Nonuniqueness and its treatment in the boundary integral equations and boundary element method

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June 20, 2003

Outlines

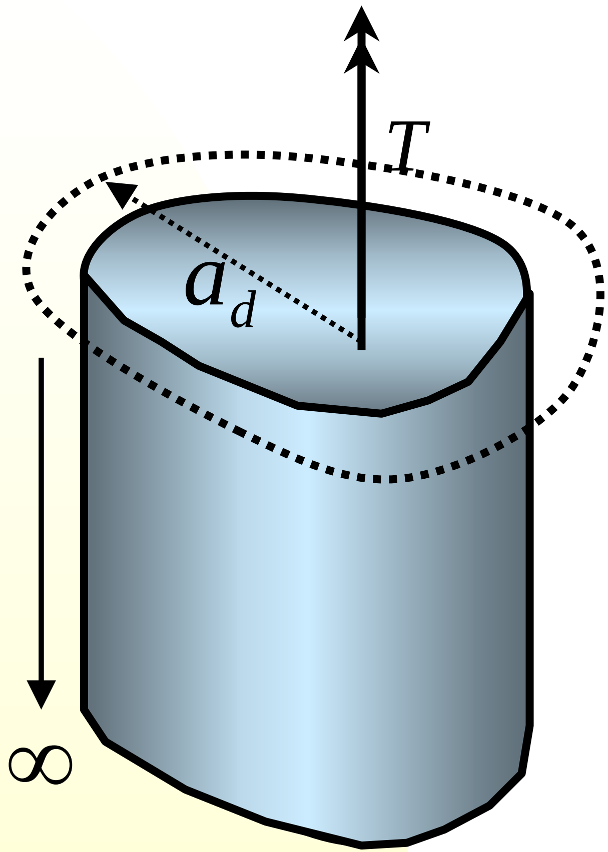
- **Overview of BIE and BEM**
- **Mathematical tools**
  - **Hypersingular BIE**
  - **Degenerate kernel**
  - **Circulants**
  - **SVD updating term**
  - **SVD updating document**
  - **Fredholm alternative theorem**
- **Nonuniqueness and its treatments**
  - **Degenerate scale**
  - **Degenerate boundary**
  - **True and spurious eigensolution (interior prob.)**
  - **Fictitious frequency (exterior acoustics)**
  - **Corner**
- **Conclusions and further research**

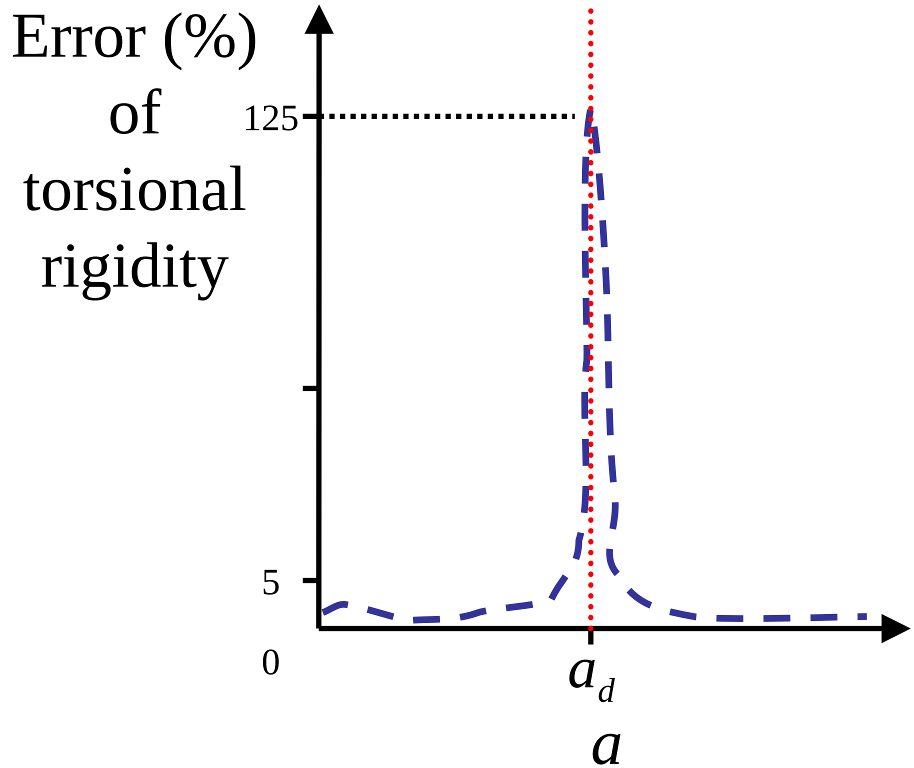
Related works since 1984

Research topics of NTOU

MSV LAB (1984-2003)

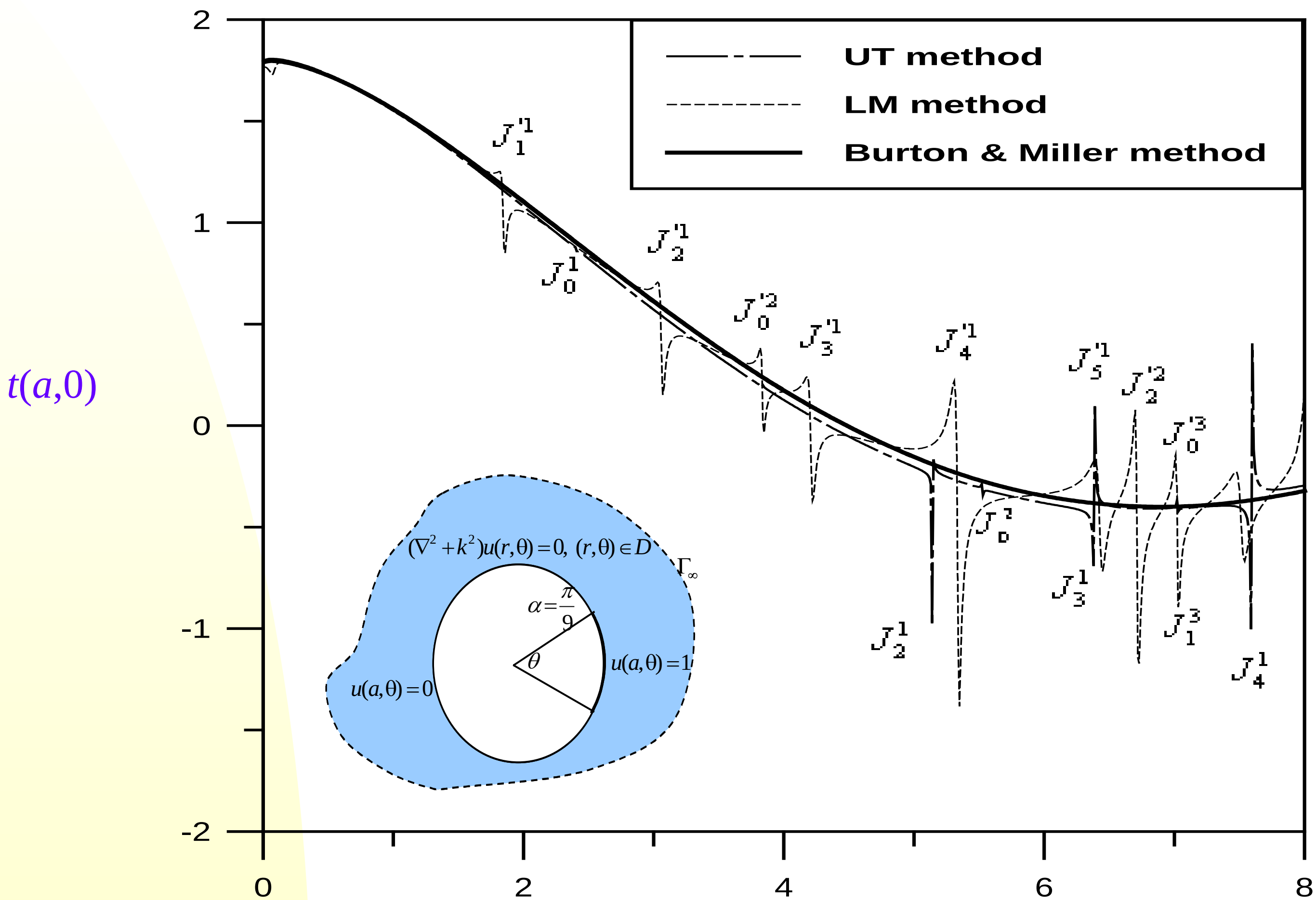
Numerical phenomena
(Degenerate scale)



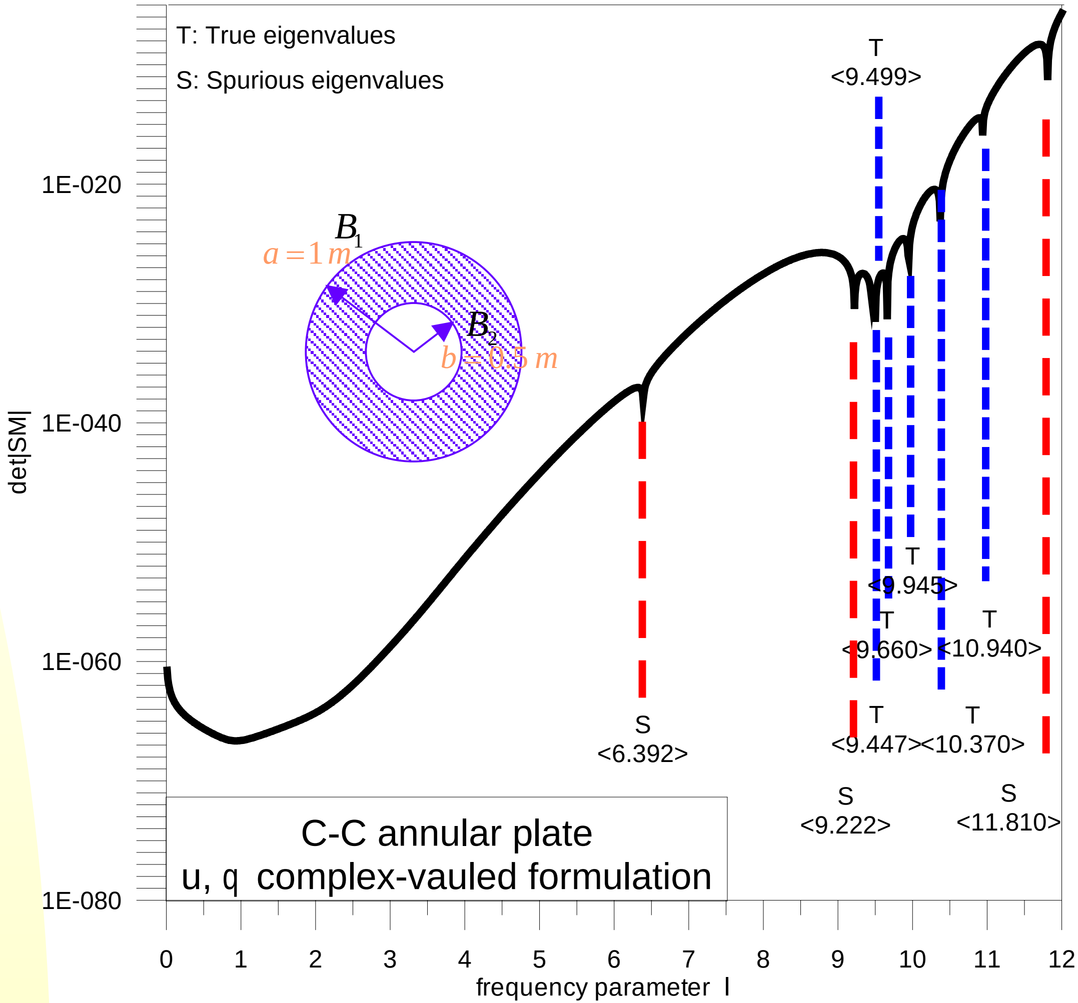


Previous approach : Try and error on a
Present approach : Only one trial

Numerical phenomena
(Fictitious frequency)

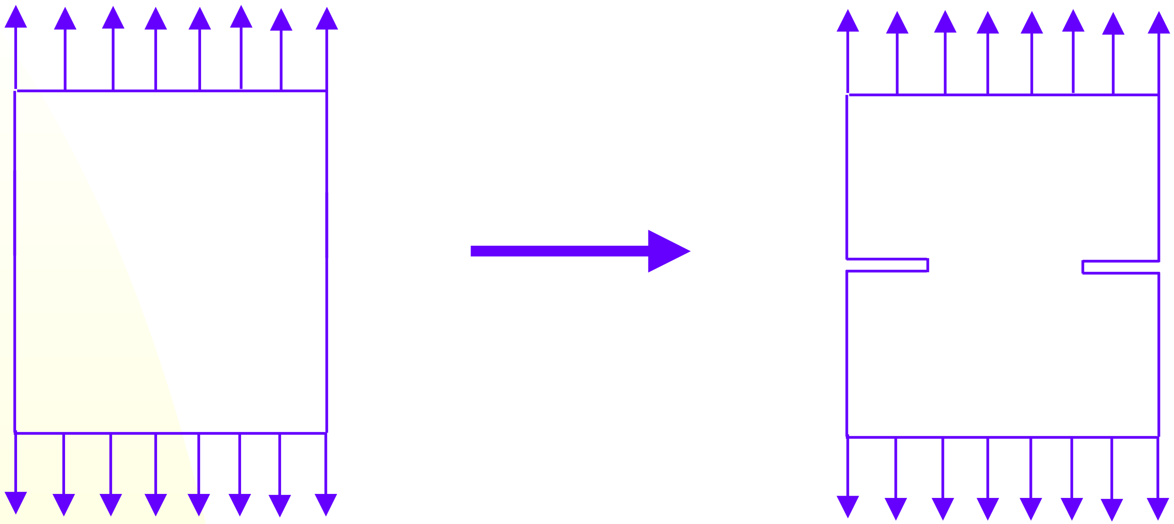


Numerical phenomena
(Spurious eigensolution)



Numerical phenomena
(Degenerate boundary)

normal boundary



degenerate boundary

Singular integral equation → Hypersingular integral equation

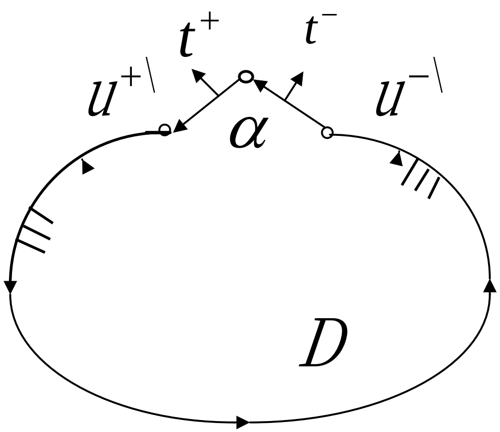
Cauchy principal value → Hadamard principal value

Boundary element method → Dual boundary element method

Numerical phenomena
(Corner)

$$\alpha u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)t(s)dB(s), \quad x \in B$$

$$\alpha t^-(x) + \sin(\alpha)t^+(x) = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)t(s)dB(s), \quad x \in B$$



Boundary

Motivation

Five pitfalls in BEM

- **Numerical instability occurs in BEM ?**

  (1) degenerate scale

  (2) degenerate boundary

  (3) fictitious frequency

  (4) corner

- **Spurious eigenvalues appear ?**

  (5) true and spurious eigenvalues

Mathematical essence—rank deficiency ?
(How to deal with ?) nonuniqueness ?

