Mathematical analysis of the true and spurious eigensolutions for free vibration of plate using real-part BEM

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Abstract

In this paper, a real-part BEM for solving the eigenfrequencies of plates is proposed for saving half effort in computation instead of using the complex-valued BEM. By employing the real-part fundamental solution, the spurious eigensolutions in conjunction with the true eigensolution are obtained for free vibration of plate. To verify this finding, the circulant is adopted to analytically derive the true and spurious eigenequation in the discrete system of a circular plate. In order to obtain the eigenvalues and boundary modes at the same time, the singular value decomposition (SVD) technique is utilized. For the continuous system, mathematical analysis for the spurious eigensolution was done by using the degenerate kernel and Fourier series. Good agreement among the analytical solutions (continuous and discrete systems) is made. The clamped circular plate is demonstrated analytically and numerically to see the validity of the present method.

實部邊界元素法之板自由振動真假特徵方程數學分析

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摘要

本文以實部邊界元素法求解一固定圓板之特徵頻率問題以節省數值運算。使用實部邊界元素法在求解過程中所伴隨而來的假根問題為此文章之討論重點。為證明假根產生之機制，本文在連續系統中採用退化核及富利葉級數來進行數學推導，於離散系統中利用退化核及循環矩陣來做一解析之動作，並使用奇異值分解法來同時獲得特徵頻率之邊界模態。本中並以一固端圓板為例來說明由本文所提出之方法不論是在連續或是離散系統中，均能得到相吻合之結果，以驗證此方法之正確性。

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1. Introduction

For the simply-connected problems of interior acoustics, either the real-part or imaginary-part BEM results in spurious eigensolutions [8]. Tai and Shaw [24] first employed BEM to solve membrane vibration using complex-valued kernel. De Mey [11, 12], Hutchinson and Wong [15] employed only the real-part kernel to solve the membrane and plate vibrations to avoid the complex-valued computation in sacrifice of occurrence of spurious eigensolutions. Kamiya *et al.* [18, 19] and Yeih *et al.* [27] linked the relation of MRM and real-part BEM independently. Wong and Hutchinson [17] have presented a direct BEM involving displacement, slope, moment and shear force. They were able to obtain numerical results for simply-connected and clamped plates by employing only the real-part BEM with obvious computational gains. However, this saving leads to the spurious eigenvalues in addition to the true ones in free vibration analysis. One has to investigate the mode shapes in order to identify and reject the spurious ones. Shaw [24] commented that only the real-part approach was incorrect since the eigensolution must satisfy the real-part and imaginary-part equations at the same time. Hutchinson [16] replied that the claim of incorrectness was perhaps a little strong since the real-part BEM can obtain all the true eigensolutions although the solution is contaminated by spurious ones according to his experience. If we need to look for the eigenmode as well as eigenvalue as usually, the sorting for the spurious eigensolutions pay a small price by identifying the mode shapes. Chen *et al*. [8] commented that the spurious modes can be reasonable which may mislead the judgement of the true and spurious ones, since the true and spurious modes may have the same nodal line for the different eigenvalues. This is the reason why Chen *et al*. have developed many systematic techniques, dual formulation [8], domain partition [4], SVD updating technique [6], CHEEF method [5], for sorting out the true and the spurious eigensolutions. Niwa *et al*. [23] also stated that ``One must take care to use the complete Green's function for outgoing waves, as attempts to use just the real or imaginary part (regular part) separately will not provide the complete spectrum". As quoted from Hutchinson [16], this criticism is not correct since the real-part BEM does not lose any true eigenvalues. The reason is that the real and imaginary-part kernels satisfy the Hilbert transform. Complete eigenspectrum is imbedded in either one, real or imaginary-part kernel. The Hilbert transform is the constraint in the frequency domain corresponding to the casual effect in the time-domain fundamental solutions. The physical meaning of the real-part kernel is the standing wave [13]. Tai and Shaw [25] claimed that spurious eigenvalues are not present if the complex-valued kernel is employed for the eigenproblem. However, it is true only for the case of problem with a simply-connected domain. For multiply-connected problems, spurious eigenequation occur even though the complex-valued BEM is utilized [9, 10].

In this paper, the spurious eigensolution for the plate eigenproblem will be studied in the real-part BEM. First of all, the true and spurious eigenvalues will be examined for the simply-connected plate using the real-part BEM. Since any two equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considered. The occurring mechanism for the spurious eigensolution in the simply-connected plate problem will be studied analytically in the continuous and discrete systems. For the continuous system, degenerate kernels for the fundamental solution and the Fourier series expansion for boundary densities will be employed to derive the true and spurious eigenequations analytically for a circular plate. For the discrete system, the degenerate kernels for the fundamental solution and circulants resulting from the circular boundary will be employed to determine the spurious eigensolution. One example will be designed to check the validity of the present formulation.

