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On the equivalence of the Trefftz method and the method of fundamental solutions for plate problem

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Abstract

In this paper, it is proved that the two approaches for biharmonic equation, known in the literature as the method of fundamental solutions (MFS) and the Trefftz method, are mathematically equivalent in spite of their essentially minor and apparent differences in the formulation. In deriving the equivalence of the Trefftz method and the MFS for plate problem, it is interesting to find that the T-complete set in the Trefftz method for 1-D, 2-D, 3-D Laplace and Helmholtz problems are imbedded in the degenerate kernels of the MFS. The unknown coefficients of each method for plate problems correlate by a mapping matrix after considering the degenerate kernels for the fundamental solutions in the MFS and the T-complete function in the Trefftz method. The mapping matrix is composed of a rotation matrix and a geometric matrix which depends on the source location. Also, the occurring mechanism of the degenerate scale for the plate problems is examined in this paper.

Keywords: biharmonic equation, method of fundamental solutions, Trefftz method, T-complete set, degenerate kernel, mapping matrix, degenerate scale

1. Introduction

Since 1926, the Trefftz method was introduced for solving boundary value problems by the superposition of the functions satisfying the governing equation, although various versions of Trefftz method, e.g., direct and indirect formulations have been developed. The unknown coefficients are determined so that the approximate solution matches the boundary condition. Many applications to the Helmholtz equation [6], the Navier equation [10,12] and biharmonic equation [11] were done.

In potential theory, it is well known that the method of fundamental solution (MFS) can solve potential problems when a fundamental solution is known. This method was attributed to Kupradze [12], extensive applications in solving a broad range of problems such as potential problems [6,13], acoustics [15], biharmonic problems [11] have been studied. The MFS can be seen as an indirect boundary element method with concentrated sources. The initial idea is to approximate the solution by a linear combination of fundamental solution with sources located outside the domain of the problem. Moreover, it has certain advantages over BEM, e.g., no singularity and no boundary integrals. It can also be applied to acoustics [6], elasticity [10,12] and plate problems [11].

However, the link between the Trefftz method and the MFS was not discussed in detail to the authors' best knowledge. A similar case to link the DRBEM and the method of particular integral was done by Polyzos *et al.* [16]. In this paper, we will construct the relationship of the two methods through the T-complete functions and the degenerate kernel. Then, we will examine the bases in the two methods for the Laplace and the Helmholtz equations and extend it to biharmonic equation. By designing a biharmonic circular problem, we will prove the mathematical equivalence between the Trefftz method and MFS. Two mathematical tools are required. One is the degenerate kernel for the closed-form fundamental solution, the other is the Fourier series expansion for the boundary density. The occurring mechanism of the degenerate scale [1,2,3,4] using the MFS will be addressed in this paper.

2. On the independent bases in the Trefftz method and the MFS

2.1 Trefftz method

In the Trefftz method, the field solution $u(x)$ is superimposed by the T-complete functions, $u_j(x)$ as follows:

$$u(x) = \sum_{j=1}^{N_T} g_j u_j(x) \tag{1}$$

where N_T is the number of T-complete functions, g_j is the unknown coefficient, $u_j(x)$ is the T-complete function which satisfies the governing equation. The solution of the problem can be approximated by the superposition of the functions satisfying the governing equation.

2.2 Method of fundamental solutions (MFS)

In the method of fundamental solutions, the field solution $u(x)$ is superimposed by $U(x,s_j)$ as follows:

$$u(x) = \sum_{j=1}^{N_M} v_j U(x, s_j), \quad s_j \in D^e \tag{2}$$

where N_M is the number of source points in the MFS, v_j is the unknown coefficient, s and x are the source point and collocation point, respectively, D^e is the complementary domain and $U(x,s_j)$ is the corresponding fundamental solution.

2.3 On the complete set of the Trefftz method and the MFS using the degenerate kernel

By expanding the fundamental solution in the MFS, we have the general form as follows shown in Fig. (1), and

$$U(x,s) = \begin{cases} U^I(x,s) = \sum_j^{\infty} A_j(x) B_j(s), & x \in \Omega^I \\ U^E(x,s) = \sum_j^{\infty} A_j(s) B_j(x), & x \in \Omega^E \end{cases} \tag{3}$$

where the superscripts of "*I*" and "*E*" denote the interior and exterior domains, respectively. It is interesting to find that all the T-complete sets in the Trefftz method are imbedded in $A_j(x)$ and $B_j(x)$ for the interior and exterior problems, respectively. To demonstrate this point, we summarize the T-complete sets in the Trefftz method and degenerate kernels for MFS in Table 1 for 1-D, 2-D and 3-D Laplace and Helmholtz problems.

