On the equivalence of method of fundamental solutions and Trefftz method for Laplace equation

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Abstract

In this paper, it is proved that the two approaches for Laplace problems, known in the literature as the method of fundamental solution (MFS) and the Trefftz method, are mathematically equivalent in spite of their essentially minor and apparent differences in the formulation. It is interesting to find that the T-complete set in the Trefftz method for the interior and exterior problems are imbedded in the degenerate kernels of MFS. By designing a circular-domain problem, the unknown coefficients of each method correlate by a mapping matrix after considering the degenerate kernels for the fundamental solutions in the MFS and the T-complete function in the Trefftz method. The mapping matrix is composed of a rotation matrix and a geometric matrix which depends on the source location. The degenerate scale for the Laplace equation appears using the MFS when the geometric matrix is singular. The ill-posed problem in MFS also stems from the geometric matrix when the fictitious source is distributed far away from the real boundary. Finally, the efficiency of MFS is compared with the Trefftz method under the same number of degrees of freedom.

Trefftz 法與基本解法之等效性－以拉普拉斯方程為例

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摘要

本文主要以 Trefftz 法與基本解法來探討兩者在數學上之等效性， 並由文中可得知 Trefftz 的完整解集合不論是在內域問題或外域問題皆可由基本解法中的退化核函數中求得。文中設計一個圓形範例做說明，利用退化核函數展開基本解所得到之係數矩陣與 Trefftz 法中所得到之係數矩陣相互比較後，可產生一映射矩陣。此映射矩陣根據源點的位置分佈所構成並可分解為一旋轉矩陣與幾何矩陣。在拉普拉斯問題中，我們可藉由幾何矩陣之奇異性說明在基本解法中退化尺度問題所產生的機制，並且當虛擬源點與實際邊界相差甚遠時所發生矩陣病態行為，亦可由此看出。最後，本文在相同自由度數目的情況下，對於兩者的收斂速率的優劣亦作探討。

1. Introduction

Since 1926, Trefftz first presented the Trefftz method for solving boundary value problem by the superposition of the functions satisfying the governing equation, although various versions of Trefftz method, e.g., direct formulation and indirect formulation methods have been developed. The unknown coefficients are determined so that the approximate

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solution satisfies the boundary condition. Many applications to Helmholtz equation [3], Navier equation [7,14] and biharmonic equation [8] were done. Until recent years, the ill-posed nature in the method was noticed [1].

In potential theory, it is well known that the method of fundamental solution (MFS) can solve potential problems when a fundamental solution is known. This method was attributed to Kupradze (1964) [14] in Russia, extensive applications in solving a broad range of problems such as potential problems [3,9], acoustics [11], biharmonic problems [8] have been found. The MFS can be reviewed as an indirect boundary element method. The initial idea is to approximate the solution by a linear combination of fundamental solution with source located outside the domain of the problem. Moreover, it has certain advantages over BEM, e.g., no singularity and no boundary integrals. It can also be applied to acoustics [3], elasticity [7,14] and plate problems [8].

However, the link between the Trefftz method and the MFS was not discussed in detail to the authors' best knowledge. A similar case to link the DRBEM and the method of particular integral was done by Polyzos *et al.* [12]. In this paper, we will design a circular domain problem for the Laplace equation and prove the mathematical equivalence of the Trefftz method and MFS. Two mathematical tools are required. One is degenerate kernels for the expansion of the closed-form fundamental solution, the other is the Fourier series expansion for the boundary density. The degenerate scale and the ill-posed behavior of the MFS will be addressed. Also, the efficiency between the Trefftz method and MFS will be compared with under the same number of degrees of freedom.