Mathematical tools

Hypersingular BIE

Degenerate kernel

Circulants

SVD updating term

SVD updating document

Fredholm alternative theorem

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Dual integral equations for a boundary point

Singular integral equation

$$\pi u(x) = C.P.V.\int_B T(s,x)u(s)\,dB(s) - R.P.V.\int_B U(s,x)t(s)\,dB(s),\ \ x \in B$$

Hypersingular integral equation

$$\pi t(x) = H.P.V.\int_B M(s,x)u(s)\,dB(s) - C.P.V.\int_B L(s,x)t(s)\,dB(s),\ \ x \in B$$

where $U(s,x)$ **is the fundamental solution.**

$$T(s,x) \equiv \frac{\partial U}{\partial n_s} \qquad L(s,x) \equiv \frac{\partial U}{\partial n_x} \qquad M(s,x) \equiv \frac{\partial^2 U}{\partial n_s \partial n_x}$$

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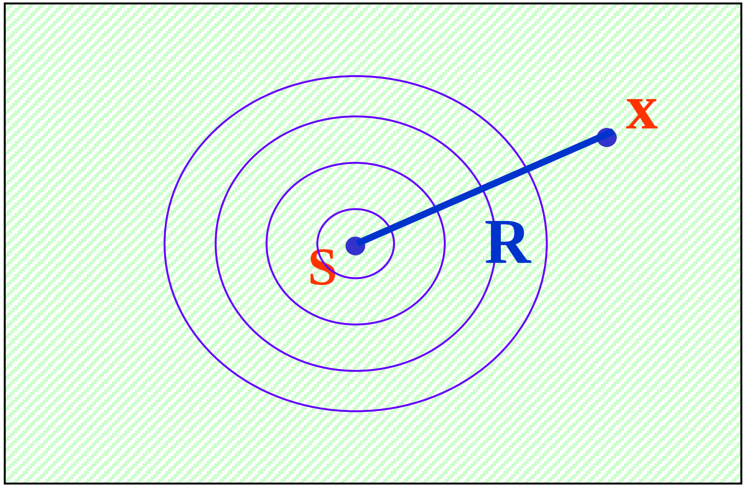
Degenerate kernel (step1)

Step 1

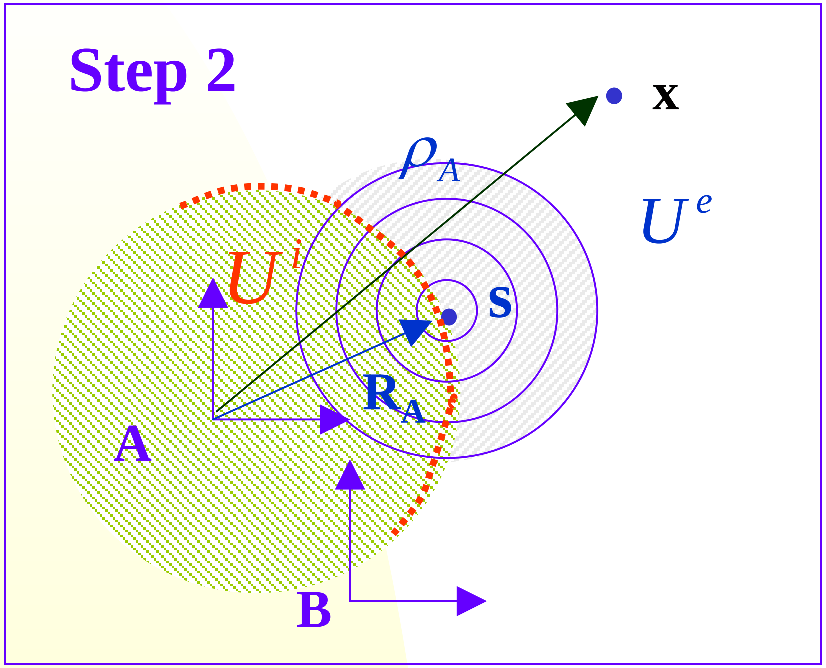
$$U(s,x) = \ln(R) = \ln\left|\underline{s} - \underline{x}\right|$$

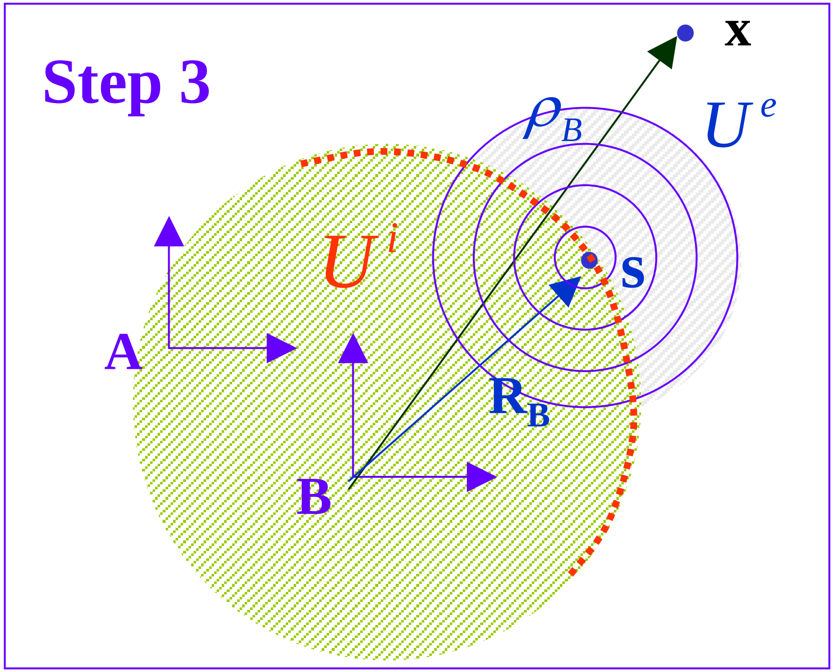


x: variable

s: fixed

Degenerate kernel (step2, step3)





$$U^e(R,\theta,\rho,\phi) = \ln(\rho) - \sum_{m=1}^{\infty} \frac{1}{m}(\frac{R}{\rho})^m \cos(m(\theta-\phi)), \quad R > \rho$$

$$U^i(R,\theta,\rho,\phi) = \ln(R) - \sum_{m=1}^{\infty} \frac{1}{m}(\frac{\rho}{R})^m \cos(m(\theta-\phi)), \quad R < \rho$$

Mathematical tools

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Circulant

$$[U] = \begin{bmatrix} Z_0 & Z_1 & Z_2 & \text{☹} & Z_{2N-1} \\ Z_{2N-1} & Z_0 & Z_1 & \text{☹} & Z_{2N-2} \\ Z_{2N-2} & Z_{2N-1} & Z_0 & \text{☹} & Z_{2N-3} \\ \text{💣} & \text{💣} & \text{💣} & \text{⚐} & \text{💣} \\ Z_1 & Z_2 & Z_3 & Z_{2N-1} & Z_0 \end{bmatrix}_{2N \times 2N}$$

$$z_m = \int_{(m-\frac{1}{2})\Delta\bar{\phi}}^{(m+\frac{1}{2})\Delta\bar{\phi}} [-U(a,\bar{\phi},a,\phi)]a\, d\bar{\phi} \approx -U(a,\bar{\phi}_m,a,\phi)a\, \Delta\bar{\phi},$$

$$m = 0,1,2,\ldots,2N-1$$

Mathematical tools

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Fredholm alternative theorem

SVD $A = \Phi\Sigma\Psi^H$

Diagonal matrix

$$[\Sigma] = \begin{bmatrix} \sigma_1 & 0 & \text{☹} & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \text{⚐} & \text{💣} \\ 0 & 0 & \text{☹} & \sigma_n \\ \text{💣} & \text{💣} & \text{☹} & \text{💣} \\ 0 & 0 & \text{☹} & 0 \end{bmatrix}_{m\times n}$$

Unitary matrix $[\Phi]_{m\times m}$, $[\Psi]_{n\times n}$

SVD updating terms

Direct method for Dirichlet B. C. :

Singular equation (UT method)

$$[T^E]\,\underset{\sim}{u} = [U^E]\,\underset{\sim}{t} = 0,$$

Hypersingular equation (LM method)

$$[M^E]\,\underset{\sim}{u} = [L^E]\,\underset{\sim}{t} = 0.$$

$$\underset{\sim}{t} = \{\psi_j\}$$

$$\begin{bmatrix} U^E \\ L^E \end{bmatrix} \{\psi_j\} = 0$$

SVD updating terms

Mathematical tools

Hypersingular BIE

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Fredholm alternative theorem

SVD updating documents

For double-layer potential approach:

$$\underbrace{\begin{Bmatrix} u(x) \\ t(x) \end{Bmatrix}}_{b} = \underbrace{\begin{bmatrix} [T(s,x)] \\ [M(s,x)] \end{bmatrix}}_{A} \underbrace{\{\psi\}}_{x}$$

⇓

$$A^T\{\phi\} = 0 \quad \text{or} \quad \{\phi\}^T A = 0$$

⇓

$$\begin{bmatrix} [T]^T \\ [M]^T \end{bmatrix}\{\phi\} = 0 \quad \text{or} \quad \underline{\underline{\{\phi\}^T\big[[T] \quad [M]\big] = 0}}$$

Mathematical tools

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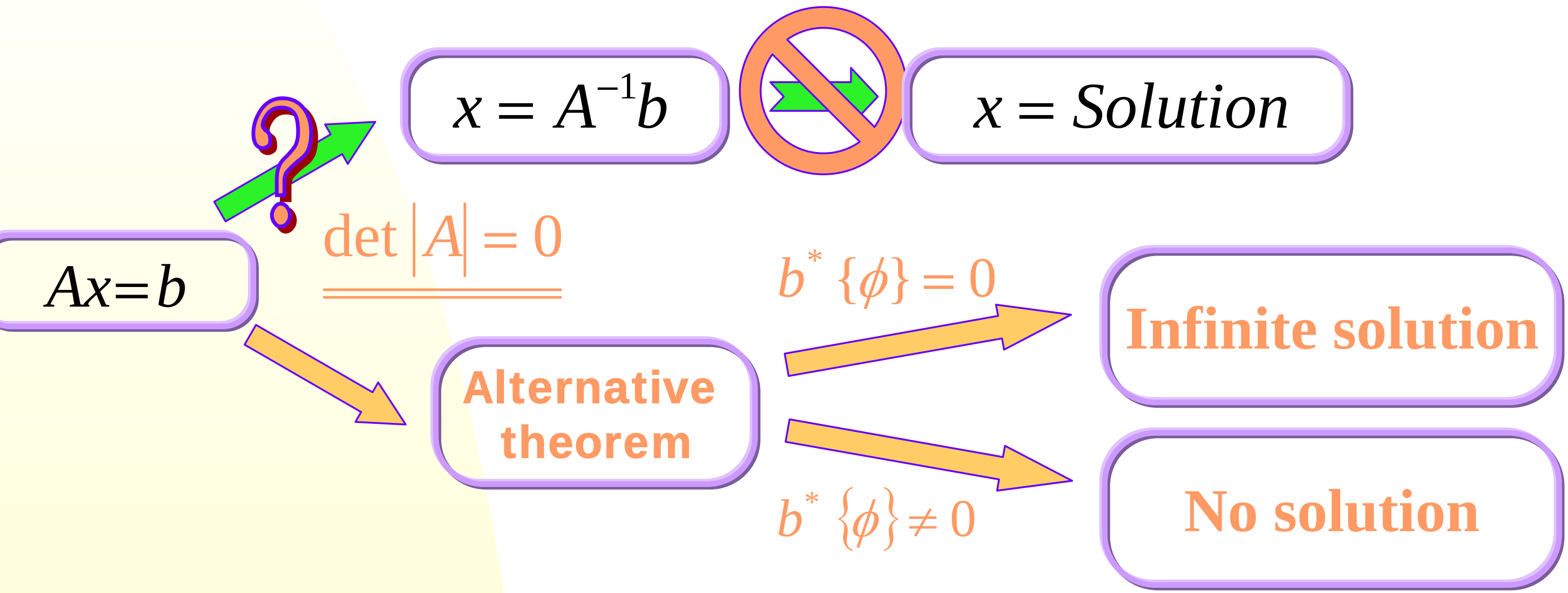
Fredholm alternative theorem

Fredholm alternative theorem

Fredholm's alternative theorem:

For solving an algebraic system: $Ax = b$



A^* : the transpose conjugate matrix of A

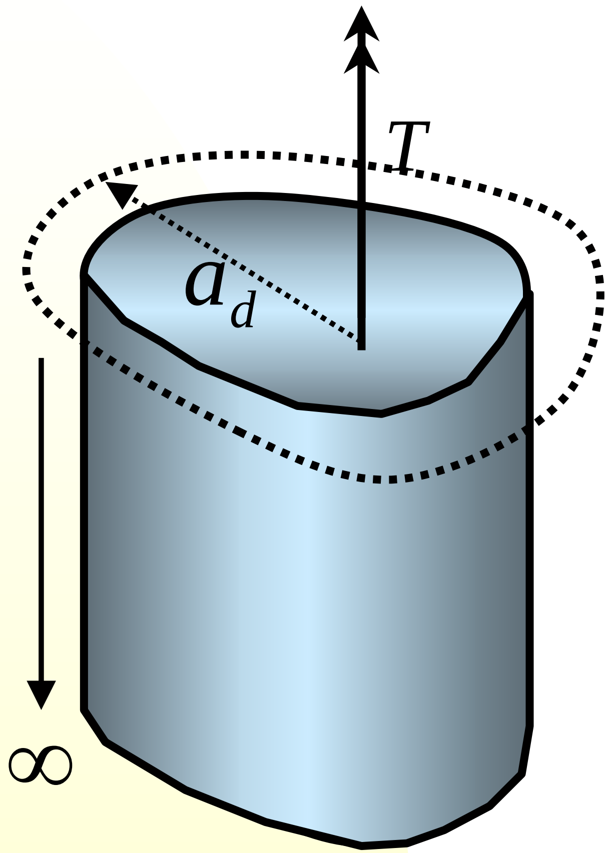
$A^* = A^T$ if A is real

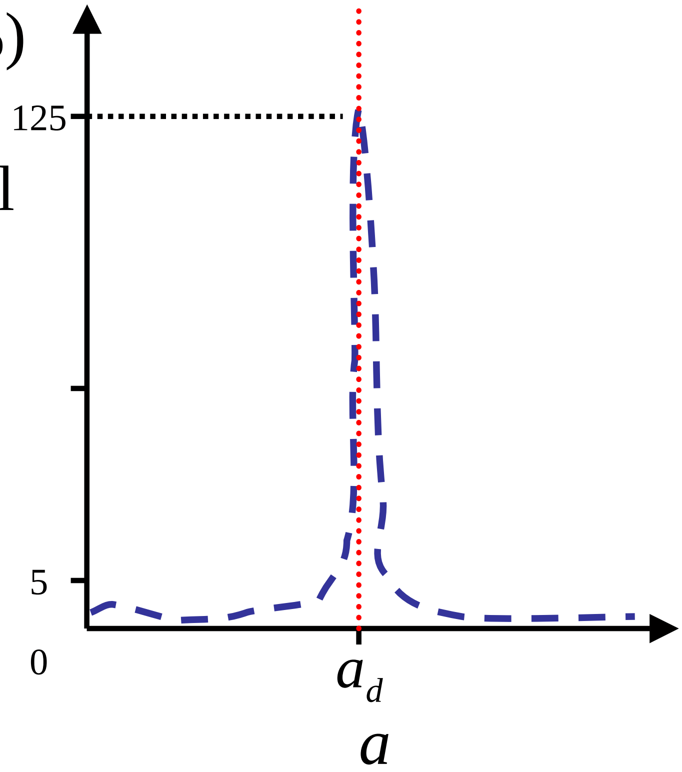
where f satisfies $A^* \{\phi\} = 0$

Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner

The degenerate scale for torsion bar using BEM





Previous approach : Try and error on a
Present approach : Only one trial

Determination of the degenerate scale by trial and error

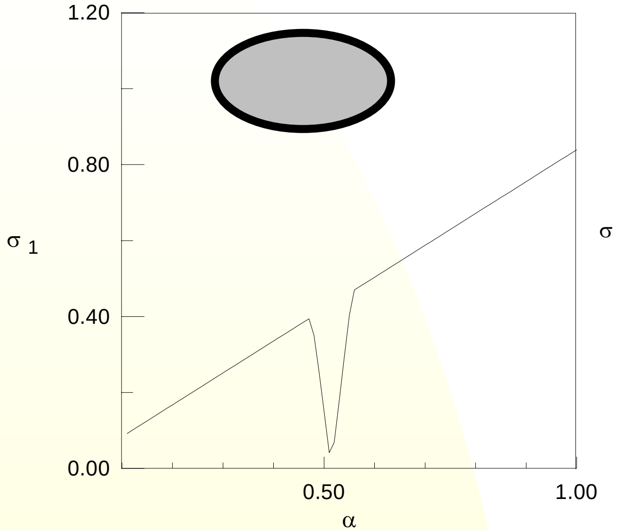


Fig.2-13 The minimum singular value σ_1 of [U] versus semiaxes α for the interior potential problem with an elliptic domain.

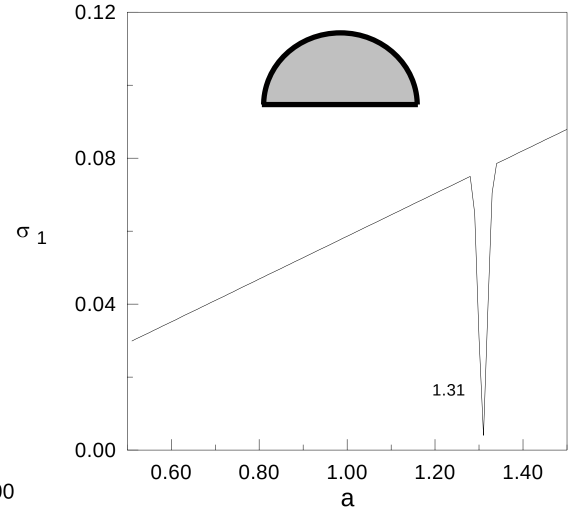


Fig.2-19 The minimum singular value σ_1 of [U] versus radius a for the interior potential problem with a semicircular domain.

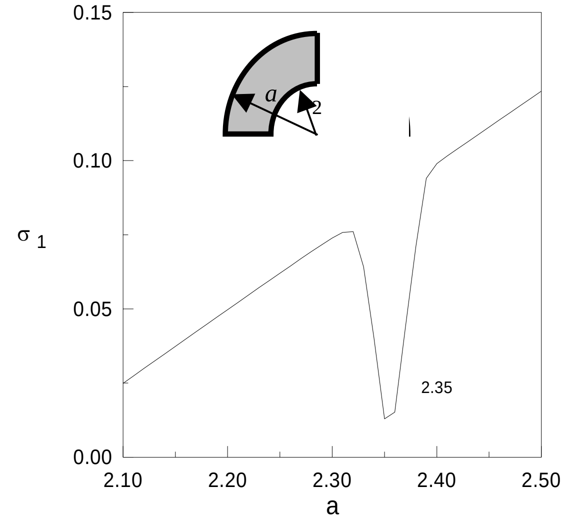
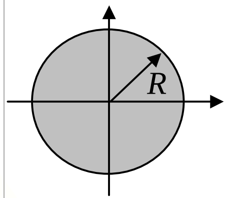
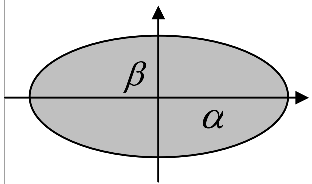
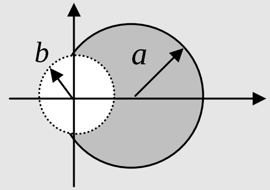
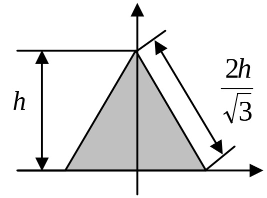
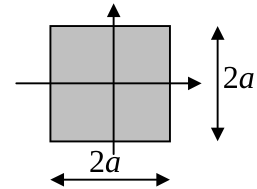


Fig.2-20 The minimum singular value σ_1 of [U] versus parameter a for the interior potential problem with a sector domain.

