The method of fundamental solutions for two-dimensional exterior acoustics

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ABSTRACT

In this paper, the method of fundamental solutions is applied to solve for to exterior acoustic radiation problem. By using the fundamental solution, the coefficients of influence matrices are easily determined according to a two-point function. This method also results in the irregular frequency as well as the boundary element method does. The position of irregular frequency depends on the location where sources are located. To avoid this numerical instability, Burton & Miller technique is employed to deal with the problem. Based on the circulant properties and degenerate kernels, an analytical study in discrete sy stem of a cylinder radiator is demostrated.

基本解法解二維外域聲場問題

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摘要

本文中以提出基本解法來解二維外域輻射聲場的問題。藉由基本解，影響係數矩陣由二點函數輕易的得到。這個方法會有虛擬頻率的問題，這就與採用邊界元素法解外域聲場所產生的虛擬頻率問題一般。但是不規則頻率的產生位置則與所佈源點的位置有關。為了避免這個數值不穩定的問題，Burton & Miller 的技巧可用來避免虛擬頻率的發生。藉由循環矩陣及退化核函數，一個對於圓柱的輻射聲場問題，我們以離散系統成功的解析證明此論點。

1. Introduction

The method of fundamental solutions is a technique for the numerical solution of certain elliptic boundary value problems. It can be viewed as an indirect boundary element method. Like the boundary element method, it can be easily formulated when a fundamental solution of the differential equation in question is known. The basic idea is to approximate the solution by a linear combination of fundamental solutions with sources located outside the problem domain. The coefficients of the linear combination are determined so that the approximate solution satisfies the boundary conditions. Poullikkas *et al.* (2002) employed MFS to solve three-dimensional electrostatics, only a few sources were adopted. Cisilino and Sensale (2002) developed a simulated annealing algorithm for the Laplace equation to decide the optimal position of source points by using the MFS. The drawback of the method is complicated in computation and the benefit of the MFS is lost. Ramachandran (2002) adopted the SVD technique, by truncating the nearly zero singular value, to cure the ill-posed problem in the MFS. Kondapalli *et al.* (1992) applied the MFS to acoustic scattering in fluids and solids. One can consult the review paper of the MFS approach by Fairweather (1998).

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One of the problems frequently addressed in BEM is the problem of irregular (fictitious) frequencies for exterior acoustics. Kondapalli pointed out that the difficulty of fictitious frequency appears in the BEM is not present in the MFS. We may wonder whether the irregular frequency problem will occur or not in the MFS. The fictitious frequencies do not represent any kind of physical resonance but are due to the numerical method, which has not a unique solution at some eigenfrequencies for a corresponding interior problem (Ursell, 1981; Ohmatsu, 1983; Lee and Sclavounos, 1989; Dokumaci, 1990; Lee et al, 1996; Malenica and Chen,1998). It was found that BEM results in fictitious eigenvalues, which are associated with the interior frequency of the Dirichlet problem. The general derivation was provided in a continuous system (Chen,1998), and a discrete system using a circulant (Chen and Kuo, 2000). Following the retracted BEM formulation (Hwang and Chang, 1991), it was found that the position of irregular frequency depends on the source location. The MFS and the retracted BEM can be seen as the similar indirect method instead of the difference of lump source and distributed source.

In order to obtain the unique solution that is known to exist analytically, several approaches for BEM that provide additional constraints to the original system of equations have been proposed. Burton & Miller (1971) proposed an integral equation that was valid for all wave numbers by forming a linear combination of the singular integral equation and its normal derivative. However, the calculation for the hypersingular integration is required using the Burton & Miller approach. To avoid this computation, an alternative method, CHIEF, was proposed by Schenck (1968; Benthien and Schenck, 1997). Many researchers (Seybert and Rengarajan, 1989; Wu and Lobitz, 1991; Juhl, 1994; Poulin, 1997; Chen et al, 2000) applied the CHIEF method to deal with the problem of fictitious frequencies. Schenck used the CHIEF method, which employs the boundary integral equations by collocating the interior point as an auxiliary condition to make up deficient constraint condition. If the chosen point is on the node of the associated interior problem, then this method fails. This paper will focus on the study of the occurring mechanism of fictitious-frequency. An analytical study in a discrete system for a circular cylinder is conducted using the degenerate kernel and circulants. The relation between the retracted BEM and the MFS will be constructed.

2. The MFS formulation for exterior acoustics

The boundary value problem one wish to solve can be stated as follows: The acoustic pressure $u(x)$ must satisfy the Helmholtz equation,

$$\nabla^2 u(x) + k^2 u(x) = 0, \quad x \in D, \tag{1}$$

in which $k = w/c$ is the wave number and w is the angular frequency and D is the domain of interest.