2. Boundary integral equations for plate eigenproblems

The governing equation for the free flexural vibration of a uniform thin plate is written as follows:

$$\nabla^4 u(x) = \lambda^4 u(x), \quad x \in \Omega \qquad (1)$$

where u is the lateral displacement, $\lambda^4 = \dfrac{\omega^2 \rho h}{D}$, λ is the frequency parameter, ω is the circular frequency, ρ is the surface density, D is the flexural rigidity expressed as $D = \dfrac{Eh^3}{12(1-\nu^2)}$ in terms of Young's modulus E, Poisson ratio ν, the plate thickness h, and Ω is the domain of the thin plate. The integral equations for the domain point can be derived from the Rayleigh-Green identity as follows [20]:

$$u(x)=\int_B\{-U(s,x)v(s)+\Theta(s,x)m(s) - M(s,x)\theta(s)+V(s,x)u(s)\}\,dB(s),\quad x\in\Omega \tag{2}$$

$$\theta(x)=\int_B\{-U_\theta(s,x)v(s)+\Theta_\theta(s,x)m(s) - M_\theta(s,x)\theta(s)+V_\theta(s,x)u(s)\}\,dB(s),\quad x\in\Omega \tag{3}$$

$$m(x)=\int_B\{-U_m(s,x)v(s)+\Theta_m(s,x)m(s) - M_m(s,x)\theta(s)+V_m(s,x)u(s)\}\,dB(s),\quad x\in\Omega \tag{4}$$

$$v(x)=\int_B\{-U_v(s,x)v(s)+\Theta_v(s,x)m(s) - M_v(s,x)\theta(s)+V_v(s,x)u(s)\}\,dB(s),\quad x\in\Omega \tag{5}$$

where B is the boundary, u, θ, m and v mean the displacement, slope, normal moment, effective shear force, s and x are the source and field points, respectively, U, Θ, M and V kernel functions will be elaborated on later. By moving the point to the boundary, Eqs.(2)-(5) reduce to

$$\alpha\, u(x)=-P.V.\int_B U(s,x)v(s)\,dB(s)+P.V.\int_B \Theta(s,x)m(s)\,dB(s) - P.V.\int_B M(s,x)\theta(s)\,dB(s)+P.V.\int_B V(s,x)u(s)\,dB(s),\quad x\in B \tag{6}$$

$$\alpha\, \theta(x)=-P.V.\int_B U_\theta(s,x)v(s)\,dB(s)+P.V.\int_B \Theta_\theta(s,x)m(s)\,dB(s) - P.V.\int_B M_\theta(s,x)\theta(s)\,dB(s)+P.V.\int_B V_\theta(s,x)u(s)\,dB(s),\quad x\in B \tag{7}$$

$$\alpha\, m(x)=-P.V.\int_B U_m(s,x)v(s)\,dB(s)+P.V.\int_B \Theta_m(s,x)m(s)\,dB(s) - P.V.\int_B M_m(s,x)\theta(s)\,dB(s)+P.V.\int_B V_m(s,x)u(s)\,dB(s),\quad x\in B \tag{8}$$

$$\alpha\, v(x)=-P.V.\int_B U_v(s,x)v(s)\,dB(s)+P.V.\int_B \Theta_v(s,x)m(s)\,dB(s) - P.V.\int_B M_v(s,x)\theta(s)\,dB(s)+P.V.\int_B V_v(s,x)u(s)\,dB(s),\quad x\in B \tag{9}$$

where $P.V.$ denotes the principal value, and $\alpha=\frac{1}{2}$ for a smooth boundary point. The kernel function $U(s,x)$ is the real-part of the fundamental solution $U_c(s,x)$ which satisfies

$$\nabla^4 U_c(s,x)-\lambda^4 U_c(s,x)=\delta(x-s) \tag{10}$$

The three kernels, $\Theta(s,x)$, $M(s,x)$ and $V(s,x)$, are defined as follows :

$$\Theta(s,x)=K_\theta(U(s,x)) \tag{11}$$

$$M(s,x)=K_m(U(s,x)) \tag{12}$$

$$V(s,x)=K_v(U(s,x)) \tag{13}$$

where $K_\theta(\cdot)$, $K_m(\cdot)$ and $K_v(\cdot)$ mean the operators which are defined as follows:

$$K_\theta(\cdot)=\frac{\partial(\cdot)}{\partial n} \tag{14}$$

$$K_m(\cdot)=\nu\nabla^2(\cdot)+(1-\nu)\frac{\partial^2(\cdot)}{\partial n^2} \tag{15}$$

$$K_v(\cdot)=\frac{\partial\nabla^2(\cdot)}{\partial n}+(1-\nu)\frac{\partial}{\partial t}\left(\frac{\partial^2(\cdot)}{\partial n\partial t}\right) \tag{16}$$