2. Connection between the Trefftz method and the MFS for Laplace equation

Consider a two-dimensional Laplace problem with a ciucular domain of radius ρ as shown in Fig.1

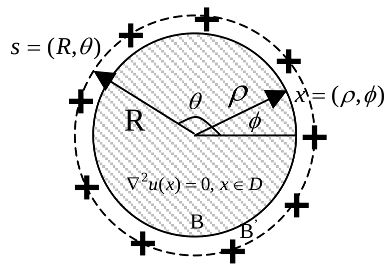

Fig.1 Definition sketch of polar coordinate for Laplace equation in the circular domain
(+ is the source location of the MFS)

the governing equation of the boundary value problem is the Laplace equation,

$$\nabla^2 u(x)=0, \quad x\in D \tag{1}$$

where ∇^2 denotes the Laplacian operator and $u(x)$ is the potential function. The boundary condition is given by the Dirichlet type

$$u(x)=\bar{u} \quad x\in B \tag{2}$$

By using the Fourier series expansion, the boundary condition $u(x)$ can be expressed as

$$u(\rho,\phi)=\bar{a}_0+\sum_{n=1}^{N}\bar{a}_n\cos(n\phi)+\sum_{n=1}^{N}\bar{b}_n\sin(n\phi) \tag{3}$$

where $\bar{a}_0$, $\bar{a}_n$, $\bar{b}_n$ are the Fourier coefficients with respect to Fourier bases, $\cos(n\phi)$ and $\sin(n\phi)$, and ϕ is the angle along the circular boundary.

2.1 Trefftz method

In the Trefftz method, the field solution $u(x)$ is superimposed by the T-complete functions, $u_j(x)$ as follows:

$$u(x)=\sum_{j=1}^{2N_T+1} w_j u_j(x) \tag{4}$$

where $2N_T+1$ is the number of complete functions, w_j is the unknown coefficient, $u_j(x)$ is the complementary set which satisfies the Laplace equation. Then, we choose $r^n\sin(n\phi)$ and $r^n\cos(n\phi)$ to be the bases of the complementary set in two-dimensional problem. Eq.(4) can be expressed by

$$u(r,\phi)=a_0+\sum_{n=1}^{N_T}a_n r^n\cos(n\phi)+\sum_{n=1}^{N_T}b_n r^n\sin(n\phi),\ 0<r<\rho \quad (5)$$

By matching the boundary condition at $r=\rho$, we have

$$u(\rho,\phi)=a_0+\sum_{n=1}^{N_T}a_n\rho^n\cos(n\phi)+\sum_{n=1}^{N_T}b_n\rho^n\sin(n\phi). \quad (6)$$

After comparing the Eq.(3) with Eq.(6), we obtain

$$a_0=\bar{a}_0, \quad (7)$$

$$a_n=\frac{\bar{a}_n}{\rho^n}, \qquad n=1,2,...N_T \quad (8)$$

$$b_n=\frac{\bar{b}_n}{\rho^n} \qquad n=1,2,...N_T \quad (9)$$

2.2 Method of fundamental solution

In the method of fundamental solution, the field solution $u(x)$ is superimposed by $U(x,s_j)$ as follows:

$$u(x)=\sum_{j=1}^{N_M}c_j U(x,s_j), \quad s_j\in D^e \quad (10)$$

where N_M is the number of source points in the MFS, c_j is the unknown coefficient, s and x are the source point and collocation point, respectively, D^e is the complementary domain and $U(x,s_j)$ is the fundamental solution with symmetry property

$$U(x,s_j)=U(s_j,x) \quad (11)$$

In order to match the boundary condition, we have

$$u(x)=\sum_{j=1}^{N_M}c_j U(s_j,x), \quad s_j\in D^e. \quad (12)$$

The fundamental solution can be expressed by the

$$U(R,\theta,\rho,\phi)=\begin{cases}U^i(R,\theta,\rho,\phi)=\ln(R)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos(m(\theta-\phi)), & R>\rho\\ U^e(R,\theta,\rho,\phi)=\ln(\rho)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{R}{\rho})^m\cos(m(\theta-\phi)), & R<\rho\end{cases} \quad (13)$$

where the superscripts " i " and " e " denote the interior expression ($R>\rho$) and the exterior expression ($R<\rho$), $s=(R,\theta)$ and $x=(\rho,\phi)$ are the polar coordinates of s and x, respectively. Eq.(12) reduces to

$$u(\rho,\phi)=\sum_{j=1}^{N_M}c_j[\ln(R)-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{R})^m\cos(m(\theta_j-\phi))] \quad (14)$$

