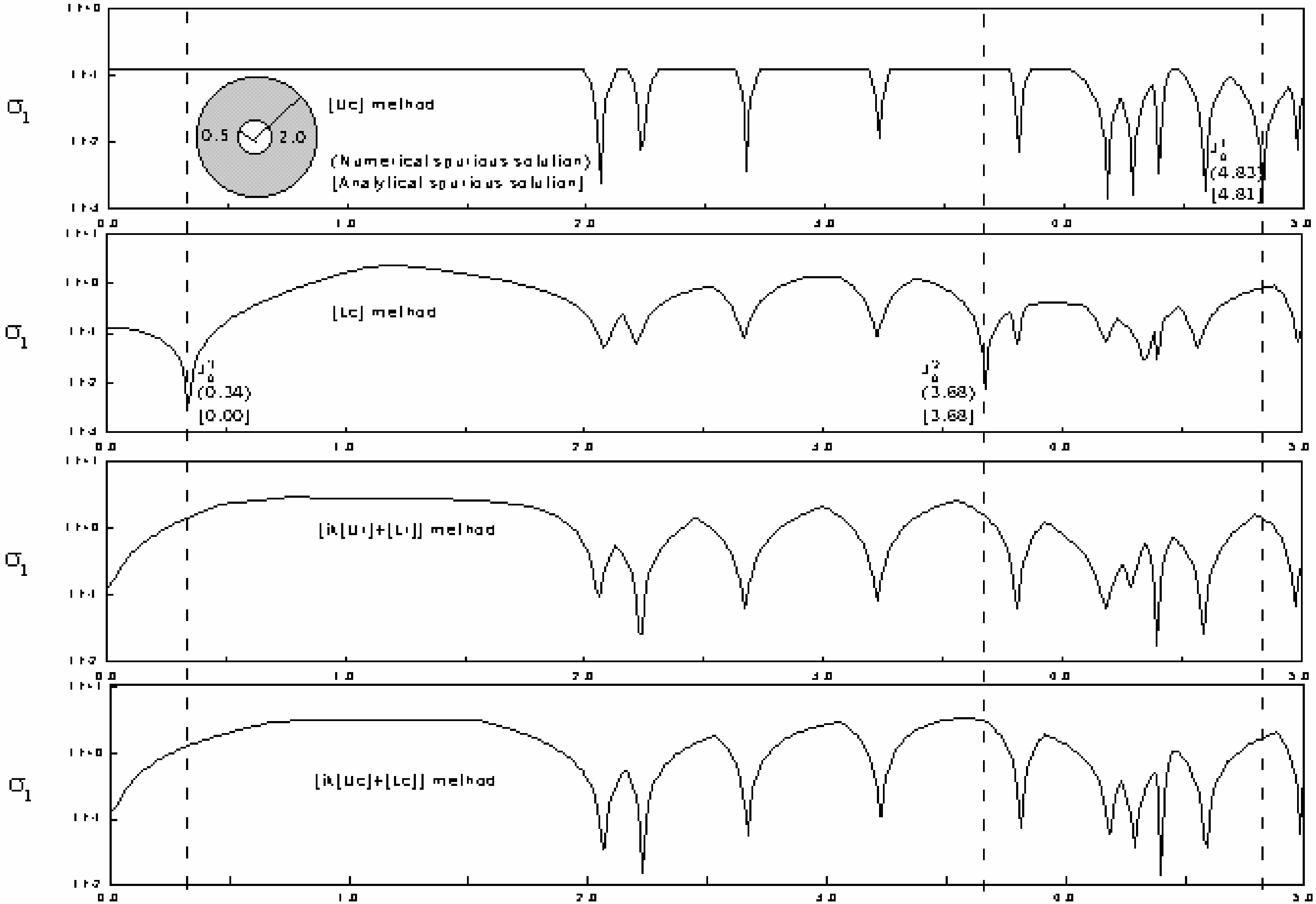
$\Rightarrow$

$$\begin{bmatrix} U \\ L \end{bmatrix}\{t\}=\{0\}$$