where n and t are the normal vector and tangential vector, respectively. The operators $K_\theta(\cdot)$, $K_m(\cdot)$ and $K_v(\cdot)$ can be applied to U, Θ, M and V kernels. The kernel functions can be expressed as:

$$U(s,x)=\mathrm{Re}\left[\frac{i}{8\lambda^2}\left(H_0^{(1)}(\lambda r)+H_0^{(2)}(i\lambda r)\right)\right] \tag{17}$$

$$\Theta(s,x)=\frac{\partial U(s,x)}{\partial n_s} \tag{18}$$

$$M(s,x)=\nu\nabla_s^2 U(s,x)+(1-\nu)\frac{\partial^2 U(s,x)}{\partial n_s^2} \tag{19}$$

$$V(s,x)=\frac{\partial\nabla_s^2 U(s,x)}{\partial n_s}+(1-\nu)\frac{\partial}{\partial t_s}\left(\frac{\partial^2 U(s,x)}{\partial n_s\partial t_s}\right) \tag{20}$$

where $H_0^{(1)}(\lambda r)$ and $H_0^{(1)}(i\lambda r)$ are the zeroth order Hankel and modified Hankel functions, $r\equiv|s-x|$ and $i^2=-1$, respectively. The displacement, slope, normal moment and effective shear force are derived by

$$\theta(x)=K_\theta(u(x)) \tag{21}$$

$$m(x)=K_m(u(x)) \tag{22}$$

$$v(x)=K_v(u(x)) \tag{23}$$

Once the field point x locates outside the domain, the null-field BIEs based on the direct method of Eqs.(2)-(5) yield

$$0=\int_B\{-U(s,x)v(s)+\Theta(s,x)m(s) - M(s,x)\theta(s)+V(s,x)u(s)\}\,dB(s),\quad x\in\Omega^e \tag{24}$$

$$0=\int_B\{-U_\theta(s,x)v(s)+\Theta_\theta(s,x)m(s) -M_\theta(s,x)\theta(s)+V_\theta(s,x)u(s)\}\,dB(s),\quad x\in\Omega^e \tag{25}$$

$$0=\int_B\{-U_m(s,x)v(s)+\Theta_m(s,x)m(s) -M_m(s,x)\theta(s)+V_m(s,x)u(s)\}\,dB(s),\quad x\in\Omega^e \tag{26}$$

$$0=\int_B\{-U_v(s,x)v(s)+\Theta_v(s,x)m(s) -M_v(s,x)\theta(s)+V_v(s,x)u(s)\}\,dB(s),\quad x\in\Omega^e \tag{27}$$

where Ω^e is the complementary domain of Ω. Note that the null-field BIEs are not singular, since x and s never coincide.

When the boundary is discretized into $2N$ constant elements, the linear algebraic equations of Eqs.(2)-(5) can be obtained as follows:

$$[U]\{v\}+[M]\{\theta\}=[\Theta]\{m\}+[V]\{u\} \tag{28}$$

$$[U_\theta]\{v\}+[M_\theta]\{\theta\}=[\Theta_\theta]\{m\}+[V_\theta]\{u\} \tag{29}$$

$$[U_m]\{v\}+[M_m]\{\theta\}=[\Theta_m]\{m\}+[V_m]\{u\} \tag{30}$$

$$[U_v]\{v\}+[M_v]\{\theta\}=[\Theta_v]\{m\}+[V_v]\{u\} \tag{31}$$

where $[U]$, $[\Theta]$, $[M]$, $[V]$, $[U_\theta]$, $[\Theta_\theta]$, $[M_\theta]$, $[V_\theta]$, $[U_m]$, $[\Theta_m]$, $[M_m]$, $[V_m]$, $[U_v]$, $[\Theta_v]$, $[M_v]$ and $[V_v]$ are the sixteen influence matrices with a dimension $2N\times 2N$, $\{u\}$, $\{\theta\}$, $\{m\}$ and $\{v\}$ are the vectors of boundary data with a dimension $2N\times 1$.

3. Mathematical analysis for the true and spurious eigensolutions

In order to obtain the true and spurious eigensolutions for plate vibration using the real-part BEM, the degenerate kernel is adopted to analytically derive the true and spurious eigenequations in the continuous and discrete systems of a circular plate. For the continuous system, mathematical analysis for the spurious eigensolution was done by using the degenerate kernel and Fourier series. For the discrete system, mathematical analysis for the spurious eigensolution was done by using the degenerate kernel and circulants. The clamped circular plate is demonstrated analytically in the continuous and the discrete systems, respectively, in the following subsections.