The acoustic field, potential and flux, can be described by linear combinations of fundamental solutions

$$u(x) = \sum_{j=1}^{2N} U(s_j, x) A(s_j), \tag{2}$$

$$t(x) = \sum_{j=1}^{2N} L(s_j, x) A(s_j), \tag{3}$$

where $U(s,x)$ is the fundamental solution which satisfies

$$\nabla^2 U(s,x) + k^2 U(s,x) = 2pd(x-s) \tag{4}$$

in which d is the Dirac delta function, and x and s are the collocation and source points, respectively, as shown in Fig.1, $L(s,x) = \frac{\partial U(s,x)}{\partial n_x}$, $t(x) = \frac{\partial u(x)}{\partial n_x}$ and $A(s_j)$ is the generalized unknowns at s_j, $2N$ is the number of collocation points. The two kernels are,

$$U(s,x) = \frac{-ip}{2} H_0^{(1)}(kr) \tag{5}$$

$$L(s,x) = \frac{-ip}{2} H_1^{(1)}(kr) \frac{y_i n_i}{r} \tag{6}$$

in which $r \equiv |s-x|$ is the distance between the source and collocation points; n_i is the ith component of the normal vector at s; $H_0^{(1)}$ denotes the first kind of the zero-th order Hankel function, and $y_i \equiv s_i - x_i$, i= 1, 2.

We consider an infinite cylinder with the Dirichlet boundary conditions

$$u(x) = \bar{u},\ x \in B, \tag{7}$$

where B is the boundary. Matching the boundary conditions for x on the $2N$ boundary points into Eq.(2), yields

$$\{\bar{u}\} = [U_B]\{A\}, \tag{8}$$

where the subscript B denotes the boundary and $\{A\}$ is the vector of undetermined coefficients. Eq.(8) can be rearranged to

$$\{A\} = [U_B]^{-1}\{\bar{u}\}. \tag{9}$$

By substituting Eq.(9) into Eq.(2), we obtain the field pressure

$$\{u\} = [U][U_B]^{-1}\{\bar{u}\} \tag{10}$$

For the Neumann boundary conditions,

$$t(x) = \bar{t},\ x \in B, \tag{11}$$

substitution the boundary conditions for x on the $2N$ boundary points into Eq.(3) yields

$$\{\bar{t}\} = [L_B]\{A\} \tag{12}$$

Eq.(12) can be rearranged to

$$\{A\} = [L_B]^{-1}\{\bar{t}\}. \tag{13}$$

By substituting Eq.(13) into Eq.(2), we obtain the field pressure for the Neumann boundary condition

$$\{u\} = [U][L_B]^{-1}\{\bar{t}\}. \tag{14}$$

3. Analytical study for the cylinder radiator using circulants in the discrete system

For the circular case, we can express $x = (r, f)$ and $s = (R, q)$ in terms of polar coordinate. The two kernels can be expressed in terms of degenerate kernels as shown below:

$$U(s,x) = \begin{cases} U^i(R,q,r,0) = \displaystyle\sum_{m=-\infty}^{\infty} \frac{-ip}{2} H_n(kR) J_n(kr)\cos(nq) & R > r \\ U^e(R,q,r,0) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{-ip}{2} H_n(kr) J_n(kR)\cos(nq) & r > R \end{cases} \tag{15}$$

$$L(s,x) = \begin{cases} L^i(R,q,r,0) = \displaystyle\sum_{m=-\infty}^{\infty} \frac{-ip}{2} H_n(kR) J'_n(kr)\cos(nq) & R > r \\ L^e(R,q,r,0) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{-ip}{2} H'_n(kr) J_n(kR)\cos(nq) & r > R \end{cases} \tag{16}$$

where the superscripts "i" and "e" denote the interior $R > r$ and exterior domains $R < r$, respectively.

Since the rotation symmetry is preserved for a circular boundary, the two influence matrices in Eqs.(2)-(3) are denoted by [U] and [L] of the circulant with the elements

$$K_{ij} = K(R, q_j; r, f_i), \tag{17}$$

where the kernel K can be U or L, f_i and q_j are the angles of observation and source points, respectively. By superimposing $2N$ lumped strength along the boundary, we have the influence matrices,

$$[K] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-1} \\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-2} \\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & a_{2N-1} & a_0 \end{bmatrix}_{2N\times 2N} \tag{18}$$

where the element of the first row can be obtained by

$$a_{j-i} = K(s_j, x_i). \tag{19}$$

The matrix [K] in Eq.(18) is found to be a circulant since the rotational symmetry for the influence coefficients is considered. By introducing the following bases for the circulants, I, $(C_{2N})^1$, $(C_{2N})^2$,…, and $(C_{2N})^{2N-1}$, we can expand [K] into

$$[K] = a_0 I + a_1(C_{2N}) + \cdots + a_{2N-1}(C_{2N})^{2N-1} \tag{20}$$

where I is a unit matrix and

$$[C_{2N}] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{2N\times 2N} \tag{21}$$