3. Connection between the Trefftz method and the MFS for plate problem

3.1 The statements of the problem

Consider a clamped plate of radius a under uniformly distributed load $w(x)$ as shown in Fig.(2), the governing equation is:

$$\nabla^4 u(x) = \frac{w(x)}{D}, \quad x \in \Omega \tag{4}$$

where $u(x)$ is the deflection of the circular plate, D is the flexure rigidity of the plate, Ω is the domain of interest. For simplicity, we set $w(x)$ is constant w. For the clamped case, the boundary condition is

$$u(x) = 0, \quad \theta(x) = 0, \quad x \in B \tag{5}$$

where B is the boundary of the domain. Since Eq.(4) contains the body source term, the problem can be reformulated as

$$\nabla^4 u^*(x) = 0, \quad x \in \Omega \tag{6}$$

and the boundary condition is changed to

$$u^*(x) = \frac{-wa^4}{64D}, \quad \theta^*(x) = \frac{-wa^3}{16D}, \quad x \in B \cdot \tag{7}$$

This new problem of Eq.(6) subject to essential boundary conditions of Eq.(7) is a biharmonic equation with the new boundary conditions. For the general form of boundary conditions,

$$u^*(a,\phi) = p_0 + \sum_{m=1}^{\infty} p_m \cos(m\phi) + \sum_{m=1}^{\infty} q_m \sin(m\phi) \cdot \tag{8}$$

$$\frac{\partial u^*(a,\phi)}{\partial n} = r_0 + \sum_{m=1}^{\infty} r_m \cos(m\phi) + \sum_{m=1}^{\infty} s_m \sin(m\phi) \tag{9}$$

we have an analytical solution for biharmonic equation

$$\begin{aligned} u^*(\rho,\phi) = a_0 &+ \sum_{m=1}^{N_T} a_m \rho^m \cos(m\phi) + \sum_{m=1}^{N_T} b_m \rho^m \sin(m\phi) + c_0(\rho^2) \\ &+ \sum_{m=1}^{N_T} c_m \rho^{m+2} \cos(m\phi) + \sum_{m=1}^{N_T} d_m \rho^{m+2} \sin(m\phi) \end{aligned} \tag{10}$$

where

$$a_0 = p_0 - \frac{a}{2} r_0 \tag{11}$$

$$a_m = \frac{m+2}{2} a^{-m} p_m - \frac{1}{2} a^{1-m} r_n, \quad m = 1, 2, 3, \ldots.. \tag{12}$$

$$b_m = \frac{m+2}{2} a^{-m} q_m - \frac{1}{2} a^{1-m} s_m, \quad m = 1, 2, 3, \ldots.. \tag{13}$$

$$c_0 = \frac{1}{2a} r_0 \tag{14}$$

$$c_m = \frac{-m}{2} a^{-m-2} p_m + \frac{1}{2} a^{-m-1} r_m, \quad m = 1, 2, 3, \ldots.. \tag{15}$$

$$d_m = \frac{-m}{2} a^{-m-2} q_m + \frac{1}{2} a^{-m-1} s_m, \quad m = 1, 2, 3, \ldots.. \tag{16}$$

3.2 Trefftz method

By using the Trefftz method for biharmonic equation, we choose 1, $\rho^m \cos(m\phi)$, $\rho^m \sin(m\phi)$, $\rho^{m+2} \cos(m\phi)$, $\rho^{m+2} \sin(m\phi)$ to be the bases of the complementary set. Eq.(1) can be expressed by

$$\begin{aligned} u^*(x) = a_0 &+ \sum_{m=1}^{N_T} a_m \rho^m \cos(m\phi) + \sum_{m=1}^{N_T} b_m \rho^m \sin(m\phi) + c_0(\rho^2) \\ &+ \sum_{m=1}^{N_T} c_m \rho^{m+2} \cos(m\phi) + \sum_{m=1}^{N_T} d_m \rho^{m+2} \sin(m\phi) \end{aligned} \tag{17}$$

$$\begin{aligned} \theta^*(x) = &\sum_{m=1}^{N_T} a_m m \rho^{m-1} \cos(m\phi) + \sum_{m=1}^{N_T} b_m m \rho^{m-1} \sin(m\phi) + c_0(2\rho) \\ &+ \sum_{m=1}^{N_T} c_m (m+2) \rho^{m+1} \cos(m\phi) + \sum_{m=1}^{N_T} d_m (m+2) \rho^{m+1} \sin(m\phi) \end{aligned} \tag{18}$$

where a_0, a_m, b_m, c_0, c_m and d_m are the coefficients of the Trefftz method. By matching the boundary conditions of Eqs.(8) and (9) at $\rho = a$, we have

$$\begin{Bmatrix} a_0 \\ a_1 \\ b_1 \\ \vdots \\ a_m \\ b_m \\ c_0 \\ c_1 \\ d_1 \\ \vdots \\ c_m \\ d_m \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \frac{-a}{2} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{3}{2}a^{-1} & 0 & \cdots & 0 & 0 & 0 & \frac{-1}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{3}{2}a^{-1} & \cdots & 0 & 0 & 0 & 0 & \frac{-1}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{m+2}{2}a^{-m} & 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2}a^{1-m} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{m+2}{2}a^{-m} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2}a^{1-m} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2a} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{-1}{2}a^{-3} & 0 & \cdots & 0 & 0 & 0 & \frac{1}{2}a^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{2}a^{-3} & \cdots & 0 & 0 & 0 & 0 & \frac{1}{2}a^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-m}{2}a^{-m-2} & 0 & 0 & 0 & 0 & \cdots & \frac{1}{2}a^{-m-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{-m}{2}a^{-m-2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}a^{-m-1} \end{bmatrix} \begin{Bmatrix} p_0 \\ p_1 \\ q_1 \\ \vdots \\ p_m \\ q_m \\ r_0 \\ r_1 \\ s_1 \\ \vdots \\ r_m \\ s_m \end{Bmatrix} \tag{19}$$