Direct searching for the degenerate scale

Trial and error---detecting zero singular value by using SVD [Lin (2000) and Lee (2001)]

Determination of the degenerate scale for the two-dimensional Laplace problems

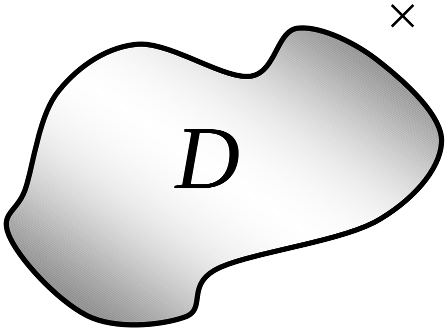
Cross Section					
Normal scale	$R = 2.0$	$\alpha = 3.0, \quad \beta = 1.0$	$a = 2.0, \quad b = \frac{2}{3}a$	$h = 3.0$	$a = 1.0$
Torsional rigidity	$G\frac{\pi}{2}R^4$	$G\frac{\pi\alpha^3\beta^3}{\alpha^2+\beta^2}$	$2Ga^4k_2$	$G\frac{\sqrt{3}}{45}h^4$	Ga^4k_1
Reference equation	$u(x) = \int_B U(s,x)\psi_1(s)dB(s)$, where , x on B, . $[U]\{\psi\} = \{1\}$				
$\Gamma = \int_B \psi_1(s)dB(s)$	1.4480 $(\frac{1}{\ln(2)})$	1.4509 $(\frac{1}{\ln(2)})$	1.5539 (N.A.)	2.6972 (N.A.)	6.1530 (6.1538)
Expansion ratio $d = e^{-\frac{1}{\Gamma}}$	0.5020 (0.5)	0.5019 (0.5)	0.5254 (N.A.)	0.6902 (N.A.)	0.8499 (0.85)
Degenerate scale	R=1.0040 (1.0)	$\alpha + \beta$ = 2.0058 (2.0)	a=1.0508 (N.A.)	h=2.0700 (N.A.)	a=0.8499 (0.85)

Note: Data in parentheses are exact solutions.

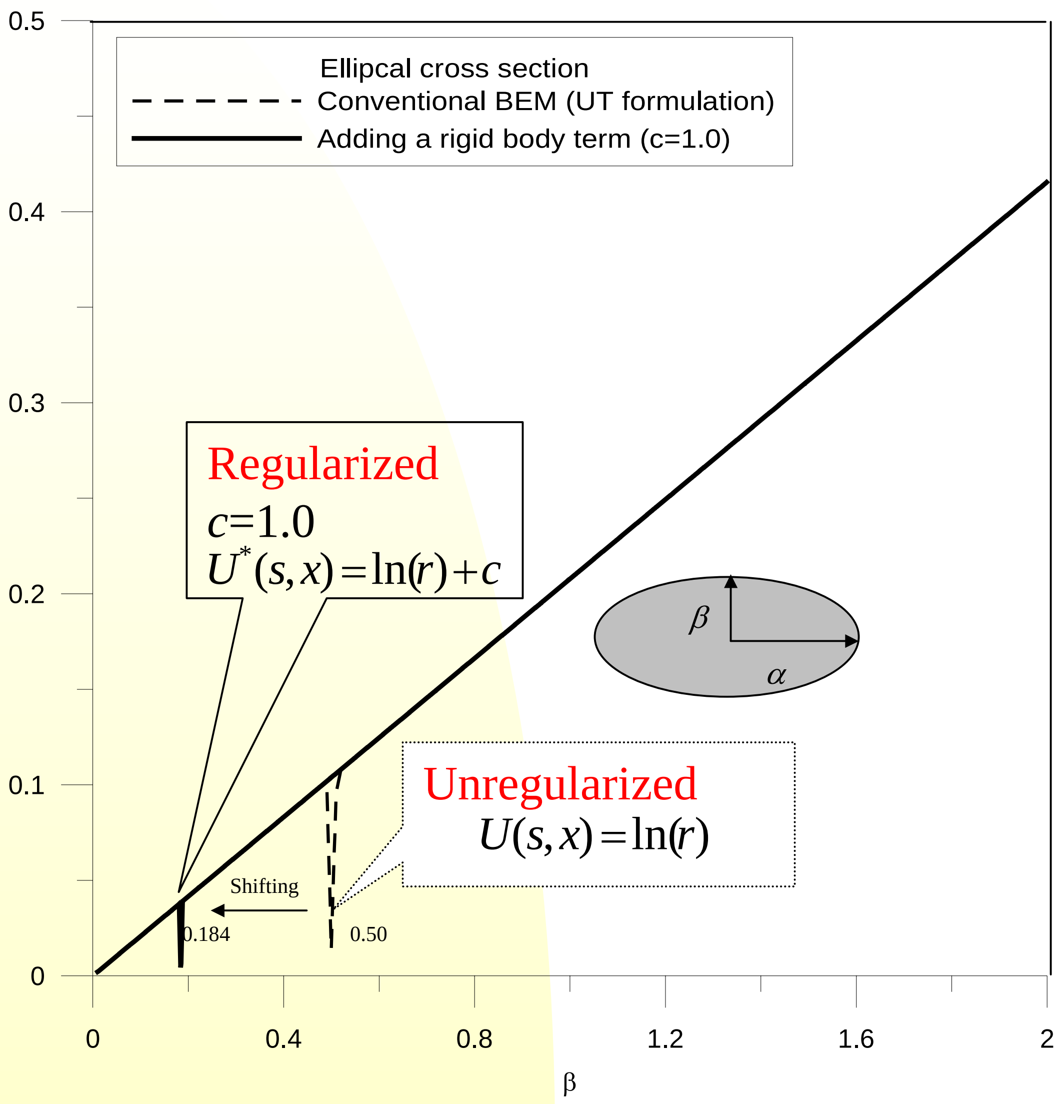
Data marked in the shadow area are derived by using the polar coordinate.

Three regularization techniques to deal with degenerate scale problems in BEM

- Hypersingular formulation (*LM* equation)
- Adding a rigid body term **($U^*(s,x)=U(s,x)+c$)**
- CHEEF concept



Degenerate scale for torsion bar problems with arbitrary cross sections



Normal scale

$\alpha = 3.0,\ \beta = 1.0\ \ (\alpha = 3\beta)$

$U(s,x)=\ln(r)$ $\quad d = e^{-\frac{1}{\Gamma}}$

Original degenerate scale

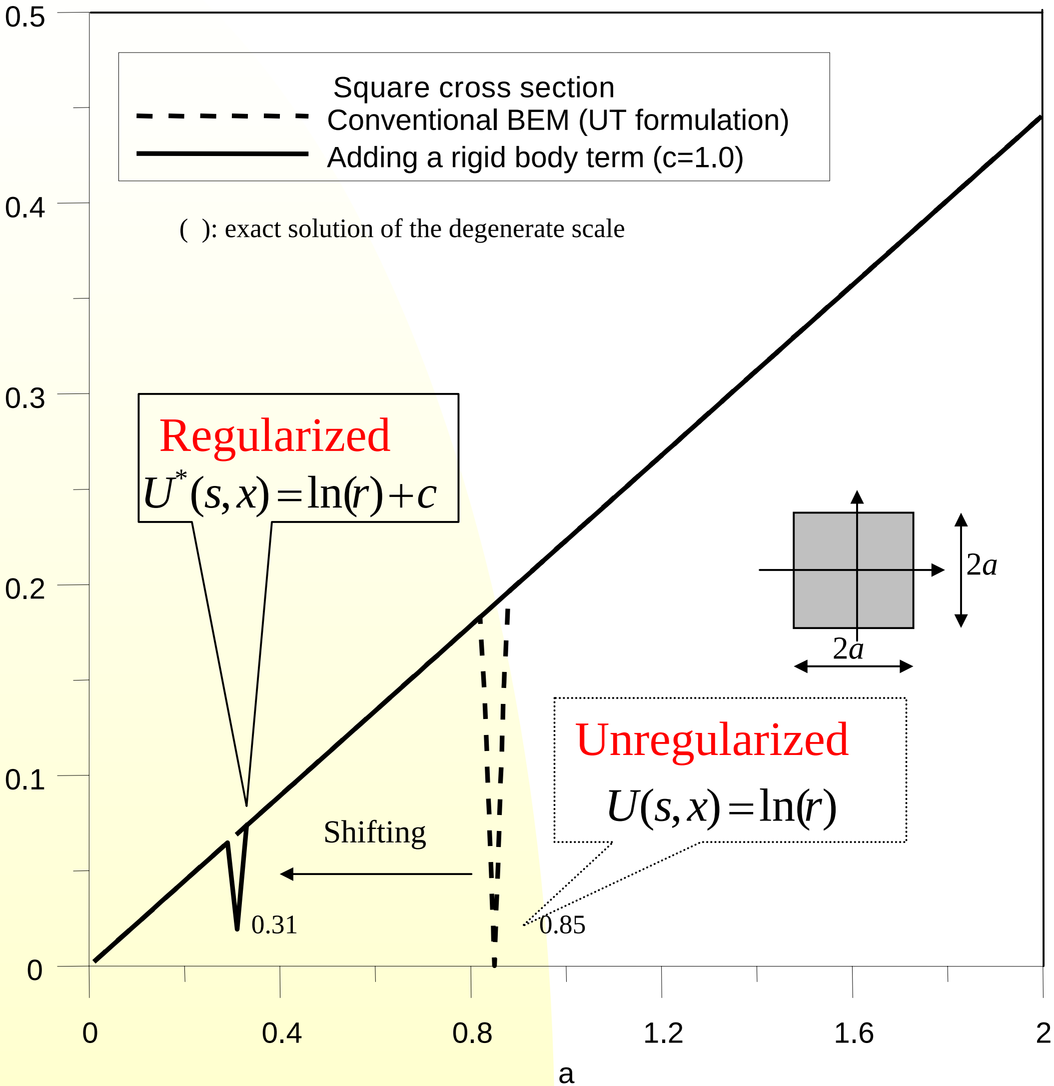
$\alpha = 1.5,\ \beta = 0.5$

$U^*(s,x)=\ln(r)+c$ $\quad d^* = e^{-c}$

New degenerate scale

$\alpha = 0.552,\ \beta = 0.184$

Degenerate scale for torsion bar problems with arbitrary cross sections



Normal scale
$a = 1.0$

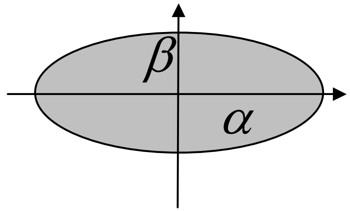
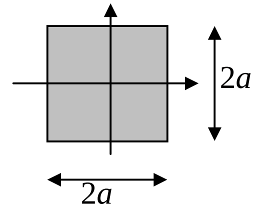
$U(s,x) = \ln(r)$ $\quad d = e^{-\frac{1}{\Gamma}}$

Original degenerate scale
$a = 0.85$

$U^*(s,x) = \ln(r) + c$ $\quad d^* = e^{-c}$

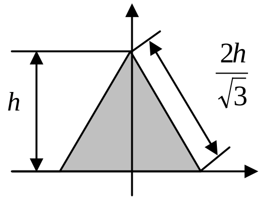
New degenerate scale
$a = 0.31$

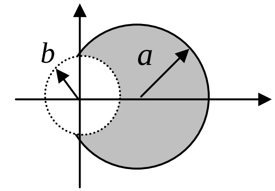
Numerical results

Torsion rigidity / cross section / method	 Ellipse		 Square	
	Normal scale (α=3.0, β=1.0)	Degenerate scale (α=1.5, β=0.5)	Normal scale (**a**=1.0)	Degenerate scale (**a**=0.85)
Analytical solution	$G\frac{\pi\alpha^3\beta^3}{\alpha^2+\beta^2}$ 8.4823	$G\frac{\pi\alpha^3\beta^3}{\alpha^2+\beta^2}$ 0.5301	$16k_1Ga^4$ 2.249	$16k_1Ga^4$ 1.174
U T Conventional BEM	8.7623 (3.30%)	-0.8911 (268.10%)	2.266 (0.76%)	2.0570 (75.21%)
L M formulation	**Regularization techniques are not necessary.**	0.4812 (9.22%)	**Regularization techniques are not necessary.**	1.1472 (2.31%)
Add a rigid body term, *c*=1.0		0.5181 (2.26%)		1.1721(0.19%)
Add a rigid body term, *c*=2.0		0.5176 (2.36%)		1.1723 (0.17%)
CHEEF concept		0.5647 (6.53%) CHEEF POINT (2.0, 2.0)		1.1722 (0.18%) CHEEF POINT (5.0, 5.0)

Note: data in parentheses denote error.

Numerical results

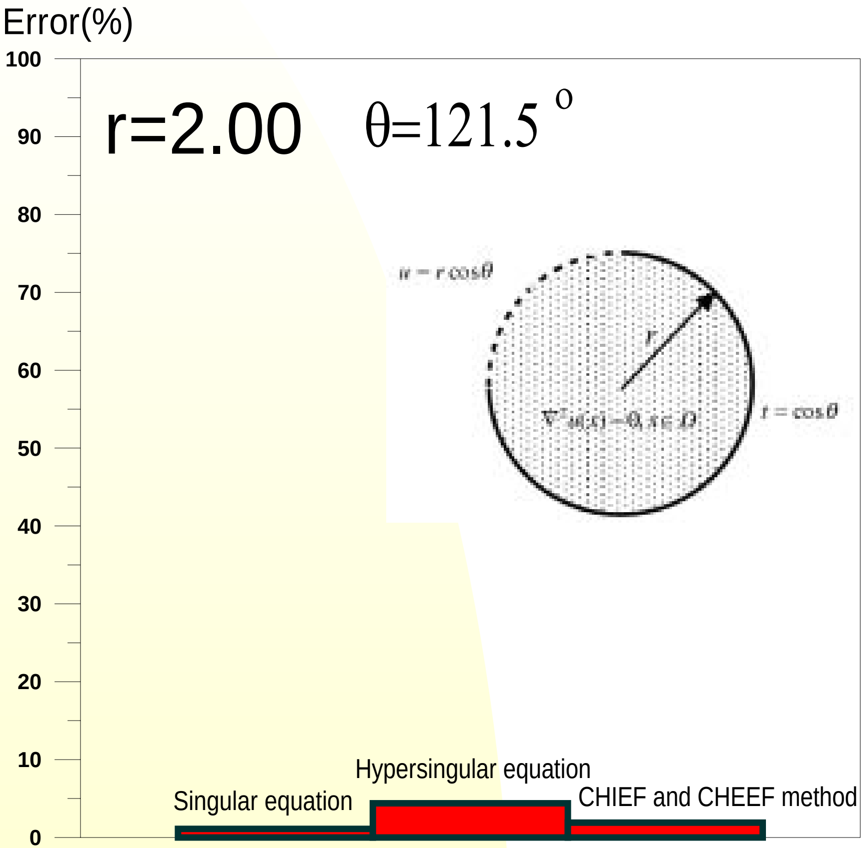


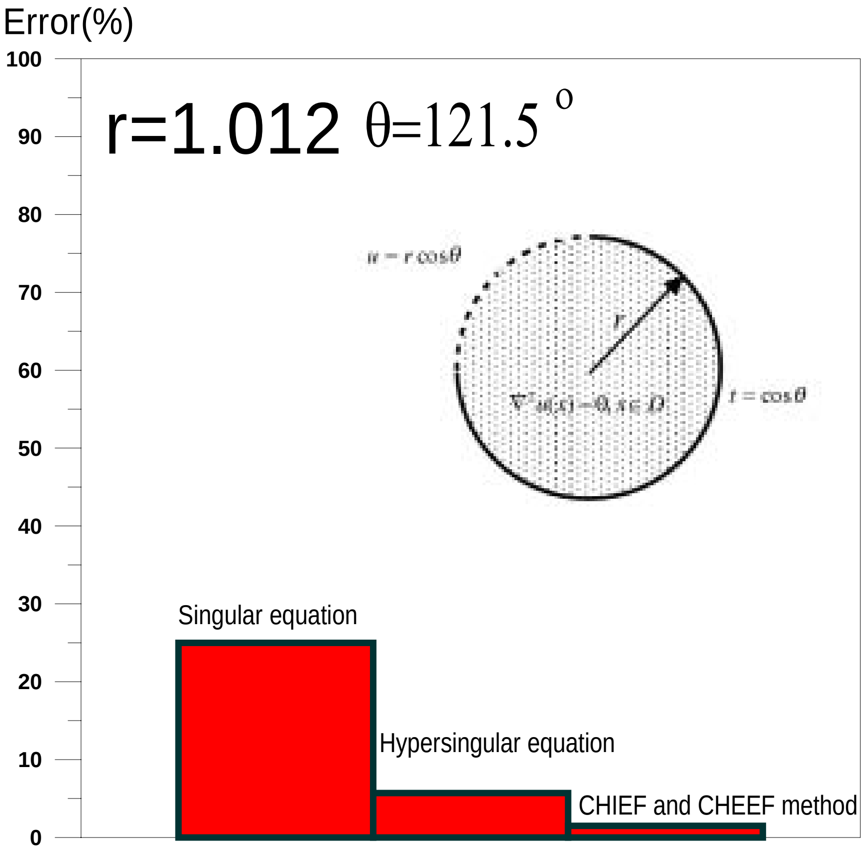


Torsion rigidity / **method** \ **cross section**		**Triangle**		**Keyway**	
		Normal scale h=3.0	Degenerate scale h=2.07	Normal scale (a=2.0)	Degenerate scale (a=1.05)
Analytical solution		3.1177 $G\frac{\sqrt{3}}{45}h^4$	0.7067 $G\frac{\sqrt{3}}{45}h^4$	12.6488 $2Ga^4k_2$	0.9609 $2Ga^4k_2$
U T Conventional BEM		3.1829 (2.09%)	1.1101 (57.08%)	12.5440 (0.83%)	1.8712 (94.73%)
L M formulation		**Regularization techniques are not necessary.**	0.6837 (3.25%)	**Regularization techniques are not necessary.**	0.9530 (0.82%)
Add a rigid body term	c=1.0		0.7035 (0.45%)		0.9876 (2.78%)
	c=2.0		0.7024 (0.61%)		0.9879 (2.84%)
CHEEF concept			0.7453 (5.46%) CHEEF POINT (15.0, 15.0)		0.9272 (3.51%) CHEEF POINT (20.0, 20.0)