By employing the property of trigonometric function, Eq.(14) can be rewritten as

$$\begin{aligned}u(\rho,\phi)=&\sum_{j=1}^{N_M}c_j\ln(R)\\&-\sum_{m=1}^{\infty}[\sum_{j=1}^{N_M}c_j\frac{1}{m}(\frac{\rho}{R})^m\cos(m\theta_j)]\cos(m\phi)\\&-\sum_{m=1}^{\infty}[\sum_{j=1}^{N_M}c_j\frac{1}{m}(\frac{\rho}{R})^m\sin(m\theta_j)]\sin(m\phi)\end{aligned} \quad (15)$$

After comparing the Eq.(3) with Eq.(15) by truncating the higher order terms, we have

$$\bar{a}_0=\sum_{j=1}^{N_M}c_j\ln(R) \quad (16)$$

$$\frac{\bar{a}_n}{\rho^n}=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(\frac{1}{R})^n\cos(n\theta_j), \quad n=1,2,...N_M \quad (17)$$

$$\frac{\bar{b}_n}{\rho^n}=-\sum_{j=1}^{N_M}c_j\frac{1}{n}(\frac{1}{R})^n\sin(n\theta_j), \quad n=1,2,...N_M \quad (18)$$

where $\theta_j=j\frac{2\pi}{N_M}$. Therefore, we can compare the coefficients with Eqs.(7)-(9) in the Trefftz method and Eqs.(16)-(18) in the MFS. The relationship between the coefficients in Trefftz method and MFS can be written as

$$a_0=\sum_{j=1}^{2N+1}c_j\ln(R) \quad (19)$$

$$a_n=-\sum_{j=1}^{2N+1}c_j\frac{1}{n}(\frac{1}{R})^n\cos(n\theta_j), \quad n=1,2,...2N+1 \quad (20)$$

$$b_n=-\sum_{j=1}^{2N+1}c_j\frac{1}{n}(\frac{1}{R})^n\sin(n\theta_j), \quad n=1,2,...2N+1 \quad (21)$$

by setting $N_T=N_M=2N+1$ under the request of the same number of degrees of freedom. After comparing the two solutions, Eqs.(7)-(9) for the Trefftz method and Eqs.(16)-(18) for the MFS, we have

$$\{\underset{\sim}{u}\} = [K]\{\underset{\sim}{v}\} \tag{22}$$

where

$$\begin{aligned} \underset{\sim}{u} &= \{a_0 \quad a_1 \quad b_1 \quad a_2 \quad b_2 \quad \cdots \quad a_N \quad b_N\}^T \\ \underset{\sim}{v} &= \{c_1 \quad c_2 \quad c_3 \quad c_4 \quad c_5 \quad \cdots \quad c_{2N} \quad c_{2N+1}\}^T \end{aligned} \tag{23}$$

$$K = \begin{bmatrix} \langle w_1 \rangle \\ \langle w_2 \rangle \\ \vdots \\ \vdots \\ \langle w_{2N+1} \rangle \end{bmatrix}_{(2N+1)\times(2N+1)} \tag{24}$$

in which

$$\begin{aligned} &\langle w_1 \rangle = \ln(R)\,[1, 1.......,1], \\ &\langle w_2 \rangle = (\frac{-1}{R})[\cos(\theta_1),\ \cos(\theta_2)....,\ \cos(\theta_{2N+1})], \\ &\langle w_3 \rangle = (\frac{-1}{R})[\sin(\theta_1),\ \sin(\theta_2)....,\ \sin(\theta_{2N+1})], \\ &\vdots \\ &\langle w_{2N} \rangle = \frac{-1}{n}(\frac{1}{R})^n\,[\cos(N\theta_1),\ \cos(N\theta_2)....,\ \cos(N\theta_{2N+1})], \\ &\langle w_{2N+1} \rangle = \frac{-1}{n}(\frac{1}{R})^n\,[\sin(N\theta_1),\ \sin(N\theta_2)....,\ \sin(N\theta_{2N+1})], \end{aligned} \tag{25}$$