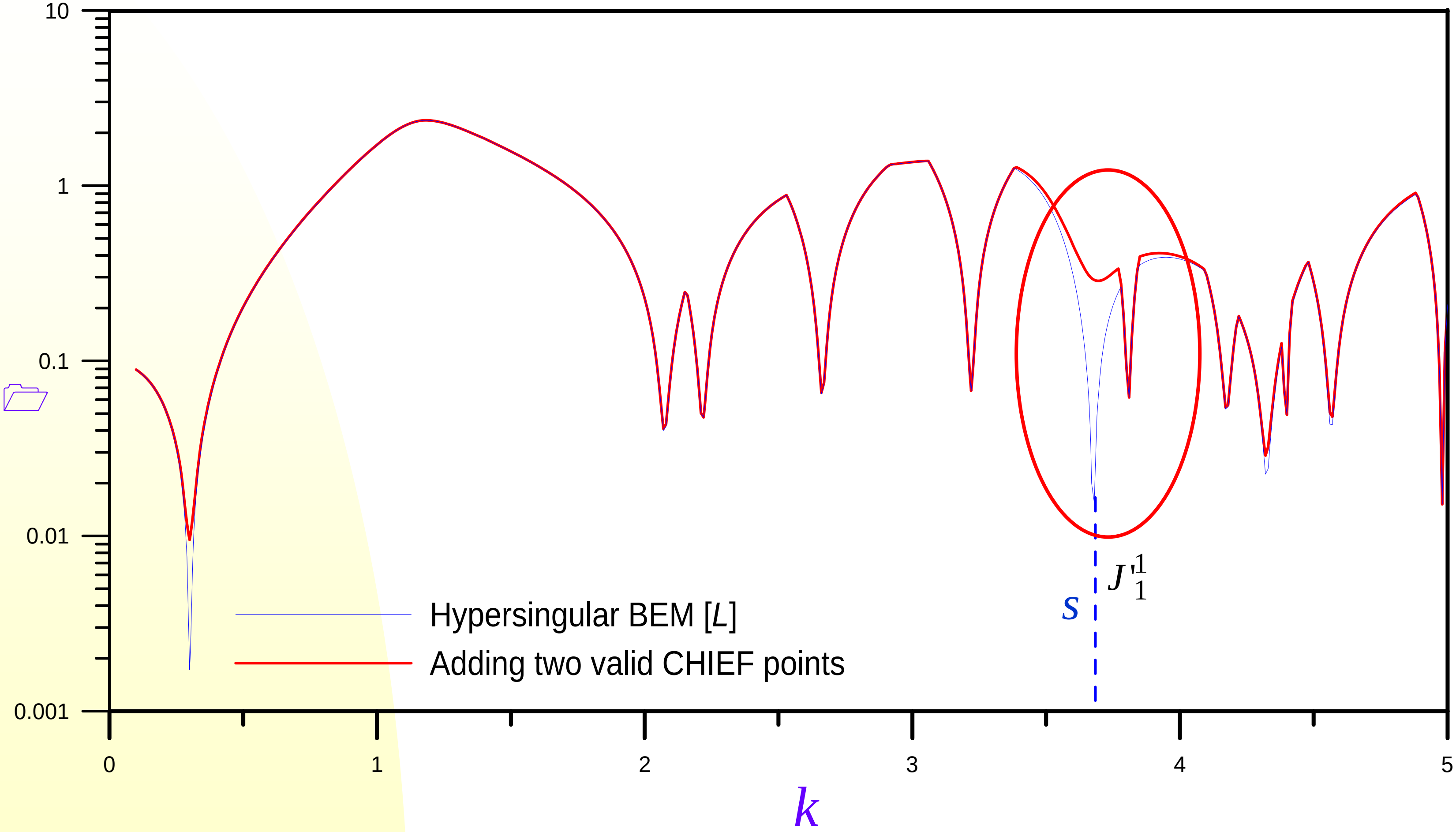
$\Rightarrow$ True eigenvalues



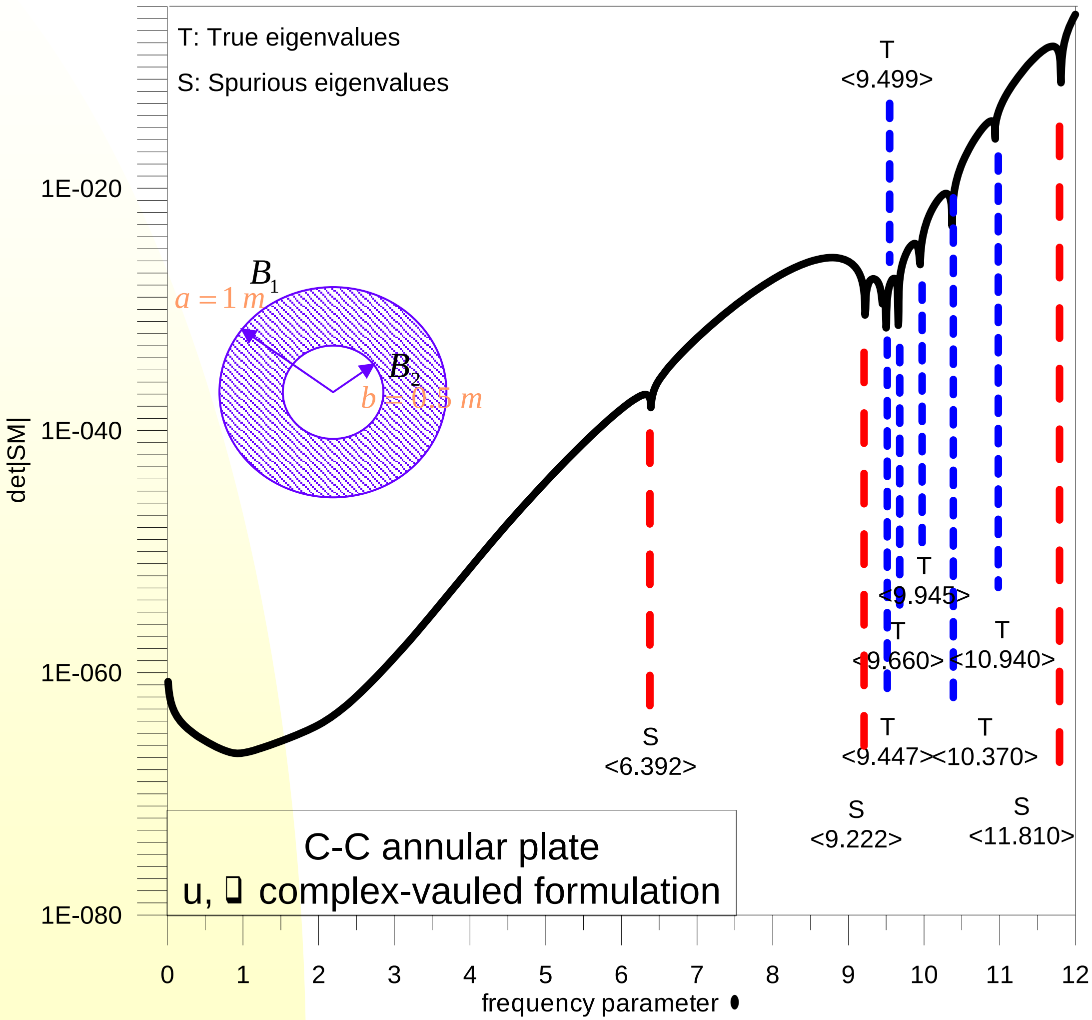
Burton & Miller method



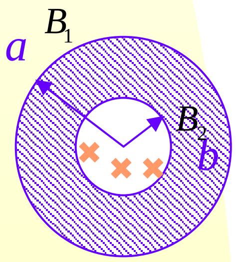
Two CHIEF points for spurious eigenvalues of multiplicity two