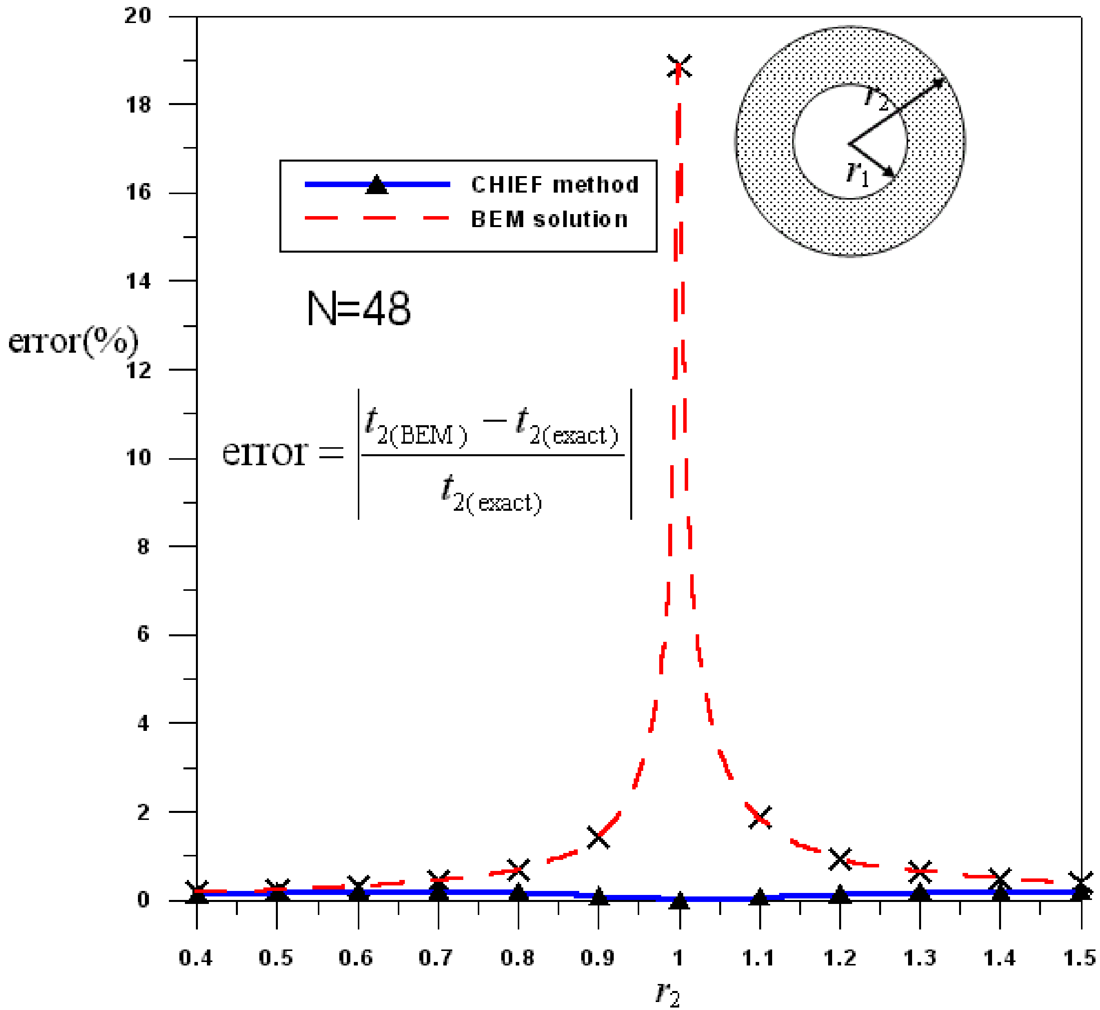
Note: data in parentheses denote error.

Error using three methods





Multiply-connected problem



Five pitfalls in BEM

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2. Degenerate boundary problems
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4. Fictitious frequency for exterior acoustics
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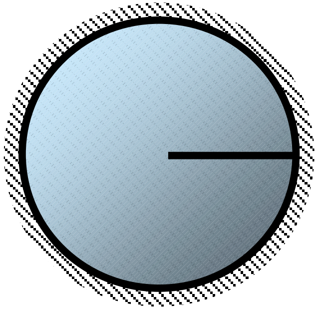
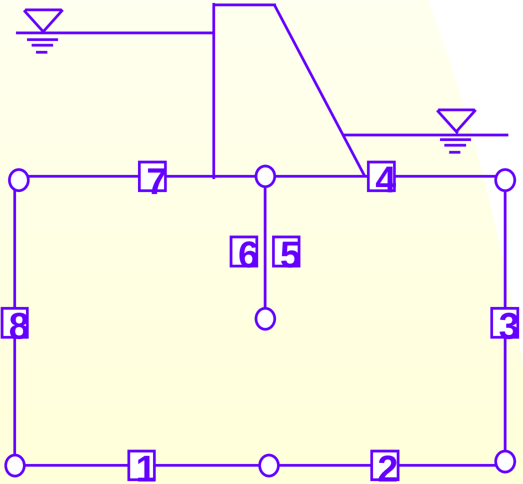


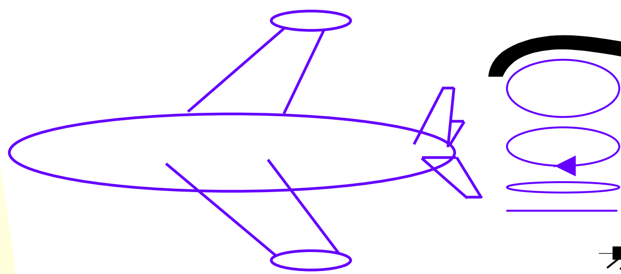
Engineering problems

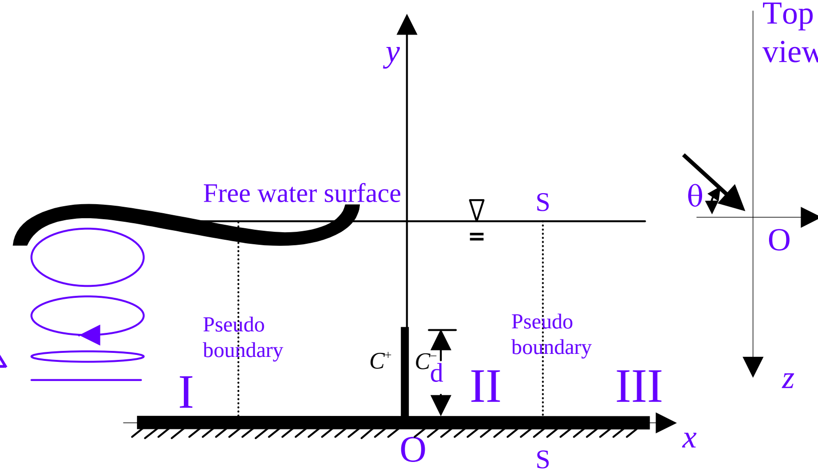
Seepage with sheetpiles

Thin-airfoill Aerodynamics

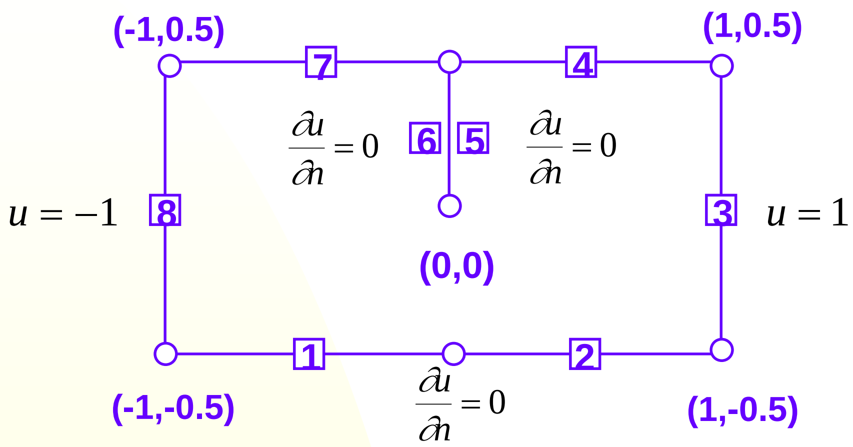
oblique incident water wave







Degeneracy of the Degenerate Boundary



○ **geometry node**

N **the Nth constant or linear element**

$$[U]\{t\} = [T]\{u\}$$

$$[L]\{t\} = [M]\{u\}$$

5(+) 6(+)

$$[U]=\begin{bmatrix} -1.693 & -0.045 & 0.471 & 0.347 & -0.054 & -0.054 & 0.039 & -0.335 \\ -0.045 & -1.693 & -0.335 & 0.039 & -0.054 & -0.054 & 0.347 & 0.471 \\ 0.445 & -0.335 & -1.693 & -0.335 & 0.019 & 0.019 & 0.445 & 0.703 \\ 0.347 & 0.039 & -0.335 & -1.693 & -0.281 & -0.281 & -0.045 & 0.471 \\ -0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 \\ -0.081 & -0.081 & 0.063 & -0.638 & -1.193 & -1.193 & -0.638 & 0.063 \\ 0.039 & 0.347 & 0.471 & -0.045 & -0.281 & -0.281 & -1.693 & -0.334 \\ -0.335 & 0.445 & 0.703 & 0.445 & 0.019 & 0.019 & -0.335 & -1.693 \end{bmatrix}$$

5(+)
6(+)

$n(s)$ 5(+) 6(-)

$$[T]=\begin{bmatrix} -\pi & 0.000 & 0.588 & 0.519 & -0.321 & 0.321 & 0.927 & 1.107 \\ 0.000 & -\pi & 1.107 & 0.927 & 0.321 & -0.321 & 0.519 & 0.588 \\ 0.219 & 1.107 & -\pi & 1.107 & 0.464 & -0.464 & 0.219 & 0.490 \\ 0.519 & 0.927 & 1.107 & -\pi & 0.785 & -0.785 & 0.000 & 0.588 \\ 0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 \\ 0.927 & 0.927 & 0.888 & 1.326 & -\pi & -\pi & 1.326 & 0.888 \\ 0.927 & 0.519 & 0.588 & 0.000 & -0.7854 & 0.785 & -\pi & 1.107 \\ 1.107 & 0.219 & 0.490 & 0.219 & -0.464 & 0.464 & 1.107 & -\pi \end{bmatrix}$$

5(+)
6(+)

5(+) 6(+)

$n(x)$

$$[L]=\begin{bmatrix} \pi & 0.000 & 0.184 & 0.519 & 0.458 & 0.458 & 0.927 & 0.805 \\ 0.000 & \pi & 0.805 & 0.927 & 0.458 & 0.458 & 0.519 & 0.184 \\ 0.612 & 0.805 & \pi & 0.805 & 0.464 & 0.464 & 0.612 & 0.490 \\ 0.519 & 0.927 & 0.805 & \pi & 0.347 & 0.347 & 0.000 & 0.184 \\ -0.511 & 0.511 & 0.888 & 1.417 & \pi & -\pi & -1.417 & -0.888 \\ 0.511 & -0.511 & -0.888 & -1.417 & -\pi & \pi & 1.417 & 0.888 \\ 0.927 & 0.519 & 0.184 & 0.000 & 0.347 & 0.347 & \pi & 0.805 \\ 0.805 & 0.612 & 0.490 & 0.612 & 0.464 & 0.464 & 0.805 & \pi \end{bmatrix}$$

5(+)
6(-)

$n(s)$ 5(+) 6(-)

$n(x)$

$$[M]=\begin{bmatrix} 4.000 & -1.333 & -0.205 & -0.061 & 0.600 & -0.600 & -0.800 & -1.600 \\ -1.333 & 4.000 & -1.600 & -0.800 & -0.600 & 0.600 & -0.061 & -0.205 \\ -0.282 & -1.600 & 4.000 & -1.600 & -0.400 & 0.400 & -0.282 & -0.236 \\ -0.061 & -0.800 & -1.600 & 4.000 & -1.000 & 1.000 & -1.333 & -0.205 \\ 0.853 & -0.853 & -0.715 & -3.765 & 8.000 & -8.000 & 3.765 & 0.715 \\ -0.853 & 0.853 & 0.715 & 3.765 & -8.000 & 8.000 & -3.765 & -0.715 \\ -0.800 & -0.062 & -0.205 & -1.333 & 1.000 & -1.000 & 4.000 & -1.600 \\ -1.600 & -0.282 & -0.235 & -0.282 & 0.400 & -0.400 & -1.600 & 4.000 \end{bmatrix}$$

5(+)
6(-)

Theory of dual integral equations

$$f(x) = (x-a)^2 Q(x) + px + q$$

$$f(a) = pa + q, \quad when\ x = a$$

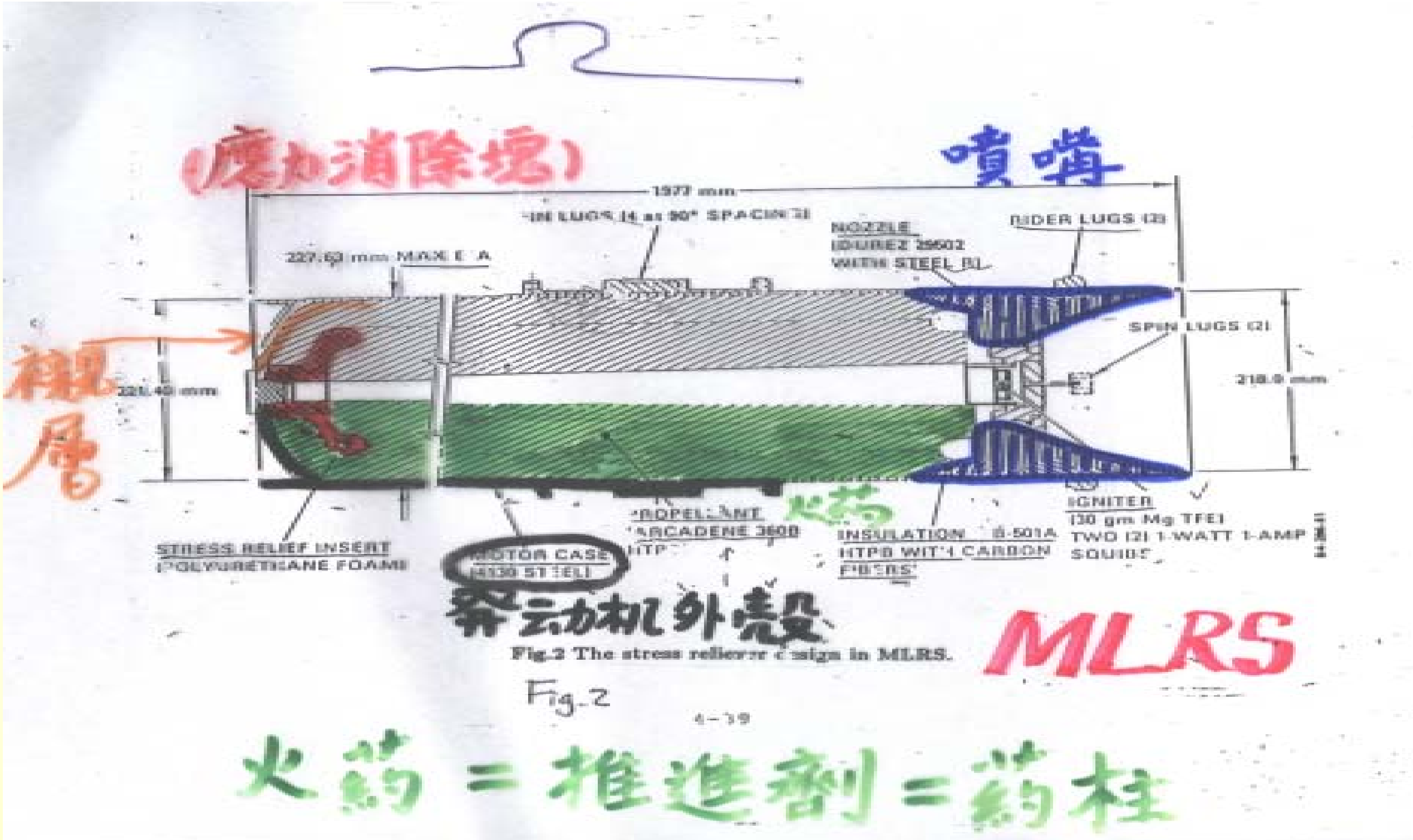
The constraint equation is not enough to determine the coefficient *p* and *q*,