Eq.(19) is found to be the same as Eqs.(11)-(16). Therefore, we can construct the analytical solution through the Trefftz method.

3.3 Method of fundamental solutions

We use the method of fundamental solutions to solve the same problem. According to the Eq.(2), the slope field can be obtained as

$$\begin{aligned} \frac{\partial u}{\partial n} = \theta(x) &= \sum_{j=1}^{N_M} v_j \frac{\partial U(x,s_j)}{\partial n_x} \\ &= \sum_{j=1}^{N_M} v_j L(s,x), \quad s_j \in D^e \end{aligned} \tag{20}$$

The fundamental solution can be expressed by using degenerate kernel as follows:

$$\begin{aligned}
U^I(\rho,\phi;R,\theta) &= r^2 \ln r \\
&= [\rho^2 + R^2 - 2\rho R\cos(\theta-\phi)]\cdot[\ln R - \sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m \cos(m(\theta-\phi))] \\
&= \rho^2(1+\ln R) + R^2\ln R - 2\rho R\ln R\cos\theta\cos\phi - 2\rho R\ln R\sin\theta\sin\phi \\
&\quad -\rho R\cos\theta\cos\phi - \rho R\sin\theta\sin\phi - \frac{1}{2}\frac{\rho^3}{R}\cos\theta\cos\phi - \frac{1}{2}\frac{\rho^3}{R}\sin\theta\sin\phi \\
&\quad -\sum_{m=2}^{\infty}\frac{\rho^m}{R^{m-2}}[\frac{\rho^2}{m(m+1)R^2} - \frac{1}{m(m-1)}]\cos m\theta\cos m\phi \\
&\quad -\sum_{m=2}^{\infty}\frac{\rho^m}{R^{m-2}}[\frac{\rho^2}{m(m+1)R^2} - \frac{1}{m(m-1)}]\sin m\theta\sin m\phi, \qquad R > \rho
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
L^I(\rho,\phi;R,\theta) &= \frac{\partial U^I(\rho,\phi;R,\theta)}{\partial n_x} \\
&= 2\rho(1+\ln R) - 2R\ln R\cos\theta\cos\phi - 2R\ln R\sin\theta\sin\phi \\
&\quad - R\cos\theta\cos\phi - R\sin\theta\sin\phi - \frac{3}{2}\frac{\rho^2}{R}\cos\theta\cos\phi - \frac{3}{2}\frac{\rho^2}{R}\sin\theta\sin\phi \\
&\quad -\sum_{m=2}^{\infty}\frac{\rho^{m+1}}{R^m}\frac{m+2}{m(m+1)}\cos m\theta\cos m\phi - \sum_{m=2}^{\infty}\frac{\rho^{m+1}}{R^m}\frac{m+2}{m(m+1)}\sin m\theta\sin m\phi \\
&\quad +\sum_{m=2}^{\infty}\frac{\rho^{m-1}}{R^{m-2}}\frac{1}{m-1}\cos m\theta\cos m\phi + \sum_{m=2}^{\infty}\frac{\rho^{m-1}}{R^{m-2}}\frac{1}{m-1}\sin m\theta\sin m\phi, \quad R > \rho
\end{aligned} \tag{22}$$

By subtituting Eqs.(21), (22) into Eqs.(2), (20), respectively, and matching the boundary conditions of Eqs.(8), (9), we have

$$\sum_{j=1}^{N_M} V_j\{R^2 \ln R\} = p_0 - \frac{a}{2} r_0 \tag{23}$$

$$-\sum_{j=1}^{N_M} V_j\{R(1+2\ln R)\}\cos\theta_j = \frac{3}{2}a^{-1}p_1 - \frac{1}{2}r_1 \tag{24}$$

$$-\sum_{j=1}^{N_M} V_j\{R(1+2\ln R)\}\sin\theta_j = \frac{3}{2}a^{-1}q_1 - \frac{1}{2}s_1 \tag{25}$$

$$-\sum_{j=1}^{N_M}\frac{1}{m(m-1)}\frac{1}{R^{m-2}}\cos m\theta_j = \frac{m+2}{2}a^{-m}p_m - \frac{1}{2}a^{1-m}r_m \tag{26}$$

$$-\sum_{j=1}^{N_M}\frac{1}{m(m-1)}\frac{1}{R^{m-2}}\sin m\theta_j = \frac{m+2}{2}a^{-m}q_m - \frac{1}{2}a^{1-m}s_m \tag{27}$$