3.1 Continuous system by using degenerate kernels and Fourier series

For the clamped circular plate ($u=0$ and $\theta=0$) with a radius a , we can obtain the eigenequation in the continuous formulation. The moment and shear force, $m(s)$ and $v(s)$ along the circular boundary, can be expanded into Fourier series by

$$m(s)=p_0+\sum_{n=1}^{\infty}(p_n\cos(n\bar{\phi})+q_n\cos(n\bar{\phi})) \tag{32}$$

$$v(s)=a_0+\sum_{n=1}^{\infty}(a_n\cos(n\bar{\phi})+b_n\cos(n\bar{\phi})) \tag{33}$$

where $\bar{\phi}$ is the angle on the circular boundary, a_n, b_n, p_n and q_n are the undetermined Fourier coefficients. Substituting Eqs.(32) and (33) into Eqs.(24) and (25) yields,

$$0=\int_B\{-U(s,x)[a_0+\sum_{n=1}^{\infty}(a_n\cos(n\bar{\phi})+b_n\cos(n\bar{\phi}))] +\Theta(s,x)[p_0+\sum_{n=1}^{\infty}(p_n\cos(n\bar{\phi})+q_n\cos(n\bar{\phi}))]dB(s) \tag{34}$$

$$0=\int_B\{-U_\theta(s,x)[a_0+\sum_{n=1}^{\infty}(a_n\cos(n\bar{\phi})+b_n\cos(n\bar{\phi}))] +\Theta_\theta(s,x)[p_0+\sum_{n=1}^{\infty}(p_n\cos(n\bar{\phi})+q_n\cos(n\bar{\phi}))]dB(s) \tag{35}$$

The kernel functions, $U(s,x)$, $\Theta(s,x)$, $U_\theta(s,x)$ and $\Theta_\theta(s,x)$, can be expanded by using the expansion formulae

$$Y_0(\lambda r)=\begin{cases}\sum_{m=-\infty}^{\infty}Y_m(\lambda\bar{\rho})J_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}>\rho\\ \sum_{m=-\infty}^{\infty}Y_m(\lambda\rho)J_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases} \tag{36}$$

$$K_0(\lambda r)=\begin{cases}\sum_{m=-\infty}^{\infty}K_m(\lambda\bar{\rho})I_m(\lambda\rho)\cos(m(\bar{\phi}-\phi)), & \bar{\rho}>\rho\\ \sum_{m=-\infty}^{\infty}K_m(\lambda\rho)I_m(\lambda\bar{\rho})\cos(m(\bar{\phi}-\phi)), & \rho>\bar{\rho}\end{cases} \tag{37}$$

where J_m and I_m denote the first kind of the *mth*-order Bessel and modified Bessel functions, Y_m and K_m denote the second kind of the *mth*-order Bessel and modified Bessel functions. The superscripts `` $i$ " and `` e " denote the interior point ($\bar{\rho}>\rho$) and the exterior point ($\bar{\rho}<\rho$),

$s=(\bar{\rho},\bar{\phi})$ and $x=(\rho,\phi)$ are the polar coordinates of s and x, respectively. In this case, $\bar{\rho}=\rho=a$ and $dB(s)=a\,d\bar{\phi}$. Similarly, the other kernels can also be expanded into degenerate forms. By using the degenerate kernels into Eqs.(34) and (35) and by employing the orthogonality condition of the Fourier series, the Fourier coefficients a_n, b_n, p_n and q_n satisfy

$$\begin{aligned}&\int_0^{2\pi}\frac{1}{8\lambda^2}\sum_{m=-\infty}^{\infty}[Y_n(\lambda a)J_n(\lambda a)-K_n(\lambda a)I_n(\lambda a)]\cos(m\bar{\phi}-\phi)\\&\qquad[a_0+\sum_{n=1}^{\infty}(a_n\cos(n\bar{\phi})+b_n\cos(n\bar{\phi}))]a\,d\bar{\phi}\\&=\int_0^{2\pi}\frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}[Y_n(\lambda a)J_n'(\lambda a)-K_n(\lambda a)I_n'(\lambda a)]\cos(m\bar{\phi}-\phi)\\&\qquad[p_0+\sum_{n=1}^{\infty}(p_n\cos(n\bar{\phi})+q_n\cos(n\bar{\phi}))]a\,d\bar{\phi}\end{aligned}\tag{38}$$

$$\begin{aligned}&\int_0^{2\pi}\frac{1}{8\lambda}\sum_{m=-\infty}^{\infty}[Y_n'(\lambda a)J_n(\lambda a)-K_n'(\lambda a)I_n(\lambda a)]\cos(m\bar{\phi}-\phi)\\&\qquad[a_0+\sum_{n=1}^{\infty}(a_n\cos(n\bar{\phi})+b_n\cos(n\bar{\phi}))]a\,d\bar{\phi}\\&=\int_0^{2\pi}\frac{1}{8}\sum_{m=-\infty}^{\infty}[Y_n'(\lambda a)J_n'(\lambda a)-K_n'(\lambda a)I_n'(\lambda a)]\cos(m\bar{\phi}-\phi)\\&\qquad[p_0+\sum_{n=1}^{\infty}(p_n\cos(n\bar{\phi})+q_n\cos(n\bar{\phi}))]a\,d\bar{\phi}\end{aligned}\tag{39}$$