Another constraint equation is required

$$f'(x) = 2(x-a)Q(x) + (x-a)^2 Q'(x) + p$$

$$f'(a) = p, \quad when\ \ x = a$$

Successful experiences



X-ray detection

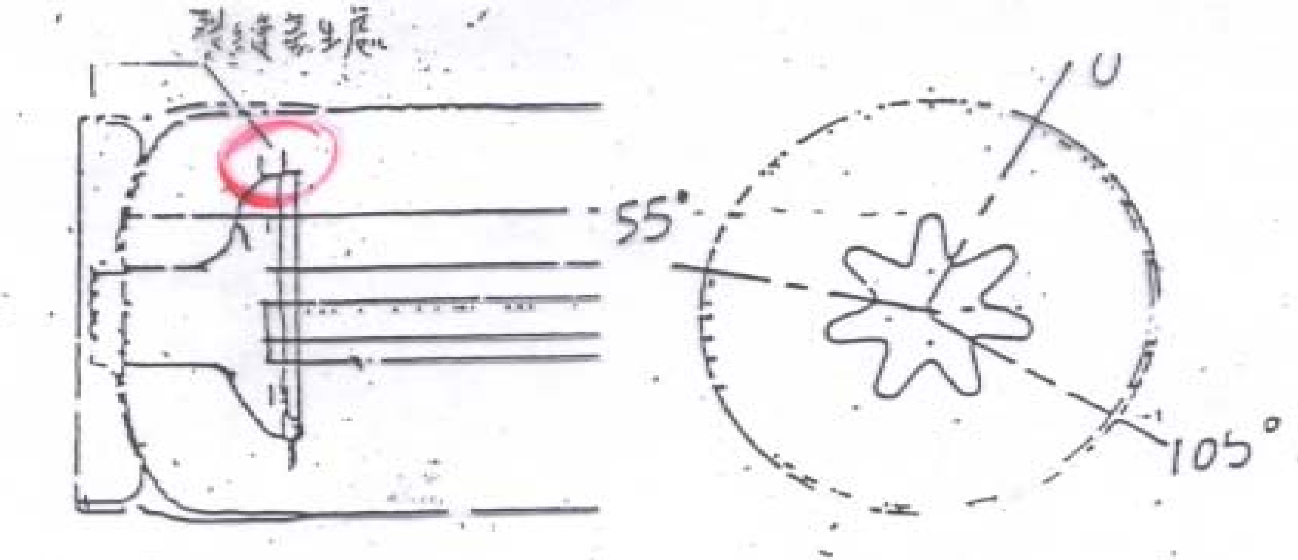
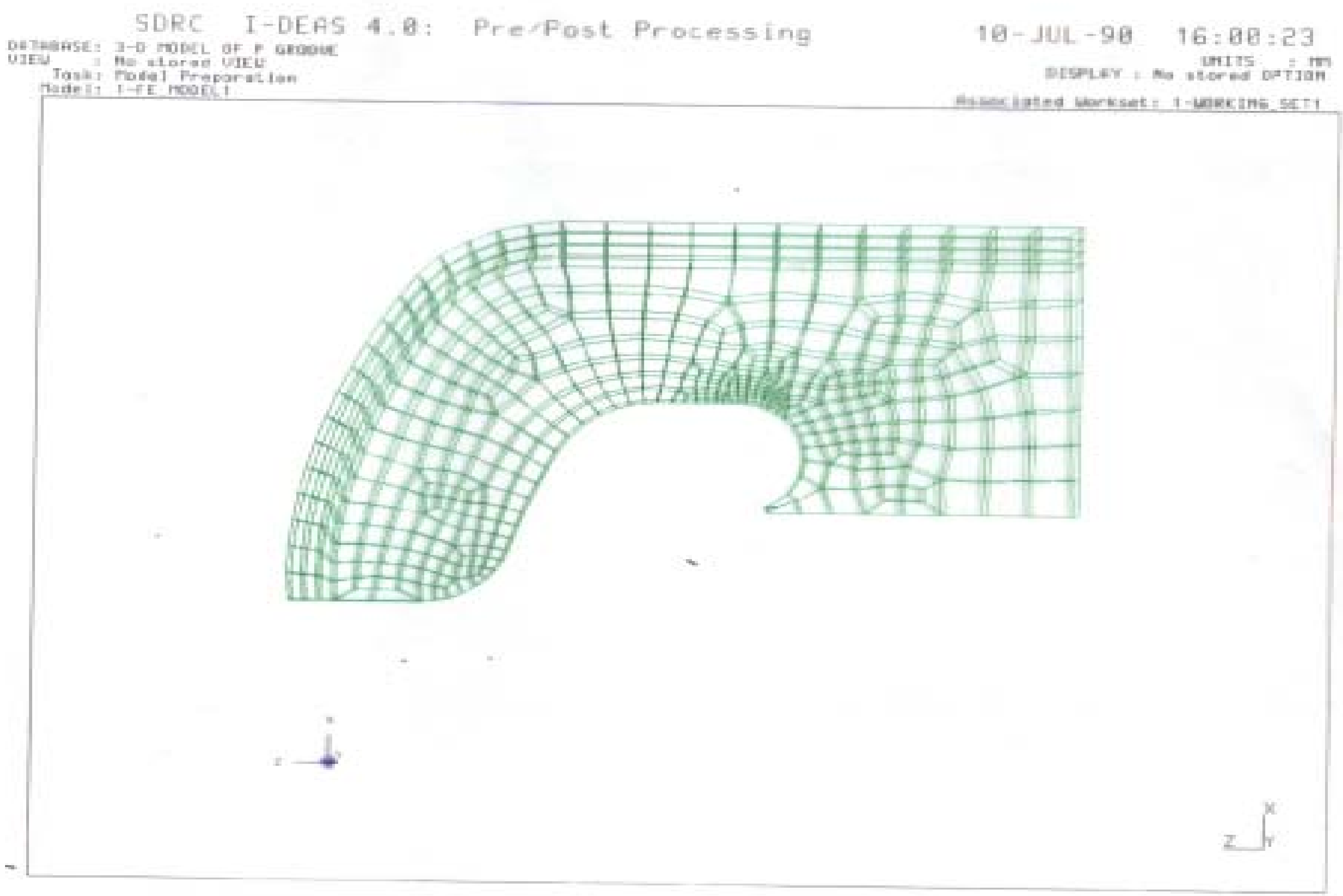


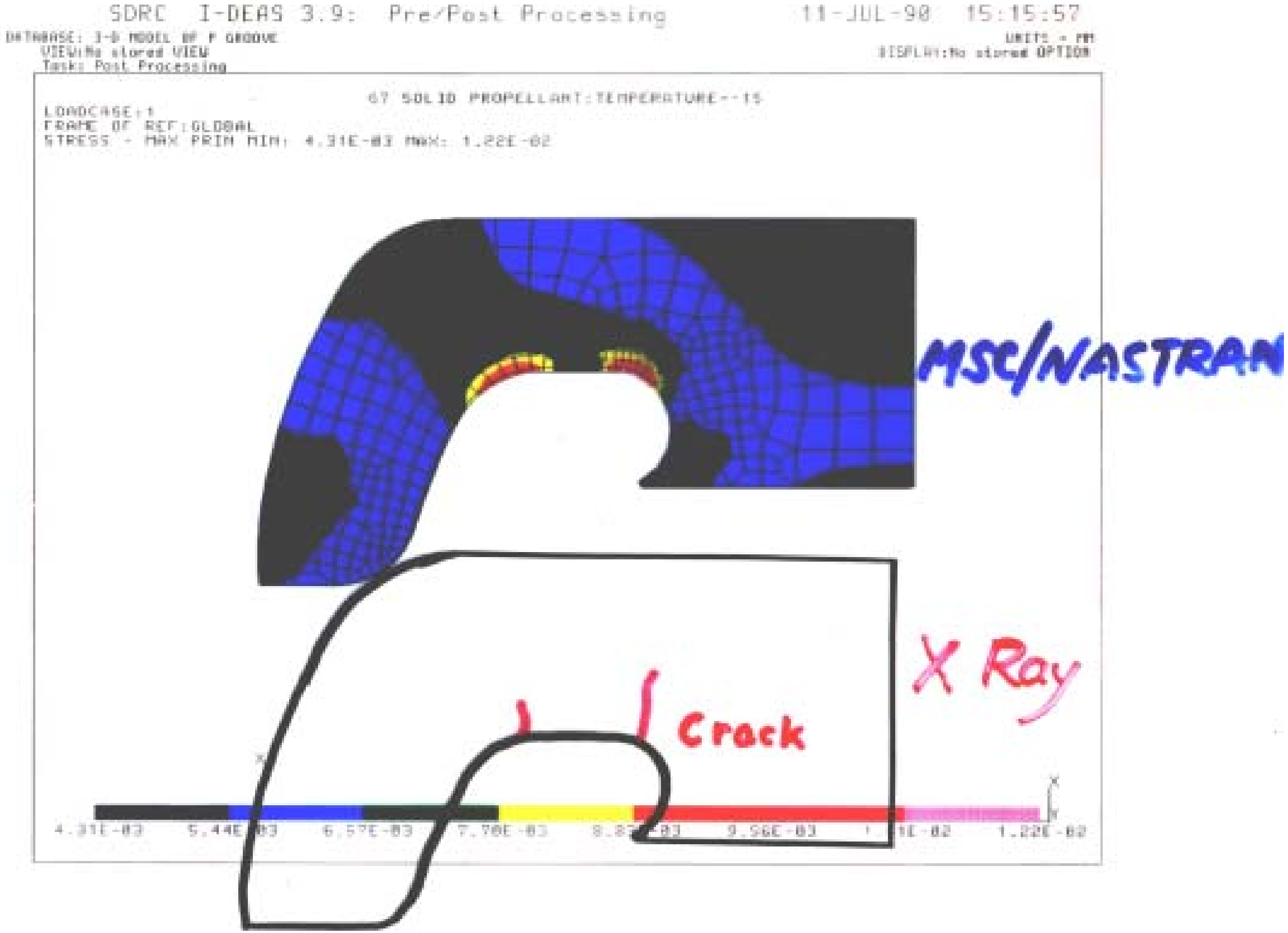
Fig. 1·DT X-ray results.

圖一應力消除塊設計圖 [12.13]

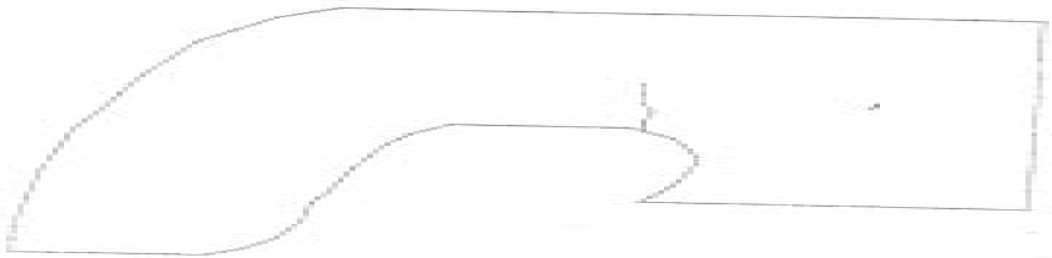
FEM simulation

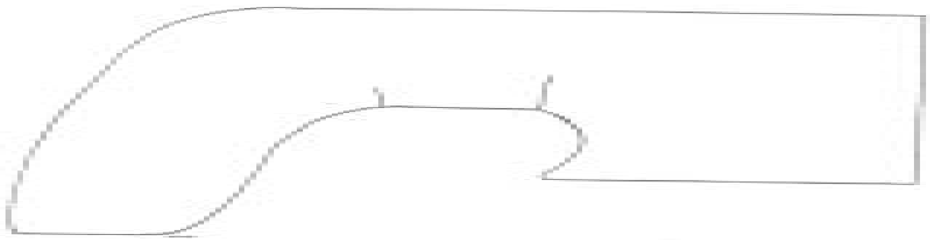


Stress analysis



BEM simulation





V-band structure (Tien-Gen missile)

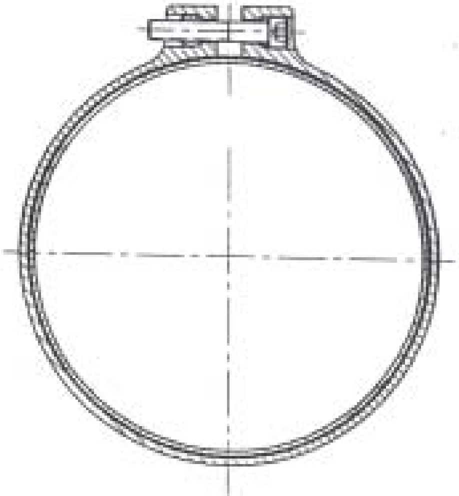
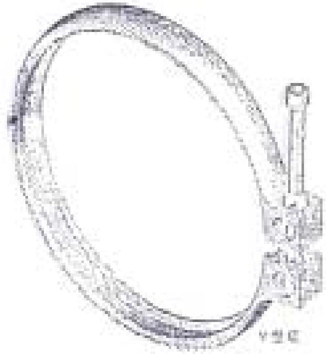


圖 1 · V型環的結構示意圖



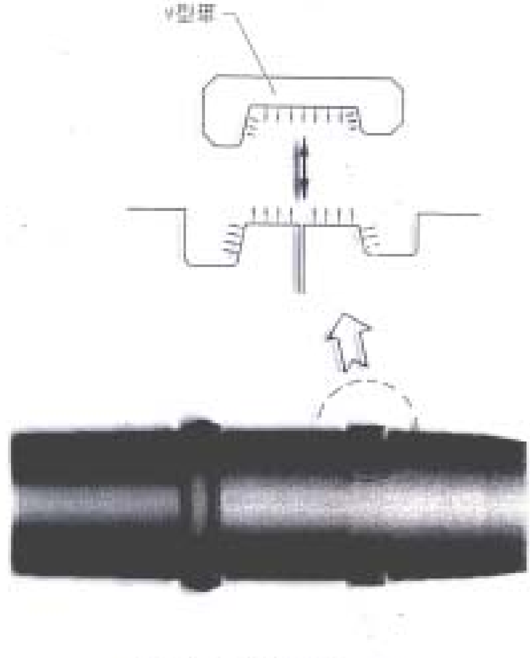
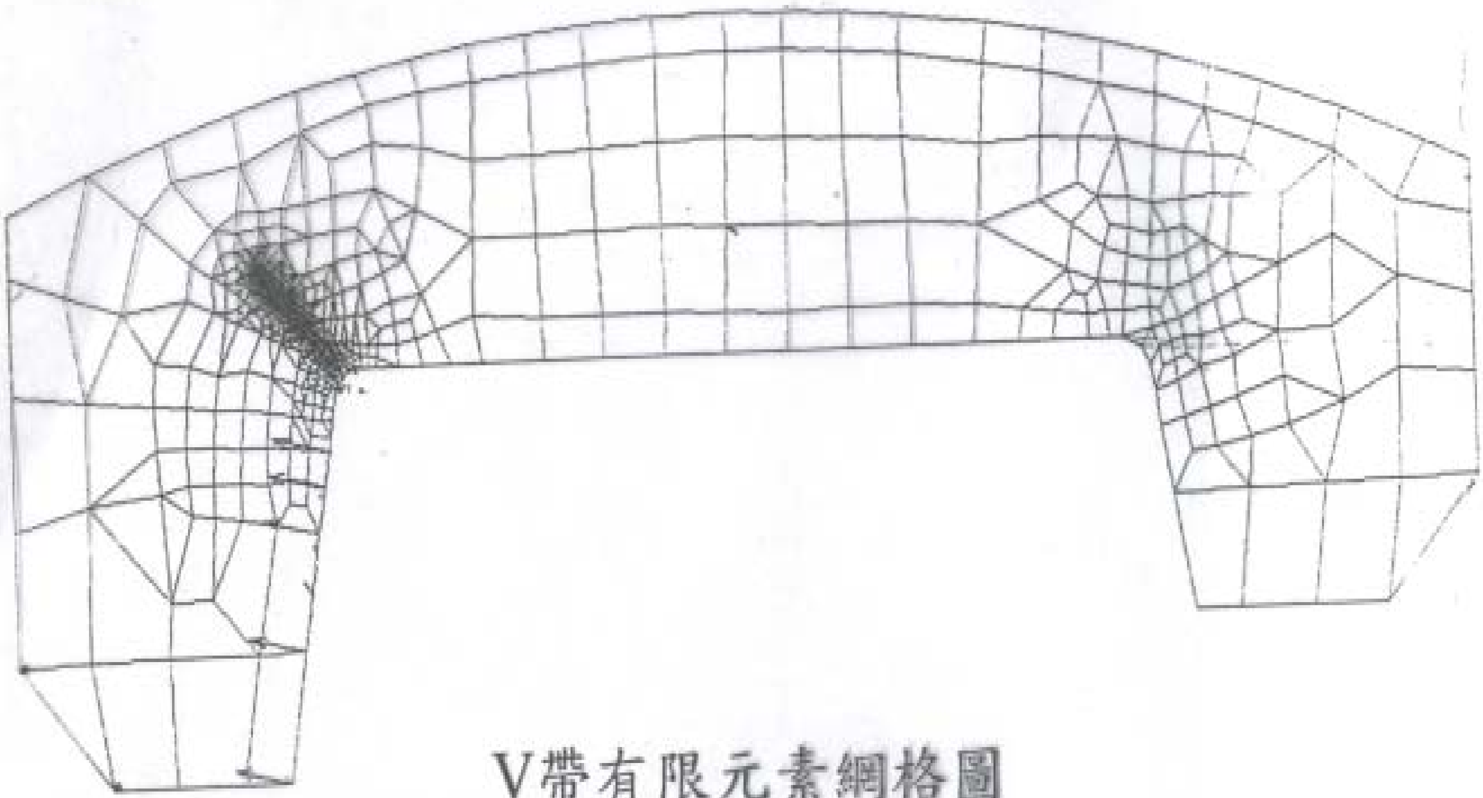