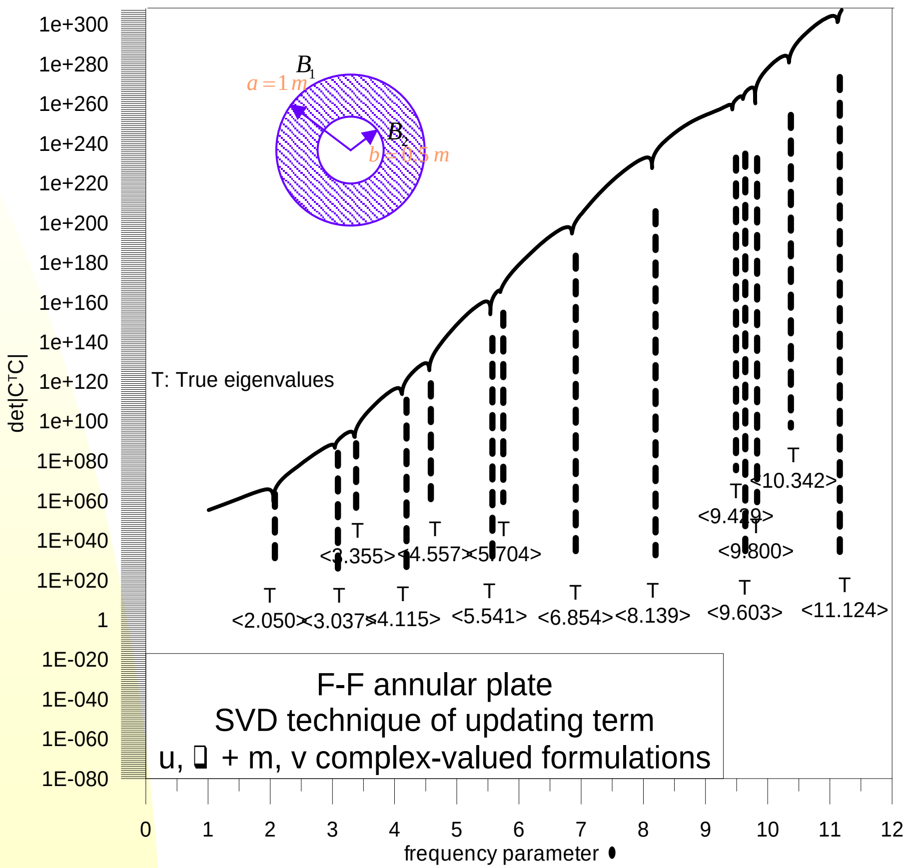
Spurious eigenvalue of plate



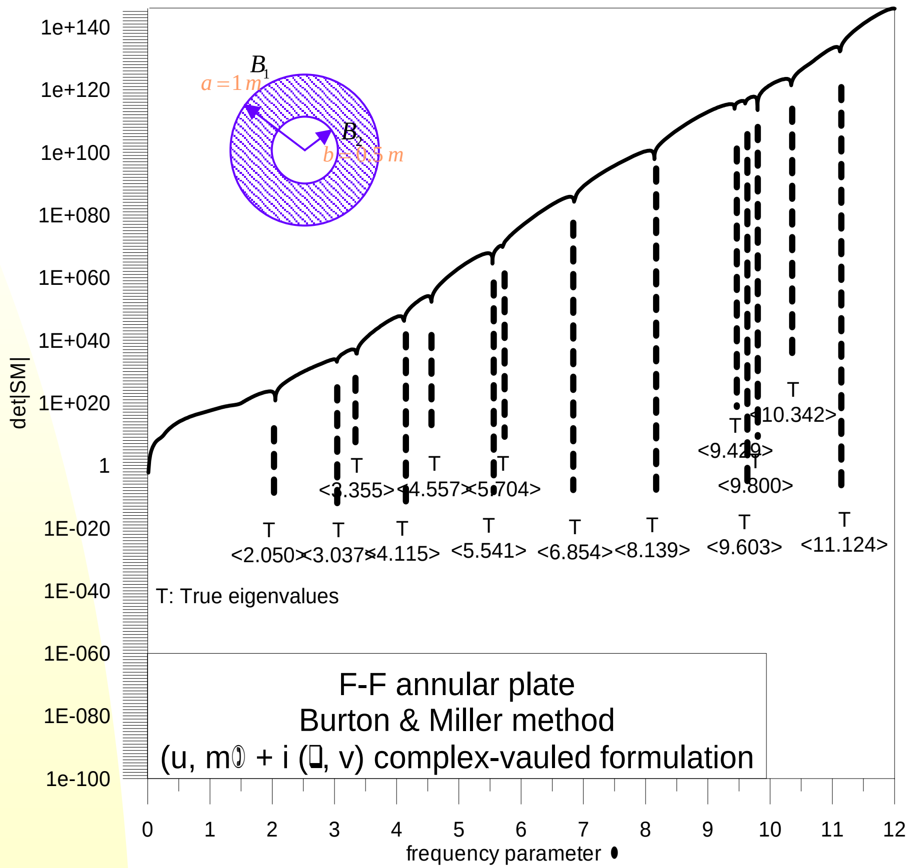
Treatments

SVD updating term	$[C]=\begin{bmatrix} SM_1^{cc} \\ SM_2^{cc} \end{bmatrix}_{16N\times 8N}$
Burton & Miller method	$[SM_1^{cc}]+i[SM_2^{cc}]$
CHIEF method 	$[C^*]=\begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \\ UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{2(4N+N_c)\times 8N}$