According to Eq.(38), we have

$$p_n=\frac{1}{\lambda}\frac{Y_n(\lambda a)J_n(\lambda a)-K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)J_n'(\lambda a)-K_n(\lambda a)I_n'(\lambda a)}a_n,\quad N=1,2,...,\tag{40}$$

$$q_n=\frac{1}{\lambda}\frac{Y_n(\lambda a)J_n(\lambda a)-K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)J_n'(\lambda a)-K_n(\lambda a)I_n'(\lambda a)}b_n,\quad N=1,2,...,\tag{41}$$

Similarly Eq.(39) yields,

$$p_n=\frac{1}{\lambda}\frac{Y_n'(\lambda a)J_n(\lambda a)-K_n'(\lambda a)I_n(\lambda a)}{Y_n'(\lambda a)J_n'(\lambda a)-K_n'(\lambda a)I_n'(\lambda a)}a_n,\quad N=1,2,...,\tag{42}$$

$$q_n=\frac{1}{\lambda}\frac{Y_n(\lambda a)J_n(\lambda a)-K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)J_n'(\lambda a)-K_n(\lambda a)I_n'(\lambda a)}b_n,\quad N=1,2,...,\tag{43}$$

To seek nontrivial data for the generalized coefficients of a_n, b_n, p_n and q_n, we can obtain the eigenequation by using either Eqs.(40) and (42) or Eqs.(41) and (43)

$$\begin{aligned}&\frac{1}{\lambda}\frac{Y_n(\lambda a)J_n(\lambda a)-K_n(\lambda a)I_n(\lambda a)}{Y_n(\lambda a)J_n'(\lambda a)-K_n(\lambda a)I_n'(\lambda a)}\\&=\frac{1}{\lambda}\frac{Y_n'(\lambda a)J_n(\lambda a)-K_n'(\lambda a)I_n(\lambda a)}{Y_n'(\lambda a)J_n'(\lambda a)-K_n'(\lambda a)I_n'(\lambda a)}\end{aligned}\tag{44}$$

After recollecting the terms, Eq.(44) can be simplified to

$$\begin{aligned}&[K_{n+1}(\lambda a)Y_n(\lambda a)-K_n(\lambda a)Y_{n+1}(\lambda a)]\\&\{I_{n+1}(\lambda a)J_n(\lambda a)+I_n(\lambda a)J_{n+1}(\lambda a)\}=0\end{aligned}\tag{45}$$

The former part in Eq.(45) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is the true eigenequation after comparing the exact solution [22].

3.2 Discrete system by using degenerate kernel and circulants

For the clamped circular plate ($u=0$ and $\theta=0$) with a radius a, Eqs.(28) and (29) can be rewritten as

$$0=[U]\{v\}+[\Theta]\{m\}\tag{46}$$

$$0=[U_\theta]\{v\}+[\Theta_\theta]\{m\}\tag{47}$$

By assembling Eqs.(46) and (47) together, we have

$$[SM]\begin{Bmatrix}v\\m\end{Bmatrix}=0\tag{48}$$

where

$$[SM]=\begin{bmatrix}U&\Theta\\U_\theta&\Theta_\theta\end{bmatrix}_{4N\times 4N}\tag{49}$$

For the existence of nontrivial solution of $\begin{Bmatrix}v\\m\end{Bmatrix}$, the determinant of the matrix versus eigenvalue must be zero, i.e.,

$$\det[SM]=0\tag{50}$$

Since the rotation symmetry is preserved for a circular boundary, the influence matrices for the discrete system are found to be circulants with the following forms into Eq.(46), we have

$$[U]=\begin{bmatrix}z_0&z_1&z_2&\cdots&z_{2N-1}\\z_{2N-1}&z_0&z_1&\cdots&z_{2N-2}\\z_{2N-2}&z_{2N-1}&z_0&\cdots&z_{2N-3}\\\vdots&\vdots&\vdots&\ddots&\vdots\\z_1&z_2&z_3&z_{2N-1}&z_0\end{bmatrix}_{2N\times 2N}\tag{51}$$