$$\sum_{j=1}^{N_M} V_j\{1+\ln R\} = \frac{1}{2a}c_0 \tag{28}$$

$$\sum_{j=1}^{N_M} V_j\frac{-1}{2R}\cos\theta_j = \frac{-1}{2}a^{-3}p_1 + \frac{1}{2}a^{-2}r_1 \tag{29}$$

$$\sum_{j=1}^{N_M} V_j\frac{-1}{2R}\sin\theta_j == \frac{-1}{2}a^{-3}p_1 + \frac{1}{2}a^{-2}r_1 \tag{30}$$

$$\sum_{j=1}^{N_M} V_j\frac{-1}{R^m}\frac{1}{m(m+1)}\cos m\theta_j = \frac{-m}{2}a^{-m-2}p_m + \frac{1}{2}a^{-m-1}r_m \tag{31}$$

$$\sum_{j=1}^{N_M} V_j\frac{-1}{R^m}\frac{1}{m(m+1)}\sin m\theta_j = \frac{-m}{2}a^{-m-2}q_m + \frac{1}{2}a^{-m-1}s_m \tag{32}$$

Eq.(23)-(32) can be rewritten as

$$[K]\{\underset{\sim}{v}\} = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \frac{-a}{2} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{3}{2}a^{-1} & 0 & \cdots & 0 & 0 & 0 & \frac{-1}{2} & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{3}{2}a^{-1} & \cdots & 0 & 0 & 0 & 0 & \frac{-1}{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{m+2}{2}a^{-m} & 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2}a^{1-m} & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{m+2}{2}a^{-m} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2}a^{1-m} \\
0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2a} & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{-1}{2}a^{-3} & 0 & \cdots & 0 & 0 & 0 & \frac{1}{2}a^{2} & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{-1}{2}a^{-3} & \cdots & 0 & 0 & 0 & 0 & \frac{1}{2}a^{2} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{-m}{2}a^{-m-2} & 0 & 0 & 0 & 0 & \cdots & \frac{1}{2}a^{-m-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{-m}{2}a^{-m-2} & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2}a^{-m-1}
\end{bmatrix}
\begin{Bmatrix} p_0 \\ p_1 \\ q_1 \\ \vdots \\ p_m \\ q_m \\ r_0 \\ r_1 \\ s_1 \\ \vdots \\ r_m \\ s_m \end{Bmatrix} \tag{33}$$

where

$$[K] = \begin{bmatrix} \langle w_1 \rangle \\ \langle w_2 \rangle \\ \vdots \\ \vdots \\ \langle w_{N_M} \rangle \end{bmatrix}_{N_M \times N_M} \tag{34}$$

$$\underset{\sim}{v} = \{v_1 \ \ v_2 \ \ v_3 \ \ v_4 \ \ v_5 \ \cdots \ v_{N_M-1} \ \ v_{N_M}\}^T \tag{35}$$

in which

$$\begin{aligned}
\langle w_1 \rangle &= R^2\ln(R)[1, 1.......,1], \\
\langle w_2 \rangle &= -R(1+2\ln R)[\cos(\theta_1), \cos(\theta_2)..., \cos(\theta_{N_M})], \\
\langle w_3 \rangle &= -R(1+2\ln R)[\sin(\theta_1), \sin(\theta_2)..., \sin(\theta_{N_M})], \\
&\vdots \\
\langle w_{\frac{N_M}{2}} \rangle &= \frac{1}{N(N-1)}(\frac{1}{R})^{N-2}[\cos(N\theta_1), \cos(N\theta_2)..., \cos(N\theta_{N_M})], \\
\langle w_{\frac{N_M}{2}+1} \rangle &= \frac{1}{N(N-1)}(\frac{1}{R})^{N-2}[\sin(N\theta_1), \sin(N\theta_2)..., \sin(N\theta_{N_M})], \\
\langle w_{\frac{N_M}{2}+2} \rangle &= (1+\ln(R))[1, 1.......,1], \\
\langle w_{\frac{N_M}{2}+3} \rangle &= \frac{-1}{2R}[\cos(\theta_1), \cos(\theta_2)..., \cos(\theta_{N_M})], \\
\langle w_{\frac{N_M}{2}+4} \rangle &= \frac{-1}{2R}[\sin(\theta_1), \sin(\theta_2)..., \sin(\theta_{N_M})], \\
&\vdots \\
\langle w_{N_M-1} \rangle &= \frac{1}{N(N+1)}\frac{-1}{R^N}[\cos(N\theta_1), \cos(N\theta_2)..., \cos(N\theta_{N_M})], \\
\langle w_{N_M} \rangle &= \frac{1}{N(N+1)}\frac{-1}{R^N}[\sin(N\theta_1), \sin(N\theta_2)..., \sin(N\theta_{N_M})],
\end{aligned} \tag{36}$$