The relation of Eq.(22) was obtained to connect the Trefftz method and the MFS. We can decompose the matrix $[K]$ into two parts, one former matrix, T_R, depends on the radius of the fictitious source distribution; the other latter matrix, T_θ, depends on the angle of the source point (Fig.1), as follows:

$$[K] = [T_R][T_\theta] \tag{26}$$

where

$$[T_R] = \begin{bmatrix} \ln(R) & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \frac{-1}{R} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \frac{-1}{R} & 0 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \frac{-1}{2}(\frac{1}{R})^2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & \frac{-1}{N}(\frac{1}{R})^N & 0 \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{-1}{N}(\frac{1}{R})^N \end{bmatrix}_{(2N+1)\times(2N+1)} \tag{27}$$

$$[T_\theta] = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cos(\theta_1) & \cos(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(\theta_{2N+1}) \\ \sin(\theta_1) & \sin(\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(\theta_{2N+1}) \\ \cos(2\theta_1) & \cos(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(2\theta_{2N+1}) \\ \sin(2\theta_1) & \sin(2\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(2\theta_{2N+1}) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \cos(N\theta_1) & \cos(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \cos(N\theta_{2N+1}) \\ \sin(N\theta_1) & \sin(N\theta_2) & \cdots & \cdots & \cdots & \cdots & \cdots & \sin(N\theta_{2N+1}) \end{bmatrix}_{(2N+1)\times(2N+1)} \tag{28}$$

In Eq.(27), it is interesting to find

$$\det[T_\theta] = \frac{(2N+1)^{N+\frac{1}{2}}}{2^N} \tag{29}$$

due to the orthogonal property (Appendix 1) as follows:

$$[T_\theta][T_\theta]^T = \begin{bmatrix} 2N+1 & 0 & \cdots & \cdots & 0 \\ 0 & \frac{2N+1}{2} & \cdots & \cdots & 0 \\ 0 & 0 & \frac{2N+1}{2} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{2N+1}{2} \end{bmatrix}_{(2N+1)\times(2N+1)} \tag{30}$$

In the T_R matrix, it becomes singular at radius of one ($\ln 1 = 0$) which results in a degenerate scale in the MFS. When the radius of fictitious boundary R and the number of the source points N_M become large, the condition number of $[K]$ matrix deteriorates. This is the reason why the ill-posed behavior is inherent in the MFS. If the exact solution is

$$u(r,\theta) = r^{10}\cos(10\theta), \tag{31}$$

the Trefftz method needs at least 21 terms of $N_T = 10$. However, the MFS needs only a fewer terms than the Trefftz method since any one fundamental solution can be expanded to the degenerate kernels with bases $\cos(10\theta)$. It is found that the MFS can approach the exact solution more efficiently than the Trefftz method under the same number of degrees of freedom.

3. Concluding remarks

In this paper, the proof of the mathematical equivalence between the Treffz method and MFS for Laplace equation was derived. The T-complete set functions in the Trefftz method for interior and exterior problems are imbedded in the degenerate kernels of the fundamental solutions as shown in Table 1 for 1-D, 2-D and 3-D Laplace problems. The sources of degenerate scale and ill-posed behavior in the MFS are easily found in the present formulation. It is found that MFS can approach the exact solution more efficiently than the Trefftz method under the same number of degrees of freedom.