The coefficients of each element can be obtained by using degenerate kernel

$$z_m=\int_{(m-\frac{1}{2})\Delta\bar{\phi}}^{(m+\frac{1}{2})\Delta\bar{\phi}}[-U(a,\bar{\phi},a,\phi)]a\,d\bar{\phi}\approx -U(a,\bar{\phi}_m,a,\phi)a\,\Delta\bar{\phi},\quad m=0,1,2,...,2N-1\tag{52}$$

where $\Delta\bar{\phi} = \dfrac{2\pi}{2N}$, $\bar{\phi}_m = m\Delta\bar{\phi}$. By introducing the following bases for circulants, I, $[C_{2N}]^1$, $[C_{2N}]^2$, $[C_{2N}]^3$,…, $[C_{2N}]^{2N-1}$, we can expand matrix $[U]$ into

$$[U] = z_0 I + z_1[C_{2N}]^1 + z_2[C_{2N}]^2 + ... + z_{2N-1}[C_{2N}]^{2N-1} \tag{53}$$

where

$$[C_{2N}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2N\times 2N} \tag{54}$$

Based on the similar properties for the matrices of $[U]$ and $[C_{2N}]$, we have

$$\mu_\ell^{[U]} = z_0 + z_1\alpha_\ell + z_1\alpha_\ell^2 + ... + z_{2N-1}\alpha_\ell^{2N-1}, \quad \ell = 0,1,2,...,2N-1 \tag{55}$$

where $\mu_\ell^{[U]}$ and α_ℓ are the eigenvalues for $[U]$ and $[C_{2N}]$, respectively. It is easily found that the eigenvalues for the circulants $[C_{2N}]$, are the roots for $\alpha_\ell^{2N} = 1$ as shown below:

$$\alpha_\ell = e^{i\frac{2\pi\ell}{2N}}, \quad \ell = 0,\pm1,\pm2,..,\pm N-1, N \quad or \quad \ell = 0,1,2,..,2N-1 \tag{56}$$

The eigenvector for the circulant $[C_{2N}]$ is

$$\{\phi_\ell\} = \begin{Bmatrix} 1 \\ \alpha_\ell \\ \alpha_\ell^2 \\ \vdots \\ \alpha_\ell^{2N-1} \end{Bmatrix}_{2N\times 1} \tag{57}$$

Substituting Eq.(56) into Eq.(55), we have

$$\mu_\ell^{[U]} = \sum_{m=0}^{2N-1} z_m \alpha_\ell^m = \sum_{m=0}^{2N-1} z_m e^{i\frac{2\pi m\ell}{2N}}, \quad \ell = 0,\pm1,\pm2,...,\pm(N-1), N \tag{58}$$

According to the definition for z_m in Eq.(52), we have

$$z_m = z_{2N-m}, \quad m = 0,1,2,...,2N-1 \tag{59}$$

Substitution of Eq.(59) into Eq.(58) yields

$$\begin{aligned} \mu_\ell^{[U]} &= z_0 + (-1)^\ell z_N + \sum_{m=1}^{N-1}(\alpha_\ell^m + \alpha_\ell^{2N-m}) z_m \\ &= \sum_{m=0}^{2N-1}\cos(m\ell\Delta\bar{\phi}) z_m, \quad \ell = 0,\pm1,\pm2,...,\pm(N-1), N \end{aligned} \tag{60}$$

Substituting Eq.(52) into Eq.(60) for $\phi = 0$ without loss of generality, the Reimann sum of infinite terms reduces to the following integral

$$\begin{aligned} \mu_\ell^{[U]} &= \lim_{N\to\infty} \sum_{m=0}^{2N-1} \cos(m\ell\Delta\bar{\phi})[-U(a,\bar{\phi}_m,a,0)]a\,\Delta\bar{\phi} \\ &\approx \int_0^{2\pi} \cos(\ell\bar{\phi})[-U(a,\bar{\phi},a,0)]a\,d\bar{\phi} \end{aligned} \tag{61}$$

By using the degenerate kernel for $U(s,x)$ and the orthogonal conditions of Fourier series, Eq.(61) reduces to

$$\mu_\ell^{[U]} = -\frac{\pi a}{4\lambda^2}[Y_\ell(\lambda a)J_\ell(\lambda a) - K_\ell(\lambda a)I_\ell(\lambda a)] \quad \ell = 0,\pm1,\pm2,...,\pm(N-1), N \tag{62}$$