Therefore, we can compare the Eq.(19) in the Trefftz method with Eq.(33) in the MFS. By setting $4N_T + 2 = N_M = 4N + 2$ under the request of the same number of degrees of freedom, the relationship between the coefficients in the Trefftz method and the MFS can be connected by

$$\begin{Bmatrix} p_0 \\ p_1 \\ q_1 \\ \vdots \\ p_N \\ q_N \\ r_0 \\ r_1 \\ s_1 \\ \vdots \\ r_N \\ s_N \end{Bmatrix}_{(4N+2)\times 1} = [K]_{(4N+2)\times(4N+2)} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ v_{4N+1} \\ v_{4N+2} \end{Bmatrix}_{(4N+2)\times 1} \qquad (37)$$

where the left-hand side is the coefficient vector of the Trefftz method and the right-hand side is the coefficient vector of the MFS. The $[K]$ matrix in Eq.(37) can be decomposed to

$$[K]=[T_R][T_\theta] \qquad (38)$$

where

$$[T_R]=\begin{bmatrix} R^2\ln R & & & & & & & & & & & \\ & -R(1+2\ln R) & & & & & & & & & & \\ & & -R(1+2\ln R) & & & & & & & & & \\ & & & \ddots & & & & & & & & \\ & & & & \frac{1}{R^{N-2}N(N-1)} & & & & & & & \\ & & & & & \frac{1}{R^{N-2}N(N-1)} & & & & & & \\ & & & & & & 1+\ln R & & & & & \\ & & & & & & & \frac{-1}{2R} & & & & \\ & & & & & & & & \frac{-1}{2R} & & & \\ & & & & & & & & & \ddots & & \\ & & & & & & & & & & \frac{1}{R^N N(N+1)} & \\ & & & & & & & & & & & \frac{1}{R^N N(N+1)} \end{bmatrix} \qquad (39)$$

and

$$[T_\theta]=\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 1 \\ \cos\theta_1 & \cos\theta_2 & \cdots & \cdots & \cos\theta_{4N+1} & \cos\theta_{4N+2} \\ \sin\theta_1 & \sin\theta_2 & \cdots & \cdots & \sin\theta_{4N+1} & \sin\theta_{4N+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \cos N\theta_1 & \cos N\theta_2 & \cdots & \cdots & \cos N\theta_{4N+1} & \cos N\theta_{4N+2} \\ \sin N\theta_1 & \sin N\theta_2 & \cdots & \cdots & \sin N\theta_{4N+1} & \sin N\theta_{4N+2} \\ 1 & 1 & \cdots & \cdots & 1 & 1 \\ \cos\theta_1 & \cos\theta_2 & \cdots & \cdots & \cos\theta_{4N+1} & \cos\theta_{4N+2} \\ \sin\theta_1 & \sin\theta_2 & \cdots & \cdots & \sin\theta_{4N+1} & \sin\theta_{4N+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \cos N\theta_1 & \cos N\theta_2 & \cdots & \cdots & \cos N\theta_{4N+1} & \cos N\theta_{4N+2} \\ \sin N\theta_1 & \sin N\theta_2 & \cdots & \cdots & \sin N\theta_{4N+1} & \sin N\theta_{4N+2} \end{bmatrix} \qquad (40)$$

It is interesting to find that T_R is an diagonal matrix of dimension *(4N+2)* by *(4N+2)* and T_θ is an orthogonal matrix. The determinant of $[T_\theta]$ can be obtained

$$\det[T_\theta]=2(2N+1)^{2N+1} \qquad (41)$$

due to the orthogonal property as shown below:

$$[T_\theta]^T[T_\theta]=\begin{bmatrix} 4N+2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2N+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2N+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4N+2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2N+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2N+1 \end{bmatrix}_{(4N+2)\times(4N+2)} \qquad (42)$$

If the [K] matrix is nonsingular, the equivalence between the two methods is proved. The singular *[K]* matrix results in the problem of solvability using the MFS since [K] can not be invertible. This is numerically reliable instead of physical phenomenon. The degenerate scale occurs at the three locations $R=e^0$, $e^{\frac{-1}{2}}$, e^{-1} since $\ln R$, $1+\ln R$ and $1+2\ln R$ in Eq.(39) are zeros. A detailed study for the degenerate scale due to the phenomenon of the numerical nonuniqueness was elaborated on in [1,2,3,4].

4. Conclusions

In this paper, the mathematical equivalence for biharmonic equations between the Treffz method and the MFS was proved. It is interesting to find that the T-complete set in the Trefftz method for 1-D, 2-D, 3-D Laplace, Helmholtz and biharmonic equations are imbedded in the degenerate kernels of MFS. The degenerate scale occurs when the fictitious sources are located at e^0, $e^{\frac{-1}{2}}$ and e^{-1} for circular case.