Appendix 1

Using the property of a geometric series, we have

$$\sum_{n=0}^{2N} e^{in\phi_m} = \frac{1-e^{i(2N+1)\phi_m}}{1-e^{i\phi_m}} = 0, \quad m \neq 0$$

where $\phi_m = m\Delta\theta = \frac{2m\pi}{2N+1}$. Also, we have

$$\sum_{n=0}^{2N} \cos(n\phi_m) = \begin{cases} 0, & m \neq 0 \\ 2N+1, & m = 0 \end{cases}$$

$$\sum_{n=0}^{2N} \sin(n\phi_m) = 0, \quad m = 0,1,2...2N$$

$$\sum_{n=0}^{2N} \cos(\mu\phi_n)\sin(\lambda\phi_n) = \sum_{n=0}^{2N} \cos(n\mu\phi_1)\sin(n\lambda\phi_1) = 0,$$

$$\phi_n = n\Delta\theta = n\phi_1$$

$$\sum_{n=0}^{2N} \sin(\mu\phi_n)\sin(\lambda\phi_n) = \begin{cases} 0, & \lambda \neq \mu \\ \frac{1}{2}(2N+1), & \lambda = \mu \end{cases}$$

$$\sum_{n=0}^{2N} \cos(u\phi_n)\cos(\lambda\phi_n) = \begin{cases} 0, & \lambda \neq \mu \\ \frac{1}{2}(2N+1), & \lambda = \mu \neq 0 \\ 2N+1, & \lambda = \mu = 0 \end{cases}$$

where $\lambda = 0,1,2....N$ and $\mu = 0,1,2....N$

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Table 1 Equivalence of the bases between the Trefftz method and the MFS for Laplace problem

		Method of Fundamental Solution (MFS)		Trefftz Method	
		Fumdamental solution	Degenerate kernel	Interior basis	Exterior basis
Basis functions and degenerate kernels	1-D	$\frac{r}{2}$	$U(x,s)=\begin{cases}\frac{1}{2}\sum_{i=1}^{2}a_i(x)b_i(s), & x>s\\ \frac{1}{2}\sum_{i=1}^{2}a_i(s)b_i(x), & x<s\end{cases}$	$1,\ x$	$1,\ x$
			$\nabla^2 U(x,s)=\delta(x-s)$	$\nabla^2 u(x)=0$	
	2-D	$\ln(r)$	$U(x,s)=\begin{cases}U^i(x,s)=\ln\bar{\rho}-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\rho}{\bar{\rho}})^m\cos(m(\bar{\theta}-\theta)), & \rho<\bar{\rho}\\ U^e(x,s)=\ln\rho-\sum_{m=1}^{\infty}\frac{1}{m}(\frac{\bar{\rho}}{\rho})^m\cos(m(\bar{\theta}-\theta)), & \rho>\bar{\rho}\end{cases}$ $s=(\bar{\rho},\bar{\theta})$ $x=(\rho,\theta)$	$\rho^m\cos m\theta\ ,\rho^m\sin m\theta$	$\rho^{-m}\cos m\theta\ ,\rho^{-m}\sin m\theta$
			$\nabla^2 U(x,s)=2\pi\delta(x-s)$	$\nabla^2 u(\rho,\phi)=0$	
	3-D	$\frac{-1}{r}$	$U(x,s)=\begin{cases}U^i(x,s)=\frac{-1}{\bar{\rho}}-\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{\rho^n}{\bar{\rho}^{n+1}}\\ U^e(x,s)=\frac{-1}{\rho}-\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-m)!}{(n+m)!}\cos[m(\phi-\bar{\phi})]P_n^m(\cos\theta)P_n^m(\cos\bar{\theta})\frac{\bar{\rho}^n}{\rho^{n+1}}\end{cases}$ $s=(\bar{\rho},\bar{\theta},\bar{\phi})$ $x=(\rho,\theta,\phi)$	$\rho^n P_n^m(\cos\theta)\cos(m\phi)$ $\rho^n P_n^m(\cos\theta)\sin(m\phi)$	$\rho^{-(n+1)}P_n^m(\cos\theta)\cos(m\phi)$ $\rho^{-(n+1)}P_n^m(\cos\theta)\sin(m\phi)$
			$\nabla^2 U(x,s)=4\pi\delta(x-s)$	$\nabla^2 u(\rho,\theta,\phi)=0$	

where P_n^m is the associated Legendre function; $m=0,1,2,3,...$ and $n=0,1,2,3,...$