Similarly, we have

$$\mu_\ell^{[\Theta]} = \frac{\pi a}{4\lambda}[Y_\ell(\lambda a)J'_\ell(\lambda a) - K_\ell(\lambda a)I'_\ell(\lambda a)] \quad \ell = 0,\pm1,\pm2,...,\pm(N-1), N \tag{63}$$

$$\kappa_\ell^{[U]} = -\frac{\pi a}{4\lambda^2}[Y'_\ell(\lambda a)J_\ell(\lambda a) - K'_\ell(\lambda a)I_\ell(\lambda a)] \quad \ell = 0,\pm1,\pm2,...,\pm(N-1), N \tag{64}$$

$$\kappa_\ell^{[\Theta]} = \frac{\pi a}{4\lambda^2}[Y'_\ell(\lambda a)J'_\ell(\lambda a) - K'_\ell(\lambda a)I'_\ell(\lambda a)] \quad \ell = 0,\pm1,\pm2,...,\pm(N-1), N \tag{65}$$

where $\mu_\ell^{[\Theta]}$, $\kappa_\ell^{[U]}$ and $\kappa_\ell^{[\Theta]}$ are the eigenvalues of $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ matrices, respectively. Since the four matrices $[U]$, $[\Theta]$, $[U_\theta]$ and $[\Theta_\theta]$ are all symmetric circulants, they can be expressed by

$$[U] = \Phi\Sigma_U\Phi^{-1} = \Phi\begin{bmatrix} \mu_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[U]} \end{bmatrix}\Phi^{-1} \tag{66}$$

$$[\Theta] = \Phi\Sigma_\Theta\Phi^{-1} = \Phi\begin{bmatrix} \mu_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{N-1}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mu_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \mu_N^{[\Theta]} \end{bmatrix}\Phi^{-1} \tag{67}$$

$$[U_\theta] = \Phi \Sigma_{U_\theta} \Phi^{-1}$$

$$= \Phi \begin{bmatrix} \kappa_0^{[U]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[U]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[U]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{N-1}^{[U]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[U]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[U]} \end{bmatrix} \Phi^{-1} \quad (68)$$

$$[\Theta_\theta] = \Phi \Sigma_{\Theta_\theta} \Phi^{-1}$$

$$= \Phi \begin{bmatrix} \kappa_0^{[\Theta]} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \kappa_1^{[\Theta]} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \kappa_{-1}^{[\Theta]} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \kappa_{N-1}^{[\Theta]} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \kappa_{-(N-1)}^{[\Theta]} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \kappa_N^{[\Theta]} \end{bmatrix} \Phi^{-1} \quad (69)$$

where

$$\Phi = \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & \cos(\frac{2\pi}{2N}) & \sin(\frac{2\pi}{2N}) & \cdots & \cos(\frac{2\pi(2n-1)}{2N}) & \sin(\frac{2\pi(N-1)}{2N}) & \cos(\frac{2\pi N}{2N}) \\ 1 & \cos(\frac{4\pi}{2N}) & \sin(\frac{4\pi}{2N}) & \cdots & \cos(\frac{4\pi(2n-1)}{2N}) & \sin(\frac{4\pi(N-1)}{2N}) & \cos(\frac{4\pi N}{2N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cos(\frac{2\pi(2N-1)}{2N}) & \sin(\frac{2\pi(2n-1)}{2N}) & \cdots & \cos(\frac{2\pi(2N-1)(N-1)}{2N}) & \sin(\frac{2\pi(2N-1)(N-1)}{2N}) & \cos(\frac{2\pi(N-1)N}{2N}) \end{bmatrix} \quad (70)$$

By employing Eqs.(66)-(69) for Eq.(49), we have

$$[SM] = \begin{bmatrix} \Phi \Sigma_U \Phi^{-1} & \Phi \Sigma_\Theta \Phi^{-1} \\ \Phi \Sigma_{U_\theta} \Phi^{-1} & \Phi \Sigma_{\Theta_\theta} \Phi^{-1} \end{bmatrix} \quad (71)$$

Eq.(71) can be reformulated into

$$[SM] = \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} \begin{bmatrix} \Phi & 0 \\ 0 & \Phi \end{bmatrix}^{-1} \quad (72)$$

Since Φ is orthogonal ($\det|\Phi| = \det|\Phi^{-1}| = 1$), the determinant of $[SM]_{4N \times 4N}$ is

$$\det[SM] = \det \begin{bmatrix} \Sigma_U & \Sigma_\Theta \\ \Sigma_{U_\theta} & \Sigma_{\Theta_\theta} \end{bmatrix} = \prod_{\ell=-(N-1)}^{N} \mu_\ell^{[U]} \kappa_\ell^{[\Theta]} - \mu_\ell^{[\Theta]} \kappa_\ell^{[U]} \quad (73)$$

By employing Eqs.(62)-(65) for Eq.(73), we have

$$\det[SM] = \prod_{\ell=-(N-1)}^{N} \frac{-\pi a^2}{16\lambda^2} \{ [Y_\ell(\lambda a) J_\ell(\lambda a) - K_\ell(\lambda a) I_\ell(\lambda a)] [Y'_\ell(\lambda a) J'_\ell(\lambda a) - K'_\ell(\lambda a) I'_\ell(\lambda a)] - [Y_\ell(\lambda a) J'_\ell(\lambda a) - K_\ell(\lambda a) I'_\ell(\lambda a)] [Y'_\ell(\lambda a) J_\ell(\lambda a) - K'_\ell(\lambda a) I_\ell(\lambda a)] \} \quad (74)$$