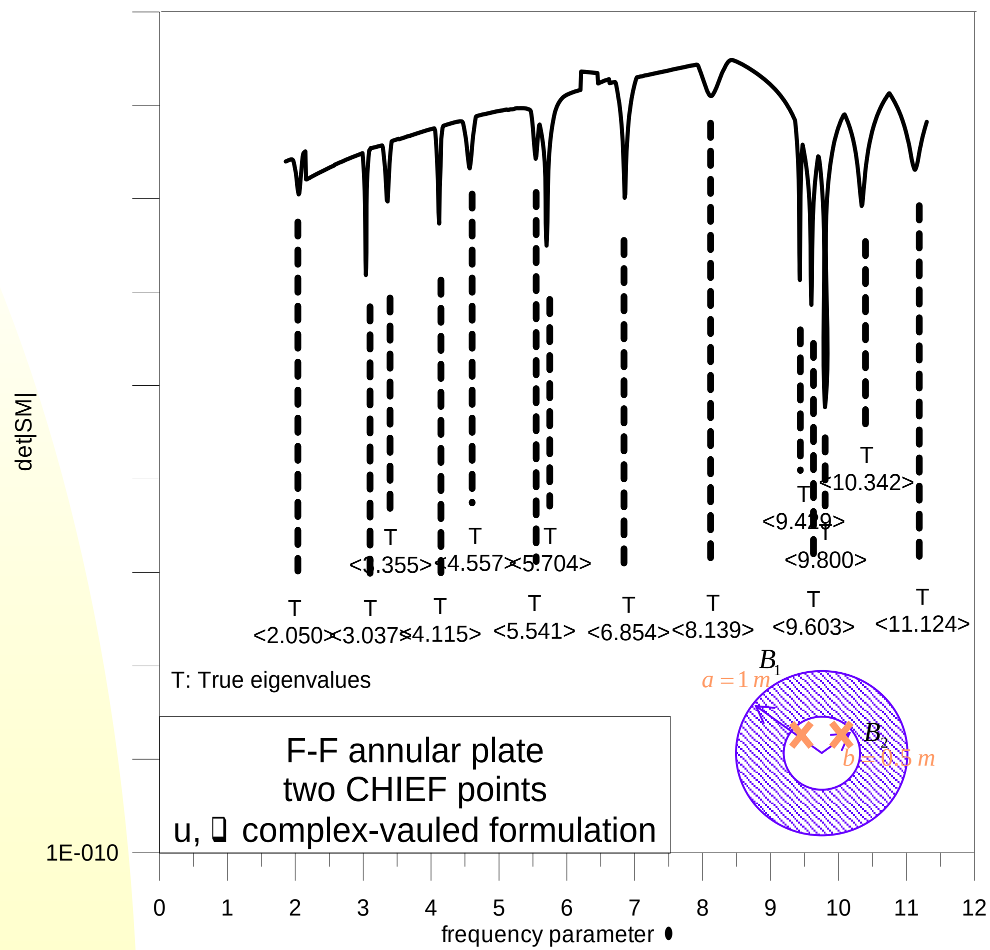
SVD updating term



The Burton & Miller concept

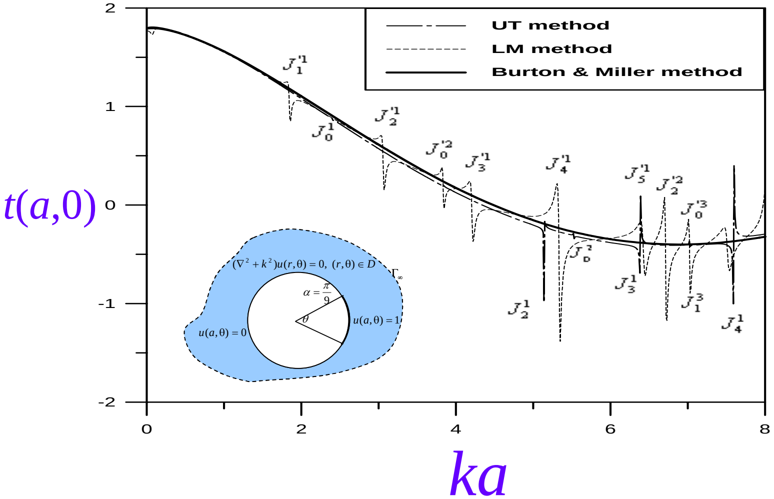


The CHIEF concept



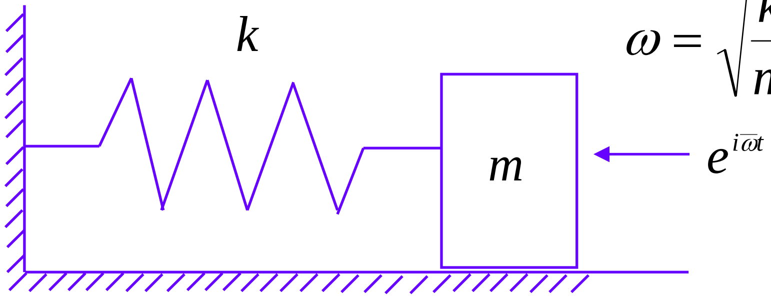
Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner



On the Mechanism of Fictitious Eigenvalues in Direct and Indirect BEM

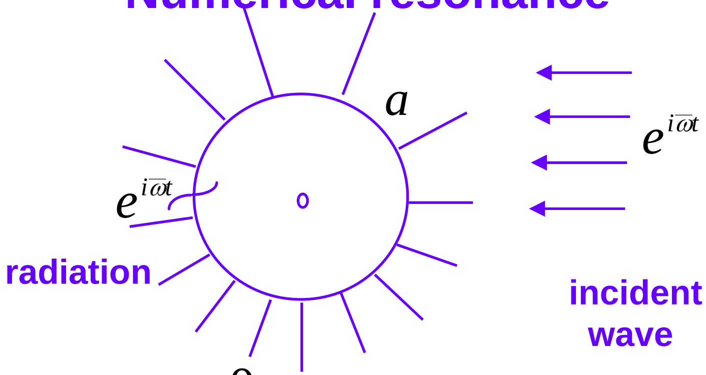
Physical resonance



$$\omega = \sqrt{\frac{k}{m}}$$

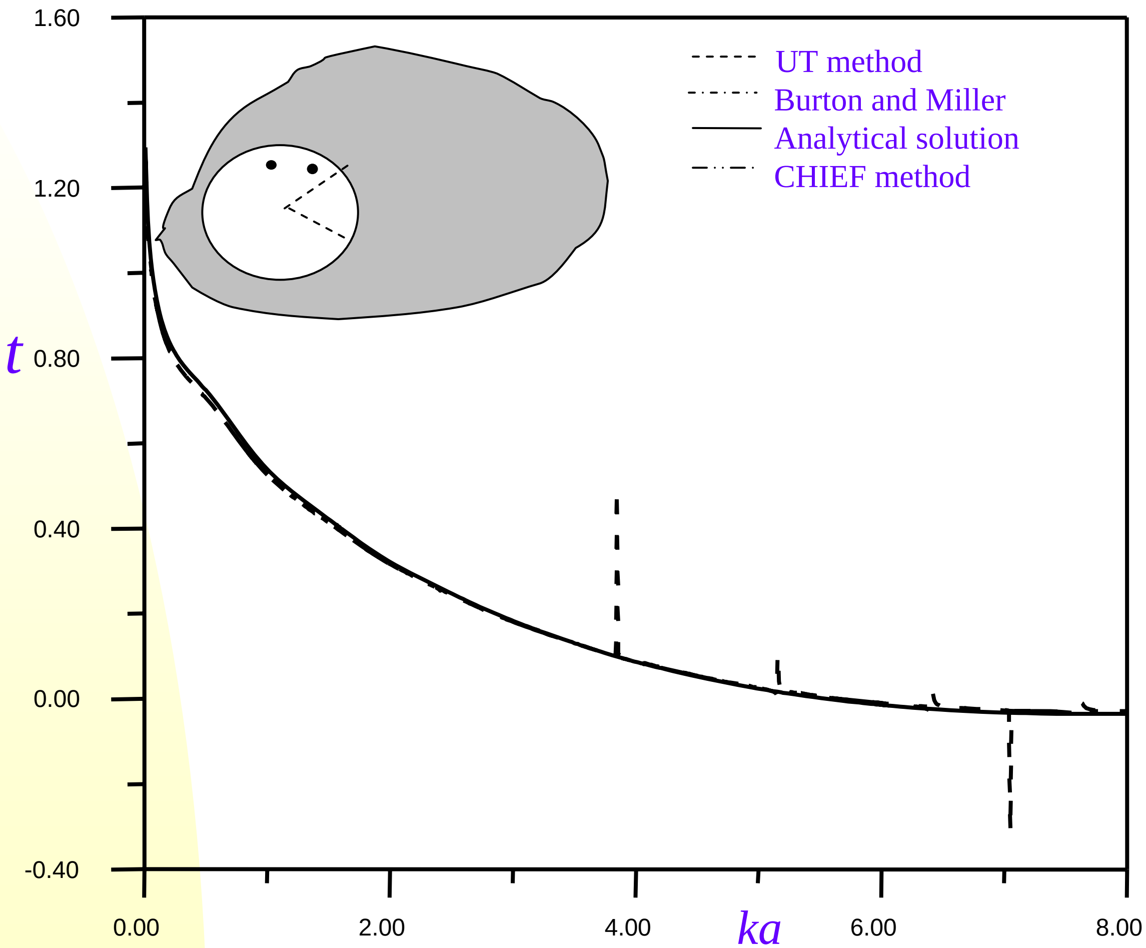
$$u = \frac{finite}{(\omega^2 - \bar{\omega}^2)} \to \infty, \text{ if } \bar{\omega} \to \omega$$