5. References

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Trefftz 法與基本解法在板問題之等效性

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摘要

本文主要探討 Trefftz 法與基本解法求解雙諧和方程問題兩者在數學上之等效性。於推導板之靜力問題前，先發現在二維及三維的拉普拉斯方程與漢姆赫茲方程中，Trefftz 的完整解集合不論是在內域問題或外域問題皆可由基本解法中的退化核函數中求得。因此，我們把兩階控制方程的成功案例拓展為四階，並設計一個圓形固端板範例做說明，利用退化核函數展開基本解所得到之係數矩陣與 Trefftz 法中所得到之係數矩陣相互比較後，可產生一映射矩陣。此映射矩陣與源點的位置分佈有關並可分解為一旋轉矩陣與幾何矩陣。透過此映射矩陣，可看出在數值中所遇到的退化尺度的發生機制。

關鍵字：雙諧和方程，基本解法，Trefftz 法，Trefftz 的完整解集合，退化核，映射矩陣，退化尺度。

(a): closed form

(b): degenerate kernel

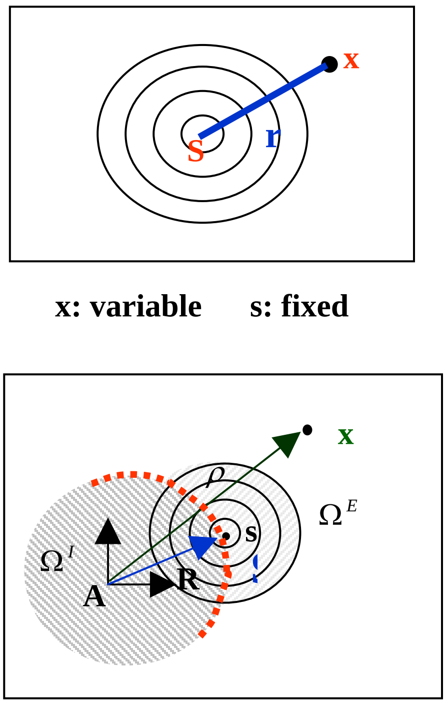

Fig. (1) Expression of fundamental solution
(a) closed form (b) degenerate kernel