To simplify Eq.(74), we have

$$\det[SM] = \prod_{\ell=-(N-1)}^{N} \frac{-\pi a^2}{16\lambda^2} [K_{\ell+1}(\lambda a) Y_\ell(\lambda a) - K_\ell(\lambda a) Y_{\ell+1}(\lambda a)] \{ I_{\ell+1}(\lambda a) J_\ell(\lambda a) + I_\ell(\lambda a) J_{\ell+1}(\lambda a) \} \quad (75)$$

Zero determinant in Eq.(75) implies that the eigenequation is

$$[K_{\ell+1}(\lambda a) Y_\ell(\lambda a) - K_\ell(\lambda a) Y_{\ell+1}(\lambda a)] \{ I_{\ell+1}(\lambda a) J_\ell(\lambda a) + I_\ell(\lambda a) J_{\ell+1}(\lambda a) \} = 0 \quad (76)$$

After comparing with the exact solution for the clamped circular plate [22], the exact eigensolution for a continuous system can be obtained by approaching N in the the discrete system to infinity. The former part in Eq.(76) inside the middle bracket is the spurious eigenequation while the latter part inside the big bracket is the true eigenequation. The result of Eq.(76) in the discrete system matches well with Eq.(45) in the continuous system.

Since any two equations in the plate formulation (Eqs(2)-(5)) can be chosen, 6 (C_2^4) options can be considerd. If we choose different formulae for the clamped circular plate, we can obtain the same true eigensolution but different spurious eigensolution. The occurrence of spurious eigensolution only depends on the formulation instead of the boundary condition. True eigensolution depends on the boundary condition instead of the formulation. All the resluts are shown in Table 1.

4. Conclusions

A real-part formulation has been derived for the eigenproblem of the clamped plate. For a circular plate, the true and spurious eigenvalues and eigenequations were derived analytically by using the degenerate kernel, Fourier series and circulants in continuous and discrete systems. Since any two equations in the plate formulation (4 equations) can be chosen, 6 (C_2^4) options can be considerd. The occurrence of spurious eigensolution only depends on the formulation instead of the boundary condition, while the true eigensolution is independent of the formulation and is relevant to the boundary condition. All the results are shown in Table 1. The clamped circular plate cases were demonstrated analytically and numerically to see the validity of the present method.

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Table 1. Spurious eigenequations using the real-part BEMs

Eqs. number	Spurious eigenequation using the real-part BEM
u, θ Eqs.(2) and (3)	$K_{\ell+1}Y_{\ell}-K_{\ell}Y_{\ell+1}=0$
u,m Eqs.(2) and (4)	$(1-\nu)(K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})-2\lambda\rho K_{\ell}Y_{\ell}=0$
u,v Eqs.(2) and (5)	$\ell^2(1-\nu)(K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})-2\lambda\rho K_{\ell}Y_{\ell}$ $+\lambda^2\rho^2(K_{\ell+1}Y_{\ell}+K_{\ell}Y_{\ell+1})=0$
θ,m Eqs.(3) and (4)	$\ell^2(1-\nu)(K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})-2\lambda\rho\,\ell\, K_{\ell}Y_{\ell}$ $+\lambda^2\rho^2(K_{\ell+1}Y_{\ell}+K_{\ell}Y_{\ell+1})=0$
θ,v Eqs.(3) and (5)	$2\lambda\rho(\ell^2K_{\ell}Y_{\ell}+\lambda^2\rho^2K_{\ell+1}Y_{\ell+1})-2\lambda^2\rho^2\,\ell\,(K_{\ell+1}Y_{\ell}+K_{\ell}Y_{\ell+1})$ $+[2\ell-(3-\nu)\ell^2-2\lambda\rho\,\ell\,(1-\ell)](K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})=0$
m,v Eqs.(4) and (5)	$4\ell\lambda\rho(-1+\ell)[1-\ell(1-\nu)-\lambda\rho]K_{\ell}Y_{\ell}$ $+[\ell^4(1-\nu)^2+\lambda^4\rho^4-2\ell(1-\nu)(-1+\lambda\rho)](K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})$ $-\ell^2[3-4\nu+\nu^2-2\lambda\rho(1-\nu)](K_{\ell}Y_{\ell+1}-K_{\ell+1}Y_{\ell})$ $-2\lambda^2\rho^2(1-\nu)(\ell-\ell^2)(K_{\ell}Y_{\ell+1}+K_{\ell+1}Y_{\ell})+2\lambda^3\rho^3(1-\nu)K_{\ell+1}Y_{\ell+1}=0$

where $\ell=0,\pm1,\pm2,\pm3,......$