Based on the circulant theory, the eigenvalues for the influence matrix, [K], are found as follows:

$$\begin{aligned} l_l &= a_0 + a_1 a_l + \cdots + a_{2N-1} a_l^{2N-1}, \\ l &= 0, \pm 1, \cdots, \pm (N-1), N \end{aligned} \tag{22}$$

where l_ℓ and a_ℓ are the eigenvalues for [K] and $[C_{2N}]$, respectively. It is easily found that the eigenvalues for the circulant $[C_{2N}]$ are the roots for $a^{2N} = 1$ as shown below:

$$a_l = e^{i\frac{2pl}{2N}}, \quad l = 0,\pm1,\pm2,..,\pm N-1, N \qquad or \quad l = 0,1,2,..,2N-1 \tag{23}$$

Substituting Eq.(23) into Eq.(22), we have

$$l_l = \sum_{m=0}^{2N-1} a_m a_l^{\,m} = \sum_{m=0}^{2N-1} a_m e^{i\frac{2pml}{2N}}, \quad l = 0,\pm1,\pm2,...,\pm(N-1), N \tag{24}$$

According to the definition for a_m in Eq.(19), we have

$$a_m = a_{2N-m}, \quad m = 0,1,\cdots,2N-1 \tag{25}$$

Substitution of Eq.(25) into Eq.(24) yields

$$\begin{aligned} l_l &= a_0 + (-1)^l a_N + \sum_{m=1}^{N-1} (a_l^{\,m} + a_l^{\,2N-m}) a_m \\ &= \sum_{m=0}^{2N-1} \cos(ml\Delta q) a_m,\ l = 0,\pm1,...,\pm(N-1), N \end{aligned} \tag{26}$$

Substituting Eq.(19) into Eq.(26) for the case U of K for $f = 0$ without loss of generality, the Reimann sum of infinite terms reduces to the following integral

$$\begin{aligned} l_l &= \lim_{N\to\infty} \sum_{m=0}^{2N-1} \cos(ml\Delta\bar{f})[U(m\Delta q, 0)] \\ &\approx \frac{1}{R\Delta q} \int_0^{2p} \cos(lq)[U(q,0)]\, R dq \end{aligned} \tag{27}$$

where $\Delta q = \frac{2p}{2N}$. By using the degenerate kernel for $U(s,x)$ in Eq.(15) and the orthogonal conditions of Fourier series, Eq.(27) reduces to

$$l_l = -iNp^2 H_l^{(1)}(kr) J_l(kR) \qquad l = 0,\pm1,\pm2,...,\pm(N-1), N \tag{28}$$

Similarly, we have

$$m_l = -iNp^2 H_l'^{(1)}(kr) J_l(kR) \qquad l = 0,\pm1,\pm2,...,\pm(N-1), N \tag{29}$$

where m_l is the eigenvalues of [L] matrix. The determinants for the two matrices are obtained by multiplying all the eigenvalues as shown below:

$$\det[U] = l_0 (l_1 \cdots l_{N-1})^2 l_N \tag{30}$$

$$\det[L] = m_0 (m_1 \cdots m_{N-1})^2 m_N \tag{31}$$

Since the two matrices [U] and [L] are all symmetric circulants, they can be expressed by

$$[U] = \Phi \begin{bmatrix} l_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & l_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & l_{-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & l_{N-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & l_{-(N-1)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & l_N \end{bmatrix} \Phi^{-1} \tag{32}$$

$$[L] = \Phi \begin{bmatrix} m_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & m_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & m_{-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & m_{N-1} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & m_{-(N-1)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & m_N \end{bmatrix} \Phi^{-1} \tag{33}$$

where

$$\Phi = \frac{1}{\sqrt{2N}} \begin{bmatrix} 1 & 1 & 0 & \cdots & 1 & 0 & 1 \\ 1 & \cos(\frac{2p}{2N}) & \sin(\frac{2p}{2N}) & \cdots & \cos(\frac{2p(2n-1)}{2N}) & \sin(\frac{2p(N-1)}{2N}) & \cos(\frac{2pN}{2N}) \\ 1 & \cos(\frac{4p}{2N}) & \sin(\frac{4p}{2N}) & \cdots & \cos(\frac{4p(2n-1)}{2N}) & \sin(\frac{4p(N-1)}{2N}) & \cos(\frac{4pN}{2N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & \cos(\frac{2p(2N-1)}{2N}) & \sin(\frac{2p(2n-1)}{2N}) & \cdots & \cos(\frac{2p(2N-1)(N-1)}{2N}) & \sin(\frac{2p(2N-1)(N-1)}{2N}) & \cos(\frac{2p(N-1)N}{2N}) \end{bmatrix} \tag{34}$$