圖 2 · V型環的結合功能

V帶結構示意圖

FEM simulation



V帶有限元素網格圖

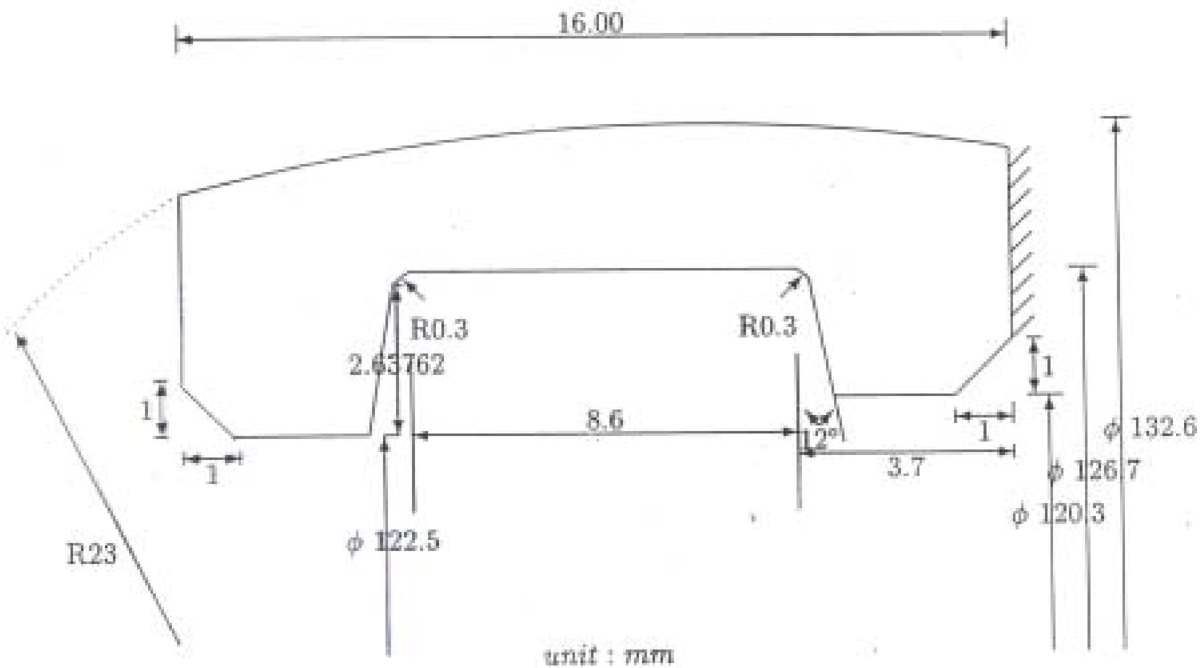
Application to V-band structure:

$E = 19950\ kgf/mm^2$, $\upsilon = 0.27$,

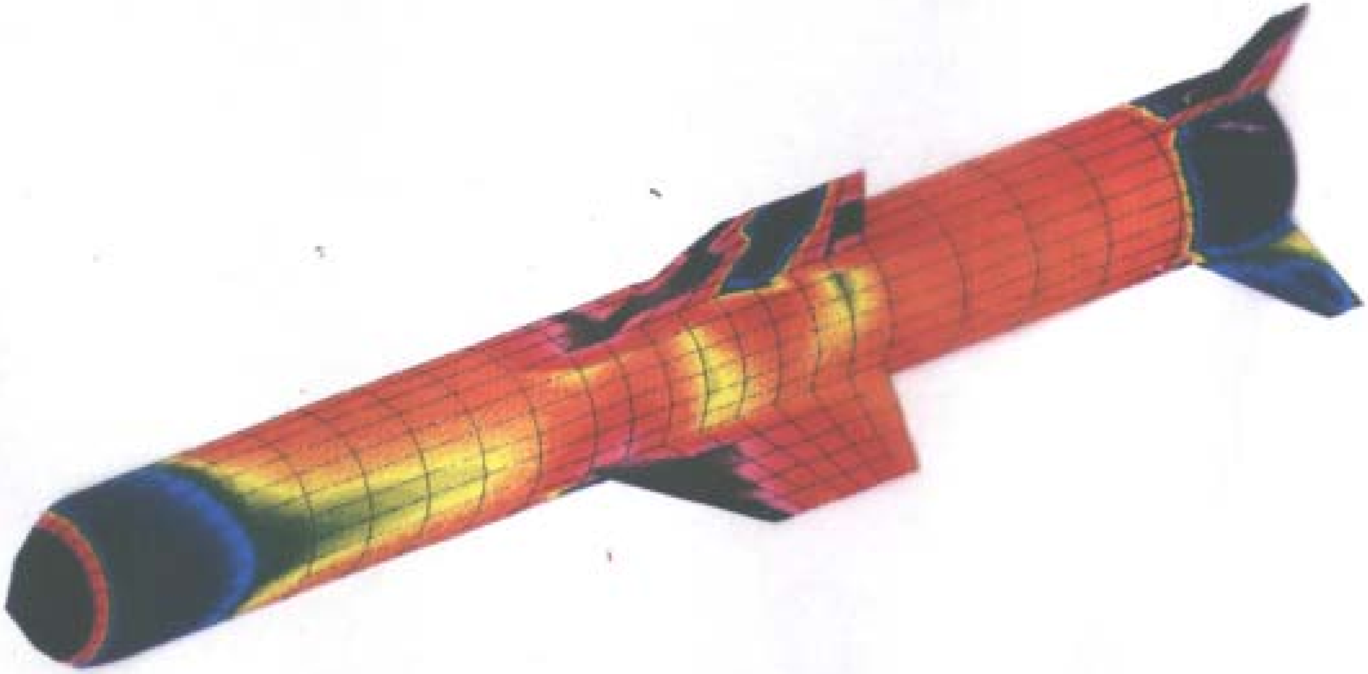
$a = 0.125$ $\sigma = 3.63 kgf/mm^2$

Pari's law: $\frac{da}{dN} = C(\Delta K)^m$

$C = 4.624 \times 10^{-12}$, $m=3.3$, $R = \frac{2}{3}$



Shong-Fon II missile



IDF

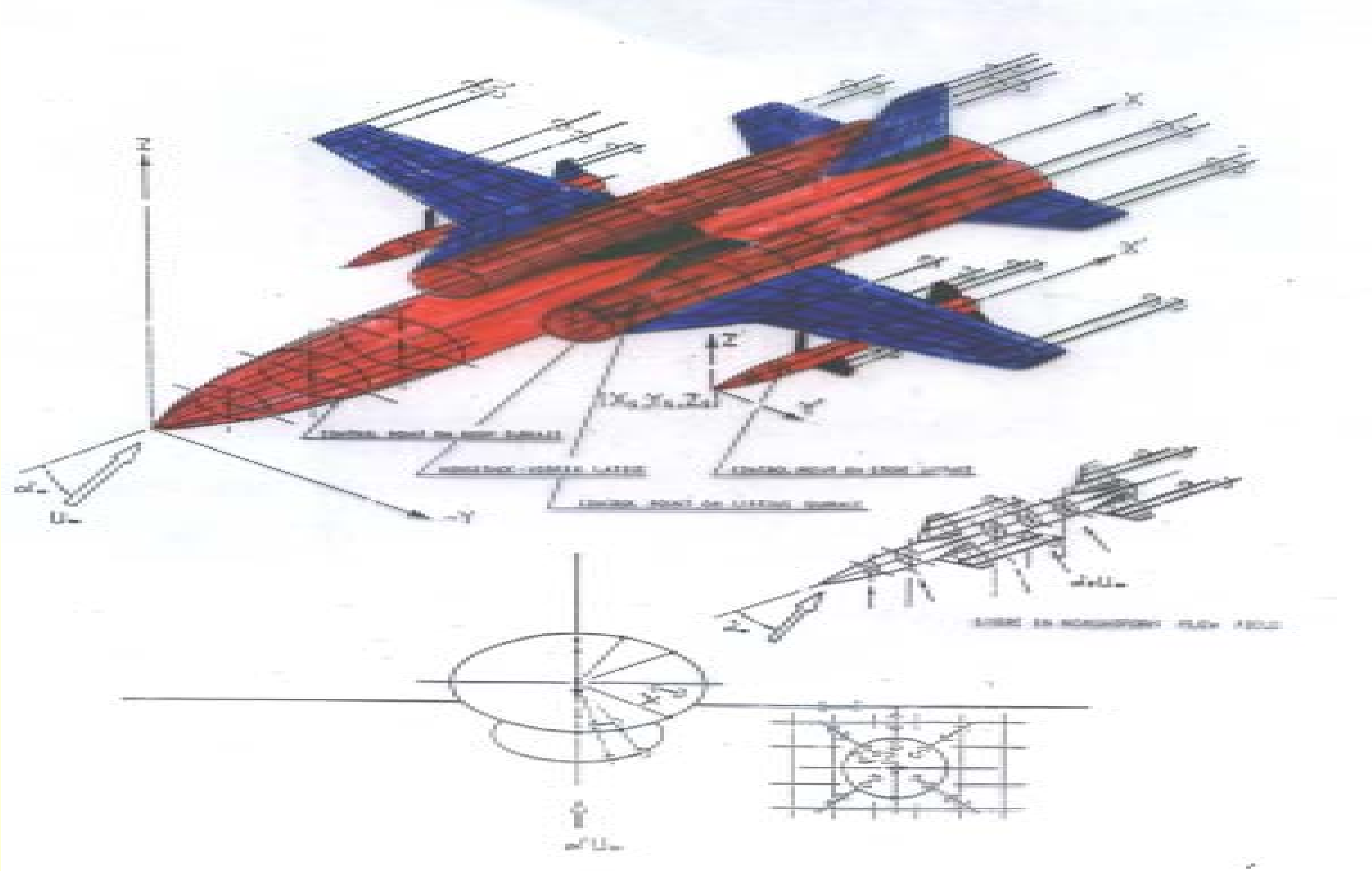


Fig.1 Image system of all the singularities
in aircraft/external store configurations.

Flow field

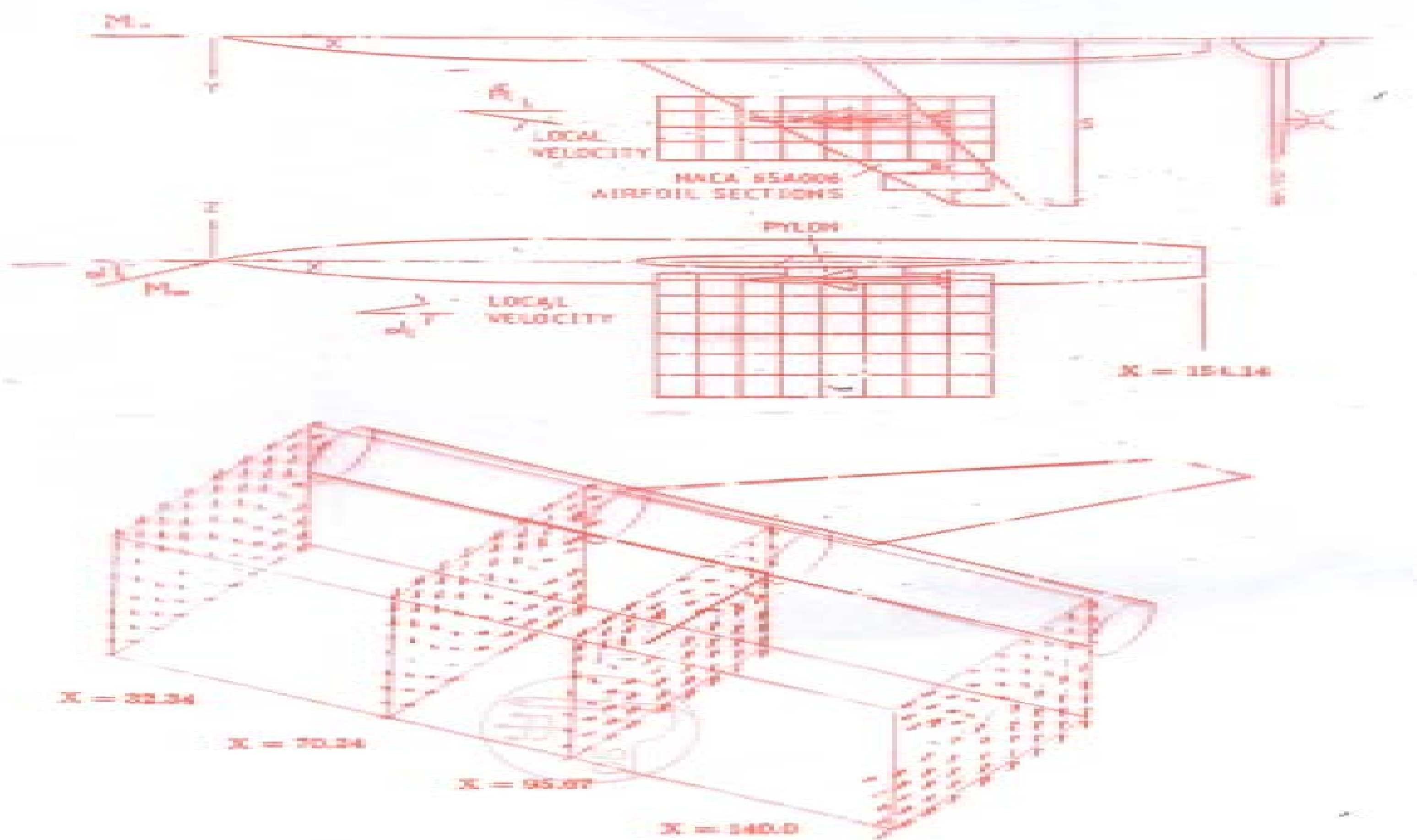


Fig.8 Cross flow velocity field on the reference planes.

Seepage flow

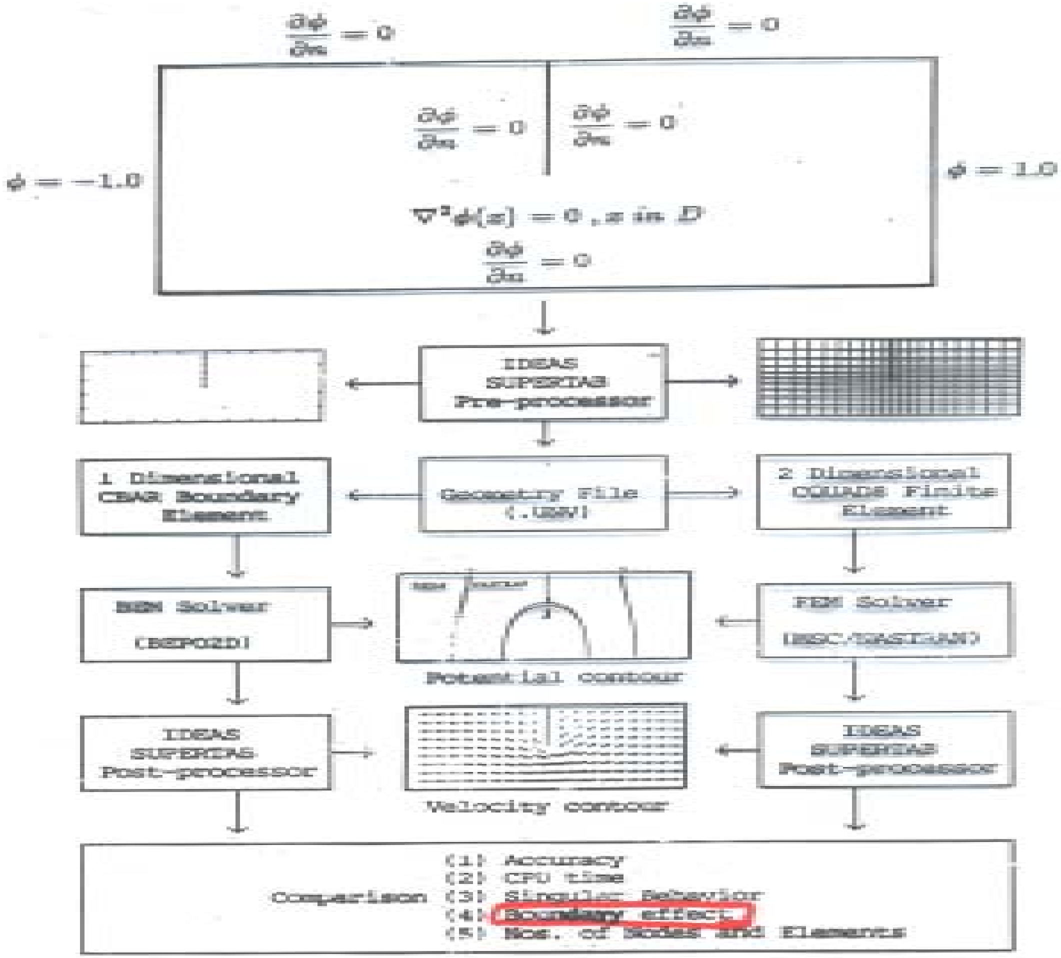
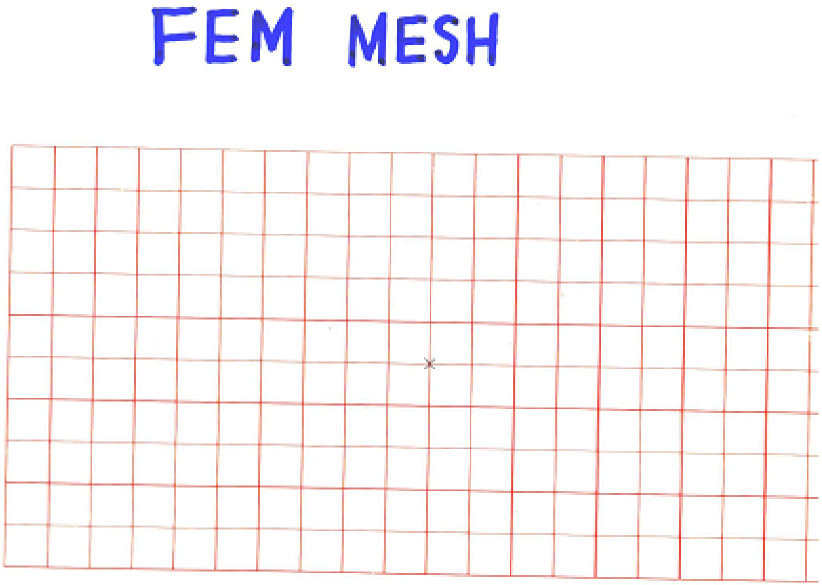
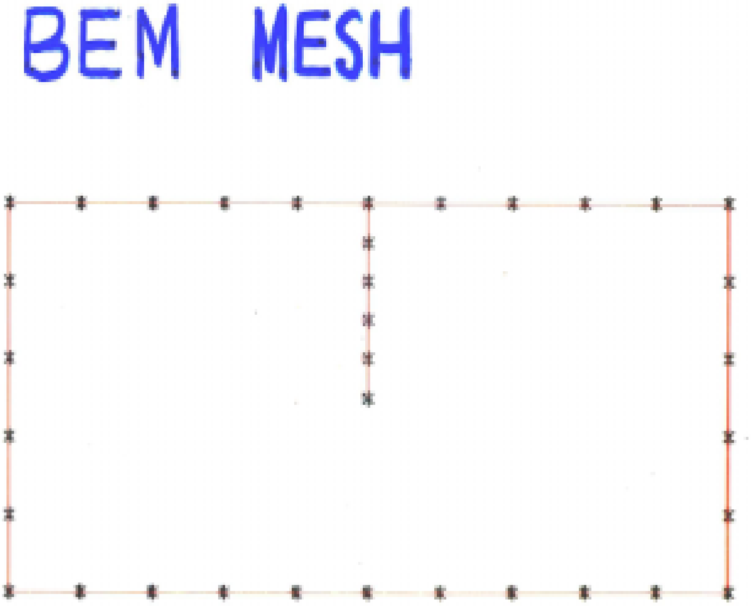