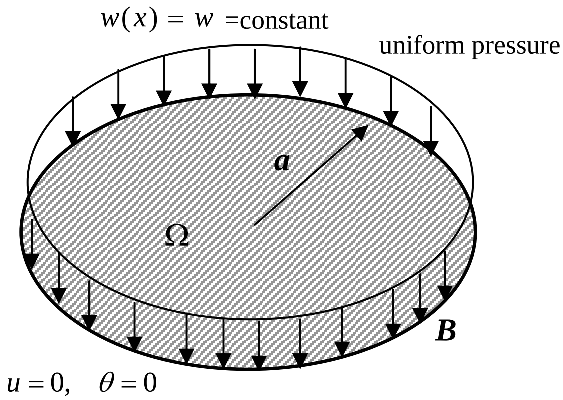

Fig.(2) A clamped plate under uniform load

Basis functions and degenerate kernels		Method of Fundamental Solution: Fundamental Solution	Method of Fundamental Solution: Degenerate kernel	Trefftz Method: Interior basis	Trefftz Method: Exterior basis
L	1	$\frac{r}{2}$	$U(x,s)=\begin{cases}\frac{1}{2}\sum_{i=1}^{2} a_i(x)b_i(s), & x>s\\ \frac{1}{2}\sum_{i=1}^{2} a_i(s)b_i(x), & x<s\end{cases}$ $T(x,s)=\begin{cases}\frac{1}{2}\sum_{i=1}^{2} a_i(x)b_i'(s), & x>s\\ \frac{1}{2}\sum_{i=1}^{2} a_i'(s)b_i(x), & x<s\end{cases}$ $L(x,s)=\begin{cases}\frac{1}{2}\sum_{i=1}^{2} a_i'(x)b_i(s), & x>s\\ \frac{1}{2}\sum_{i=1}^{2} a_i(s)b_i'(x), & x<s\end{cases}$ $M(x,s)=\begin{cases}\frac{1}{2}\sum_{i=1}^{2} a_i'(x)b_i'(s), & x>s\\ \frac{1}{2}\sum_{i=1}^{2} a_i'(s)b_i'(x), & x<s\end{cases}$ where $\begin{cases}a_1(x)=x\\ a_2(x)=-1\\ b_1(s)=1\\ b_2(s)=s\end{cases}$	**One dimensional coordinate** $1\ ,\ x$	**One dimensional coordinate** $1\ ,\ x$
L	2	$\ln(r)$ $s=(\bar{\rho},\bar{\theta})$ $x=(\rho,\theta)$	$U(x,s)=\begin{cases}U^i(x,s)=\ln\bar{\rho}-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{\bar{\rho}})^m\cos(m(\bar{\theta}-\theta)), & \rho<\bar{\rho}\\ U^e(x,s)=\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\bar{\rho}}{\rho})^m\cos(m(\bar{\theta}-\theta)), & \rho>\bar{\rho}\end{cases}$ $T(s,x)=\begin{cases}T^i(x,s)=\frac{1}{\bar{\rho}}+\sum_{m=1}^{\infty}\frac{\rho^m}{\bar{\rho}^{m+1}}\cos(m(\bar{\theta}-\theta)), & \rho<\bar{\rho}\\ T^e(x,s)=-\sum_{m=1}^{\infty}\frac{\bar{\rho}^{m-1}}{\rho^m}\cos(m(\bar{\theta}-\theta)), & \rho>\bar{\rho}\end{cases}$ $L(x,s)=\begin{cases}L^i(x,s)=-\sum_{m=1}^{\infty}\frac{\rho^{m-1}}{\bar{\rho}^m}\cos(m(\bar{\theta}-\theta)), & \rho<\bar{\rho}\\ L^e(x,s)=\frac{1}{\rho}+\sum_{m=1}^{\infty}\frac{\bar{\rho}^m}{\rho^{m+1}}\cos(m(\bar{\theta}-\theta)), & \rho>\bar{\rho}\end{cases}$ $M(x,s)=\begin{cases}M^i(x,s)=\sum_{m=1}^{\infty}\frac{m\rho^{m-1}}{\bar{\rho}^{m+1}}\cos(m(\bar{\theta}-\theta)), & \rho<\bar{\rho}\\ M^e(x,s)=\sum_{m=1}^{\infty}\frac{m\bar{\rho}^{m-1}}{\rho^{m+1}}\cos(m(\bar{\theta}-\theta)), & \rho>\bar{\rho}\end{cases}$	**Two dimensional coordinate** $\rho^m e^{im\theta}$ $m=0,1,2,\ldots$	**Two dimensional coordinate** $\rho^{-m}\cos m\theta\ ,\rho^{-m}\sin m\theta$ $m=1,2,\ldots$
L	3	$\frac{-1}{r}$ $s=(\bar{\rho},\bar{\theta},\bar{\phi})$ $x=(\rho,\theta,\phi)$	$U(x,s)=\begin{cases}U^i(x,s)=\frac{-1}{\bar{\rho}}-\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{\rho^n}{\bar{\rho}^{n+1}}\\ U^e(x,s)=\frac{-1}{\rho}-\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{\bar{\rho}^n}{\rho^{n+1}}\end{cases}$ $T(x,s)=\begin{cases}T^i(x,s)=(\frac{1}{\bar{\rho}})^2+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{(n+1)\rho^n}{\bar{\rho}^{n+2}}\\ T^e(x,s)=-2\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{n\bar{\rho}^{n-1}}{\rho^{n+1}}\end{cases}$ $L(x,s)=\begin{cases}L^i(x,s)=-2\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{n\rho^{n-1}}{\bar{\rho}^{n+1}}\\ L^e(x,s)=\frac{1}{\rho^2}+\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{(n+1)\bar{\rho}^n}{\rho^{n+2}}\end{cases}$ $M(x,s)=\begin{cases}M^i(x,s)=2\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{n(n+1)\rho^{n-1}}{\bar{\rho}^{n+2}}\\ M^e(x,s)=2\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{n(n+1)\bar{\rho}^{n-1}}{\rho^{n+2}}\end{cases}$	**Three dimensional coordinate** $\rho^n P_n^m(\cos\theta)\cos(m\phi)$ $\rho^n P_n^m(\cos\theta)\sin(m\phi)$ $m=0,1,2.....$ $n=0,1,2,......$	**Three dimensional coordinate** $\rho^{-(n+1)}P_n^m(\cos\theta)\cos(m\phi)$ $\rho^{-(n+1)}P_n^m(\cos\theta)\sin(m\phi)$ $m=0,1,2.....$ $n=0,1,2,......$