Numerical resonance



$$u = \lim_{\bar{\omega} \to \omega} \frac{0}{0} \to finite, \text{ if } \bar{\omega} \to \omega$$

CHIEF and Burton & Miller method



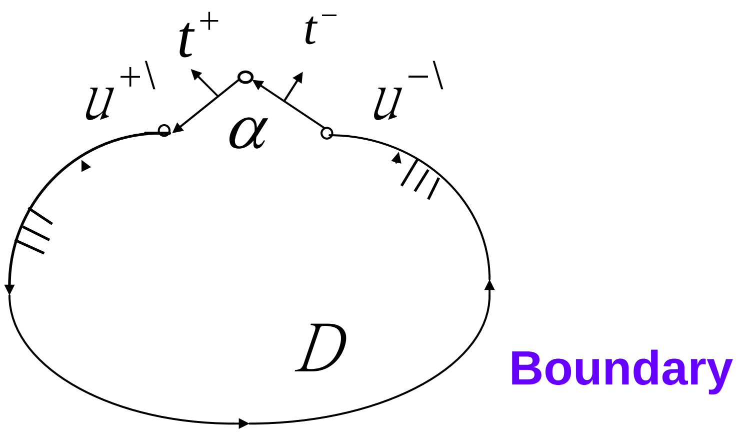
Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner

Theory of Dual Integral Equations for a Corner

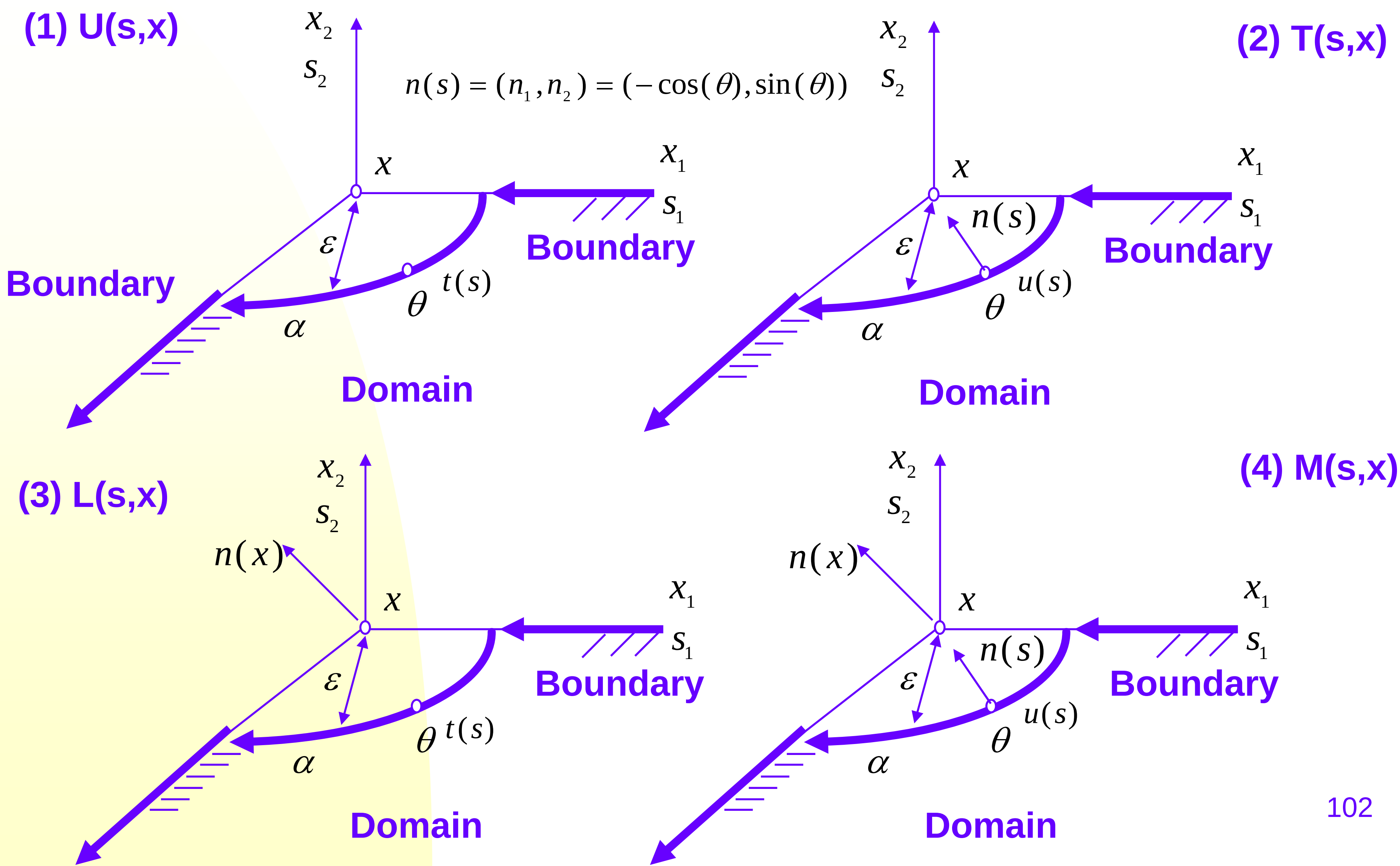
$$\alpha u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)t(s)dB(s), \quad x \in B$$

$$\alpha t^-(x) + \sin(\alpha)t^+(x) = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)t(s)dB(s), \quad x \in B$$



The related symbols around the corner

$$t(s) = -\frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta) \qquad u(s) = u(x) + \frac{\partial u}{\partial x}\varepsilon\cos(\theta) - \frac{\partial u}{\partial y}\varepsilon\sin(\theta)$$



Free terms

kernel	Laplace equation	Helmholtz equation
$U(s,x)$	$\varepsilon\ln(\varepsilon)$	$\varepsilon\left[-\frac{i\pi}{2}H_0^{(1)}(k\varepsilon)(t^{+}+t^{-})\right]$
$T(s,x)$	$-\alpha u(x)+\varepsilon(t^{+}+t^{-})$	$-\alpha u(x)+\varepsilon(t^{+}+t^{-})$
$L(s,x)$	$\frac{(-\sin(2\alpha)+2\alpha)}{4}t^{-}(x)+\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}$	$\frac{(-\sin(2\alpha)+2\alpha)}{4}t^{-}(x)+\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}$
$M(s,x)$	$-\frac{(-\sin(2\alpha)+2\alpha)}{4}t^{-}(x)$ $-\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}+B(\varepsilon)$	$-\frac{(-\sin(2\alpha)+2\alpha)}{4}t^{-}(x)$ $-\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}+B(\varepsilon)$

Conclusions

- Introduction of BEM and BIEM
- The nonuniqueness in BIEM and BEM were reviewed and its treatment was addressed.
- The role of hypersingular BIE was examined.
- The numerical problems in the engineering applications using BEM were demonstrated.
- Several mathematical tools, SVD, degenerate kernel, …, were employed to deal with the problems.

The End

Thanks for your kind attention

歡迎參觀海洋大學力學聲響振動實驗室

烘焙雞及捎來伊妹兒

http://ind.ntou.edu.tw/~msvlab/

E-mail: jtchen@mail.ntou.edu.tw