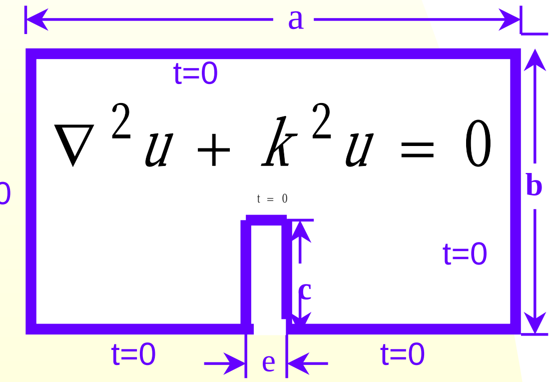
Fig.4 Flowchart of BEM and FEM solver system.

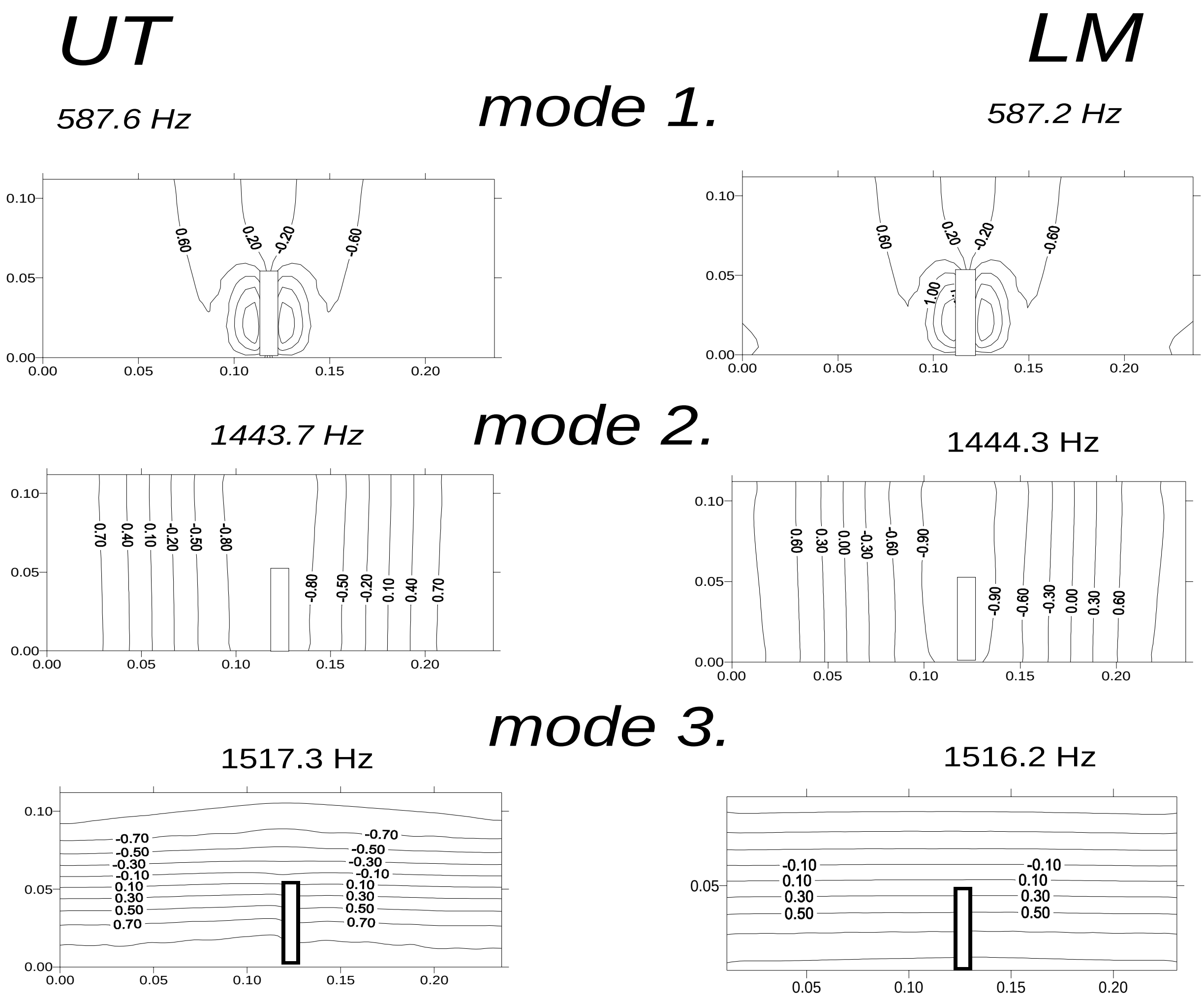
Meshes of FEM and BEM





Screen in acoustics

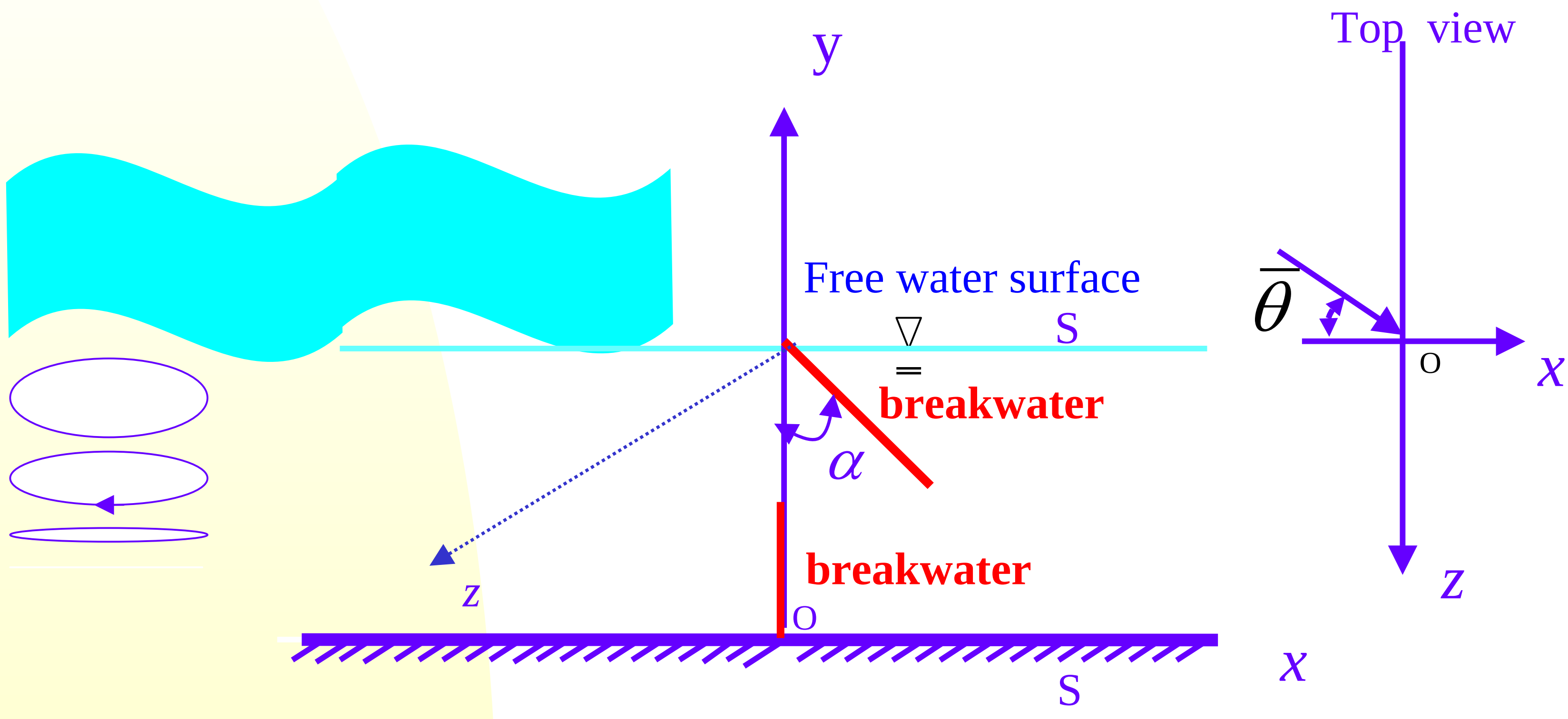




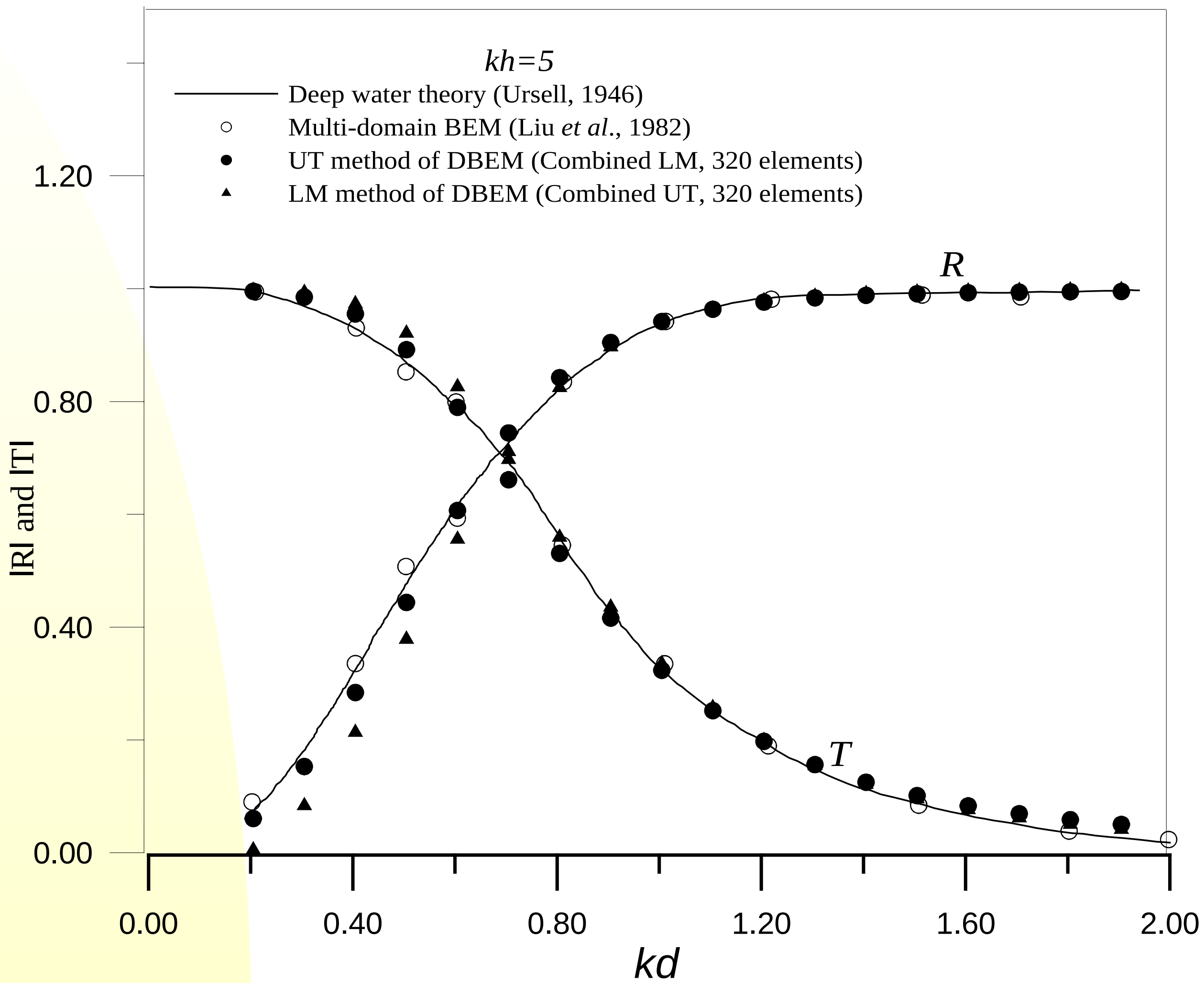
Water wave problem

oblique incident water wave

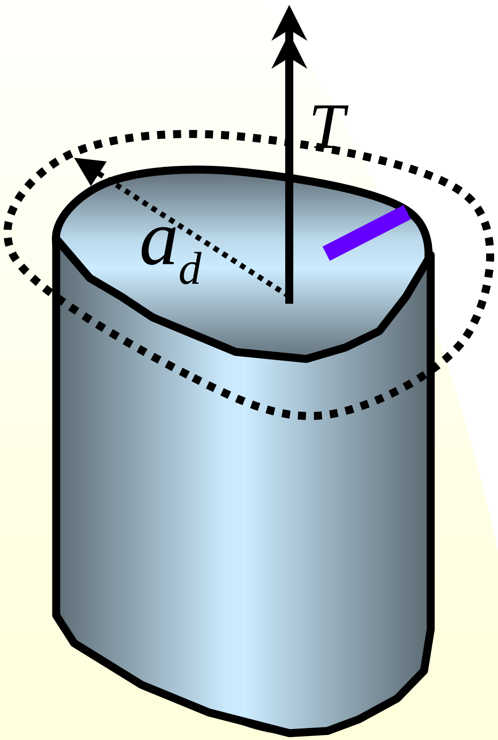
$$\nabla^2 u(\tilde{x}) - \lambda^2 u(\tilde{x}) = 0$$

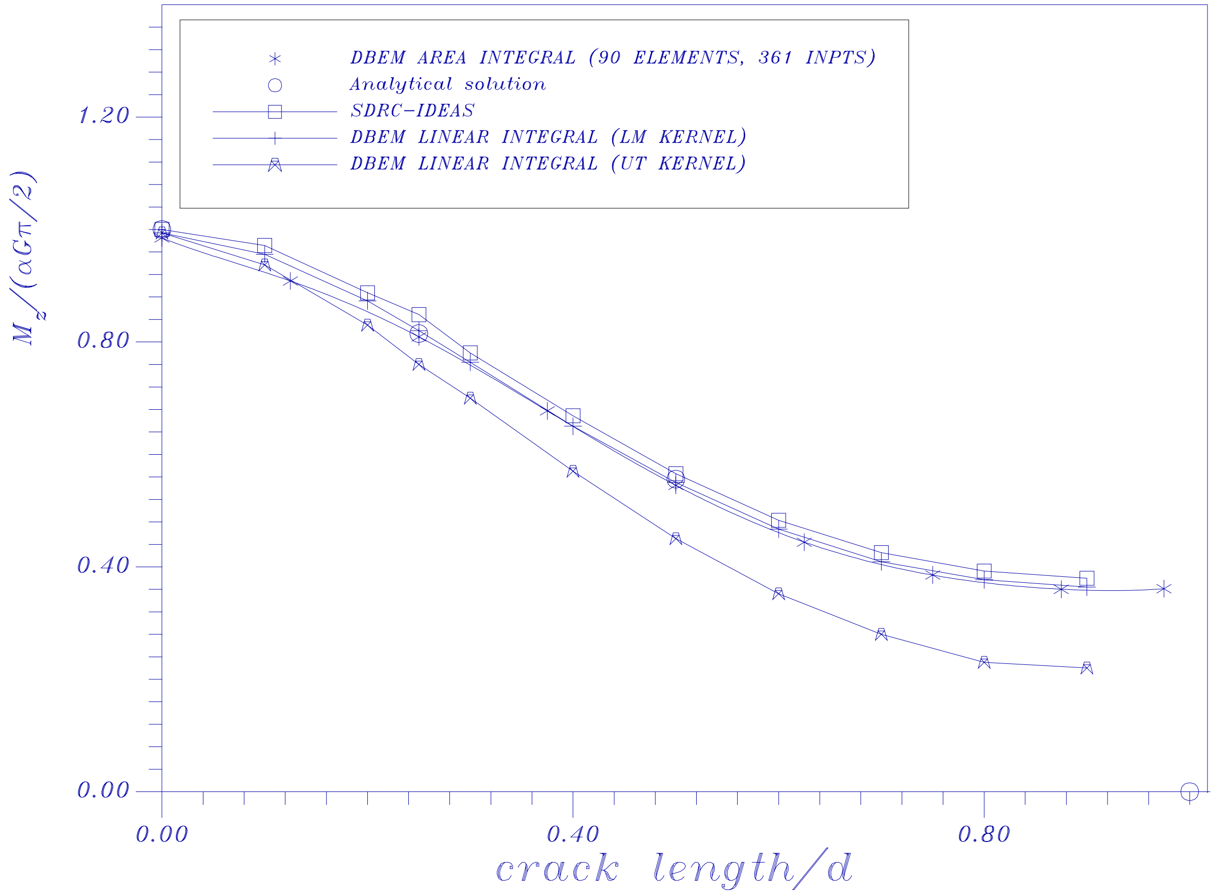


Reflection and Transmission



Cracked torsion bar





Crack Growth Analysis

There are many Finite Element software packages for crack growth analysis currently available. However, they all have a common drawback, which is the requirement for remeshing as the crack propagates. This software utilizes the state-of-the-art development in the boundary element method and for the first time removes the difficult and time consuming task of remeshing. Furthermore, it evaluates accurate stress intensity factors for which the Boundary Element Method is renowned. The software uses the established criterion for crack propagation and evaluates the residual strength as well as fatigue life calculations.