H	1	$\frac{e^{ikr}}{2ki}$	$U(x,s)=\begin{cases}\frac{1}{2ki}\sum_{i=1}^{1} a_i(x)b_i(s), & x>s\\ \frac{1}{2ki}\sum_{i=1}^{1} a_i(s)b_i(x), & x<s\end{cases}$ $T(x,s)=\begin{cases}\frac{1}{2ki}\sum_{i=1}^{1} a_i(x)b_i'(s), & x>s\\ \frac{1}{2ki}\sum_{i=1}^{1} a_i'(s)b_i(x), & x<s\end{cases}$ $L(x,s)=\begin{cases}\frac{1}{2ki}\sum_{i=1}^{1} a_i'(x)b_i(s), & x>s\\ \frac{1}{2ki}\sum_{i=1}^{1} a_i(s)b_i'(x), & x<s\end{cases}$ $M(x,s)=\begin{cases}\frac{1}{2ki}\sum_{i=1}^{1} a_i'(x)b_i'(s), & x>s\\ \frac{1}{2ki}\sum_{i=1}^{1} a_i'(x)b_i'(s), & x<s\end{cases}$ where $\begin{cases}a_1(x)=e^{ikx}\\ b_1(s)=e^{-iks}\end{cases}$	**One dimensional coordinate** $e^{ikx}\ ,\ e^{-ikx}$	**One dimensional coordinate** $e^{ikx}\ ,\ e^{-ikx}$
H	2	$\frac{-i\pi H_0^{(1)}(kr)}{2}$ $s=(\bar{\rho},\bar{\theta})$ $x=(\rho,\theta)$	$U(x,s)=\begin{cases}U^i(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}J_m(k\rho)H_0^{(1)}(k\bar{\rho})\cos(m(\bar{\theta}-\theta))\\ U^e(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}J_m(k\bar{\rho})H_0^{(1)}(k\rho)\cos(m(\bar{\theta}-\theta))\end{cases}$ $T(x,s)=\begin{cases}T^i(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}kJ_m(k\rho)H_0'^{(1)}(k\bar{\rho})\cos(m(\bar{\theta}-\theta))\\ T^e(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}kJ_m'(k\bar{\rho})H_0^{(1)}(k\rho)\cos(m(\bar{\theta}-\theta))\end{cases}$ $L(x,s)=\begin{cases}L^i(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}kJ_m'(k\rho)H_0^{(1)}(k\bar{\rho})\cos(m(\bar{\theta}-\theta))\\ L^e(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}kJ_m(k\bar{\rho})H_0'^{(1)}(k\rho)\cos(m(\bar{\theta}-\theta))\end{cases}$ $M(x,s)=\begin{cases}M^i(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}k^2J_m'(k\rho)H_0'^{(1)}(k\bar{\rho})\cos(m(\bar{\theta}-\theta))\\ M^e(x,s)=-\sum_{m=-\infty}^{\infty}\frac{\pi i}{2}k^2J_m'(k\bar{\rho})H_0'^{(1)}(k\rho)\cos(m(\bar{\theta}-\theta))\end{cases}$	**Two dimensional coordinate** $J_0(k\rho)$ $J_m(k\rho)\cos(m\theta)$ $J_m(k\rho)\sin(m\theta)$	**Two dimensional coordinate** $H_0^{(1)}(k\rho)$ $H_n^{(1)}(k\rho)\cos(m\theta)$ $H_n^{(1)}(k\rho)\sin(m\theta)$
H	3	$\frac{-e^{-ikr}}{r}$ $s=(\bar{\rho},\bar{\theta},\bar{\phi})$ $x=(\rho,\theta,\phi)$	$U(x,s)=\begin{cases}-ik\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n(k\rho)h_n^{(2)}(k\bar{\rho}), & \rho<\bar{\rho}\\ -ik\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n(k\bar{\rho})h_n^{(2)}(k\rho), & \rho>\bar{\rho}\end{cases}$ $T(x,s)=\begin{cases}-ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n(k\rho)h_n'^{(2)}(k\bar{\rho}), & \rho<\bar{\rho}\\ -ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n'(k\bar{\rho})h_n^{(2)}(k\rho), & \rho>\bar{\rho}\end{cases}$ $L(x,s)=\begin{cases}-ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n'(k\rho)h_n^{(2)}(k\bar{\rho}), & \rho<\bar{\rho}\\ -ik^2\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n(k\bar{\rho})h_n'^{(2)}(k\rho), & \rho>\bar{\rho}\end{cases}$ $M(x,s)=\begin{cases}-ik^3\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n'(k\rho)h_n'^{(2)}(k\bar{\rho}), & \rho<\bar{\rho}\\ -ik^3\sum_{n=0}^{\infty}(2n+1)\sum_{m=0}^{n}\varepsilon_m\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})j_n'(k\bar{\rho})h_n'^{(2)}(k\rho), & \rho>\bar{\rho}\end{cases}$	**Three dimensional coordinate** $j_0(k\rho)P_n(\cos\theta)$ $j_n(k\rho)P_n^m(\cos\theta)\cos(m\phi)$ $j_n(k\rho)P_n^m(\cos\theta)\sin(m\phi)$	**Three dimensional coordinate** $h_0^{(1)}(k\rho)P_n(\cos\theta)$ $h_n^{(1)}(k\rho)P_n^m(\cos\theta)\cos(m\phi)$ $h_n^{(1)}(k\rho)P_n^m(\cos\theta)\sin(m\phi)$

The equation which satisfy the basis function									
Laplace	1-D	$\nabla^2U(x,s)=\delta(x-s)$	Helmholtz	1-D	$(\nabla^2+k^2)U(x,s)=\delta(x-s)$	Laplace	$\nabla^2u(x)=0$	Helmholtz	$(\nabla^2+k^2)u(x)=0$
	2-D	$\nabla^2U(x,s)=2\pi\delta(x-s)$		2-D	$(\nabla^2+k^2)U(x,s)=2\pi\delta(x-s)$		$\nabla^2u(\rho,\phi)=0$		$(\nabla^2+k^2)u(\rho,\theta)=0$
	3-D	$\nabla^2U(x,s)=4\pi\delta(x-s)$		3-D	$(\nabla^2+k^2)U(x,s)=4\pi\delta(x-s)$		$\nabla^2u(\rho,\theta,\phi)=0$		$(\nabla^2+k^2)u(\rho,\theta,\phi)=0$

Note: where $r=\lvert x-s\rvert$; ε_m is the Neumann Factor

Table 1 The T-complete functions of the Trefftz method and degenerate kernels of the MFS