For the Dirichlet problem of the Eqs.(9) and (28), the possible fictitious frequencies occur at the position where k satisfies

$$H_l^{(1)}(kr) J_l(kR) = 0,\ l = 0,\pm1,\cdots,\pm(N-1), N \tag{35}$$

Since the term of $H_l^{(1)}(kr)$ is never zero for any value of k, the k value satisfying Eq.(35) implies

$$J_l(kR) = 0. \tag{36}$$

For the Neumann problem of the Eqs.(13) and (29), the possible fictitious frequencies occur at the position where k satisfies

$$H_l'^{(1)}(kr) J_l(kR) = 0,\ l = 0,\pm1,\cdots,\pm(N-1), N \tag{37}$$

Since the term of $H_l'^{(1)}(kr)$ is never zero for any value of k, the k value satisfying Eq.(37), implies

$$J_l(kR) = 0. \tag{38}$$

It is shown that the MFS also results in the irregular frequency no matter what the boundary condition is as well as the boundary element does. The irregular frequency also appears at the eigenvalue of interior problem where the fictitious boundary is connected by the source locations instead of the real boundary in the direct BEM.

4. Burton & Miller method

In the exterior acoustics of Helmholtz equation by using the dual BEM, Burton & Miller utilized the product of hypersingular equation with an imaginary constant to the singular equation to deal with fictitious frequency which results from the non-uniqueness solution problem. We can extend this concept to the MFS approach as shown below:

$$u(x_i) = \sum_j \left(U(s_j, x_i) + \frac{i}{k} \frac{\partial U(s_j, x_i)}{\partial n_s} \right) j(s_j) \qquad (39)$$

$$t(x_i) = \sum_j \left(\frac{\partial U(s_j, x_i)}{\partial n_x} + \frac{i}{k} \frac{\partial^2 U(s_j, x_i)}{\partial n_x \partial n_s} \right) j(s_j) \qquad (40)$$

where j is the density of mixed potential. By using the degenerate kernel and circulant, for the Dirichlet problem the possible fictitious frequencies occur at the position where *k* satisfies,

$$H_l^{(1)}(kr)(J_l(kR) + \frac{i}{k} J_l'(kR)) = 0, \quad l = 0, \pm 1, \cdots, \pm (N-1), N. \qquad (41)$$

Since the terms of $H_l^{(1)}(kr)$ and $(J_l(kR) + \frac{i}{k} J_l'(kR))$ are never zero for any value of k, the unique solution is obtained all wave numbers. Similarly, for the Neumann problem, the possible fictitious frequencies occur at the position where *k* satisfies,

$$H_l'^{(1)}(kr)(J_l(kR) + \frac{i}{k} J_l'(kR)) = 0, \quad l = 0, \pm 1, \cdots, \pm (N-1), N. \qquad (42)$$

Since the term of $H_l'^{(1)}(kr)$ is never zero for any value of *k*, the unique solution is obtained all wave numbers.

5. Conclusions

In this paper, the mechanism why fictitious frequencies occur in the MFS has been examined by considering radiation problem of a cylinder. Based on the circulant properties and degenerate kernels, an analytical scheme in discrete system of a cylinder was achieved. The results from this study indicated that the irregular frequency also appears at the eigenvalue of interior problem where the boundary is connected by the source locations instead of the real boundary in direct BEM. The position of irregular frequency depends on the source location *R*. The Burton & Miller technique was demonstrated to filter out the fictitious frequency analytically.

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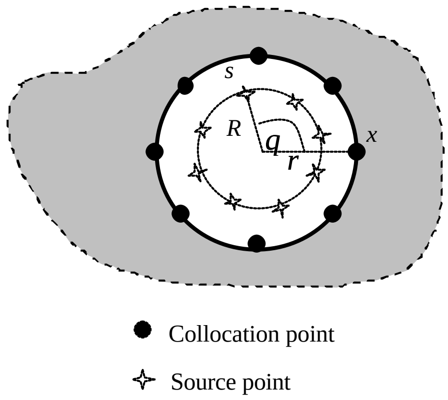

Fig.1 The located position of source and collocation point and definitions of r, q, R