MAIN FEATURES:

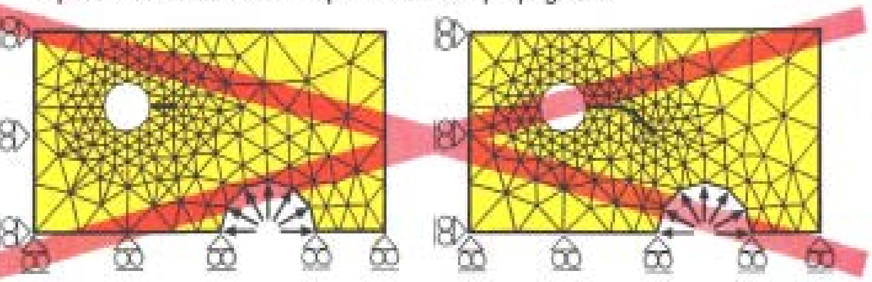
- Automatic incremental crack propagation
- Eliminates remeshing for crack growth analysis
- Accurate evaluation of stress intensity factors
- Residual strength and fatigue life computations.

MODULES IN THE SOFTWARE:

- Data generation with a minimum of input
- Plotting of the mesh
- Automatic fatigue crack growth analysis
- Plotting of the deformed configuration and principal stresses
- Plotting of the crack path

The old approach

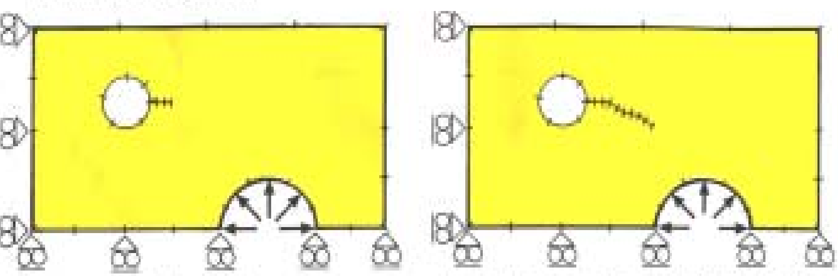
The Finite Element approach: continuous remeshing and repeated resolutions are required for crack propagation.



Starting mesh — Mesh after three crack propagation increments

The new approach

The Boundary Element approach: No remeshing is required for crack propagation.



Starting Mesh — Mesh after three crack propagation increments

Program Description

The software features include the use of quadratic continuous and discontinuous elements, evaluation of boundary stresses, displacements and tractions, element or point constraint including skew constraints and mixed-mode path independent integrals for the accurate evaluation of stress intensity factors. Automatic crack propagation algorithm is implemented utilizing an incremental crack extension which employs special solver to avoid resolution for each crack extension.

The fracture criterion is based on the maximum principal stress and the fatigue crack growth rates are calculated using established formulae.

The software package is accompanied with a user manual for data generation and the analysis program as well as a book *Boundary Elements in Crack Growth Analysis* describing the basic theory of the method. The source code in FORTRAN is included along with several example problems to demonstrate the use of the code.

The Boundary Element Method (BEM) is now widely regarded as the most accurate numerical tool for analysis of crack problems in linear elastic fracture mechanics. This software package is based on a new formulation of BEM called Dual Boundary Element Method (DBEM) developed at the Damage Tolerance Division of Wessex Institute of Technology. The Dual Boundary Element Method retains all of the important features of BEM which are: reduced set of equations, simple data preparation, accurate evaluation of stresses, strains and displacements at selected internal points as well as introducing additional improvements which include crack modelling in a single region and accurate stress intensity factors evaluation.

ORDER FORM

Please send me the following software package

Quantity	*Title/Author*	*Price*
	Crack Growth Analysis using Boundary Elements by A. Portela and M.H. Aliabadi	*£675**

**$995 for USA, Canada and Mexico - postage & packing UK £4/$7, USA £5/$9.*

Name ______

Organisation ______

Position ______

Address ______

Please indicate method of payment ______

Cheque number ______

From Portela

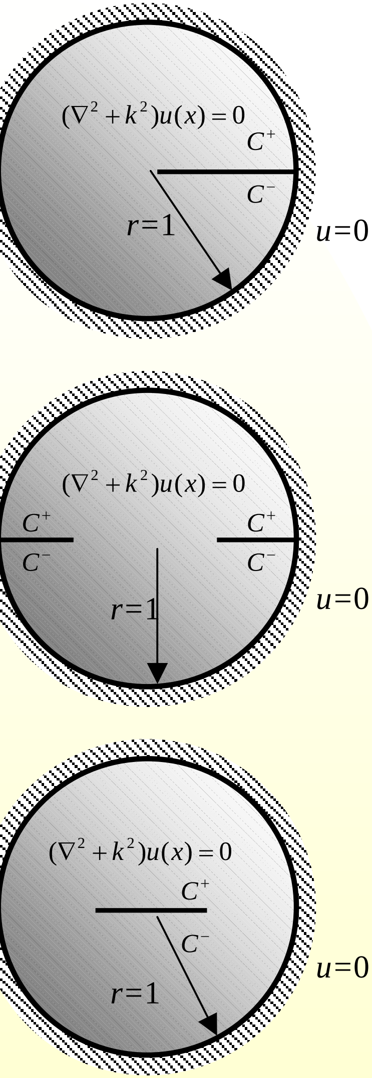
I wish to pay by Credit card ☐ *Expiry date* ______

Name ______

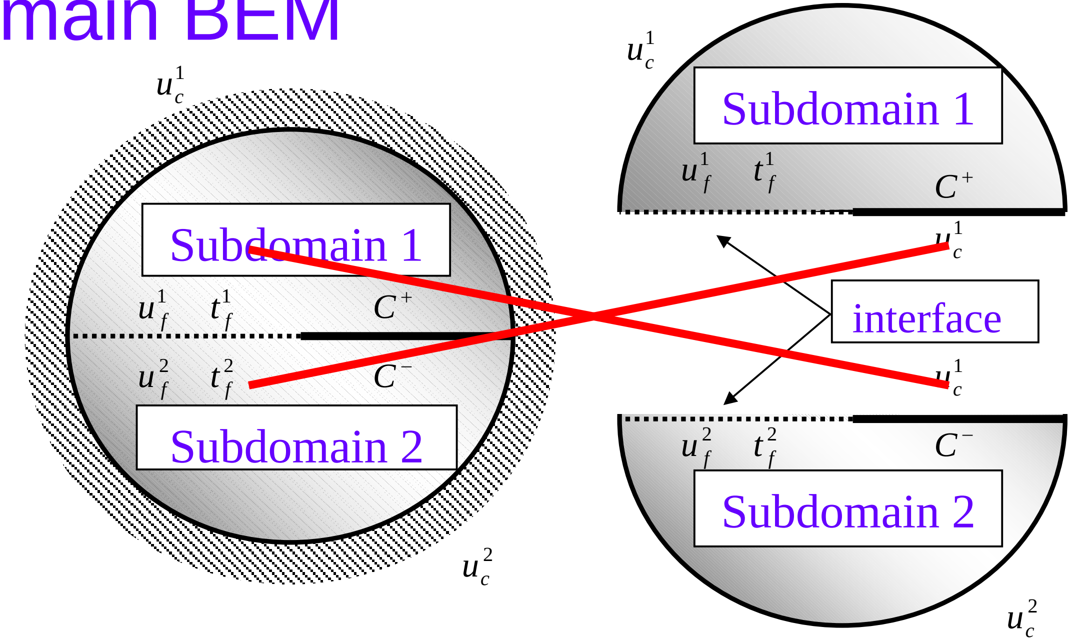
Number ______

Nov. 1993

Degenerate boundary problems



- Multi-domain BEM



- Dual BEM

$$[T]\{u\}=[U]\{t\}$$

$$[M]\{u\}=\{L\}\{t\}$$

Conventional BEM in conjunction with SVD

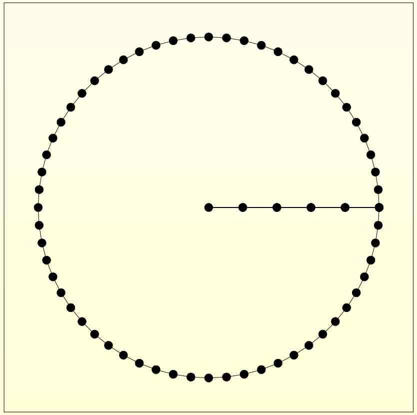
Singular Value Decomposition

$$[U]_{M\times P} = [\Phi]_{M\times M}[\Sigma]_{M\times P}[\Psi]^{H}_{P\times P}$$

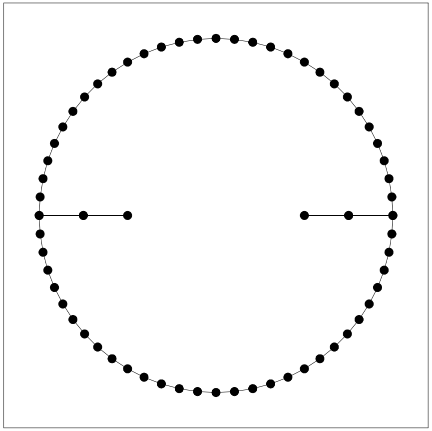
Rank deficiency originates from two sources:

(1). Degenerate boundary

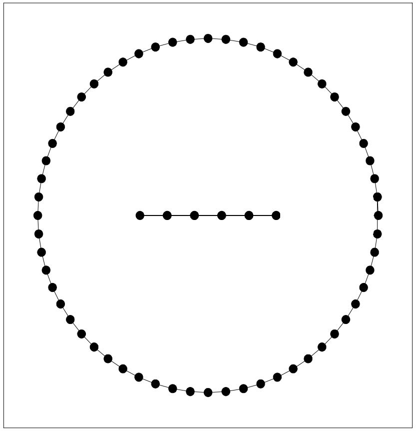
(2). Nontrivial eigensolution



$N_d=5$



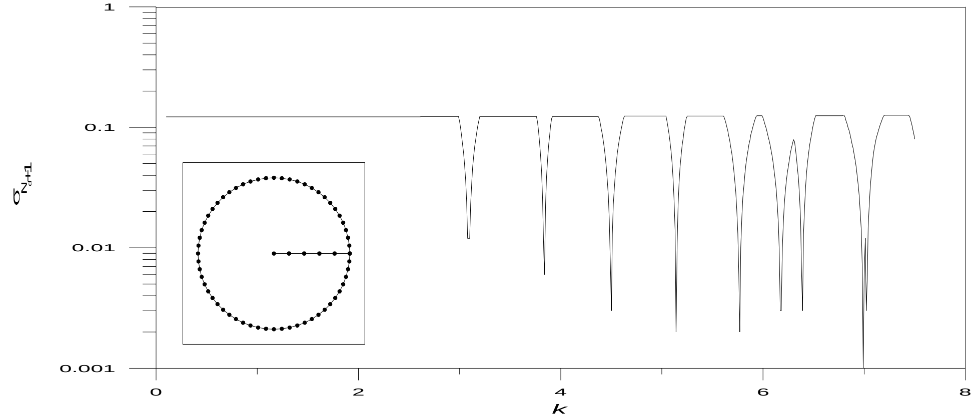
$N_d=4$



$N_d=5$

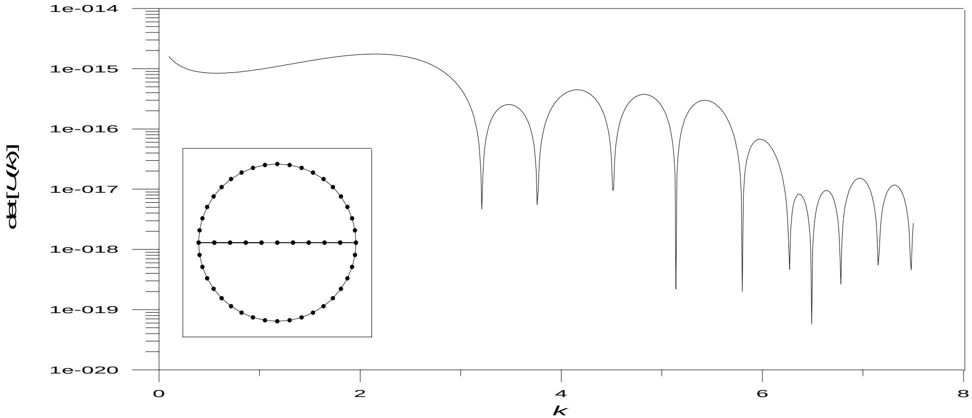
- ***UT* BEM + SVD (Present method)**

σ_{N_d+1} versus k



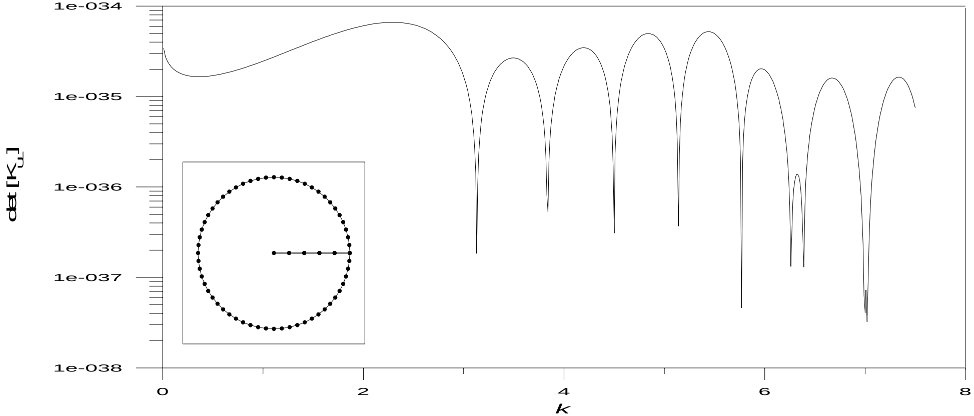
- **Multi-domain BEM**

Determinant versus k

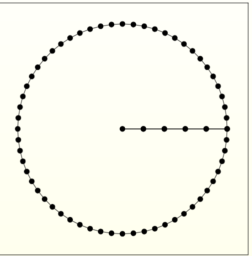


- **Dual BEM**

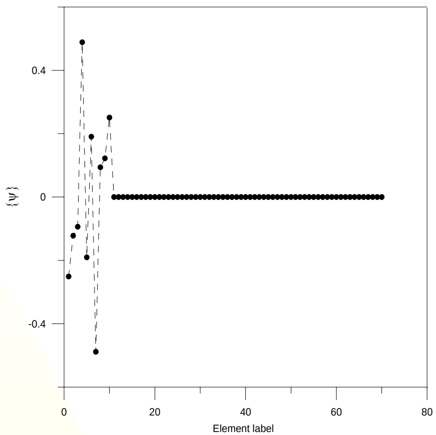
Determinant versus k



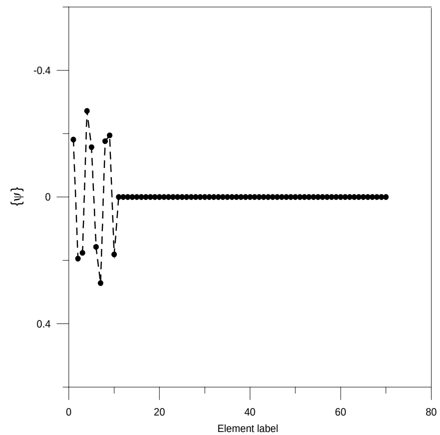
Two sources of rank deficiency (k=3.09)



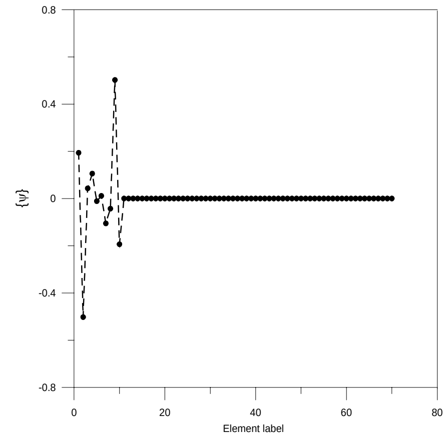
N_d=5



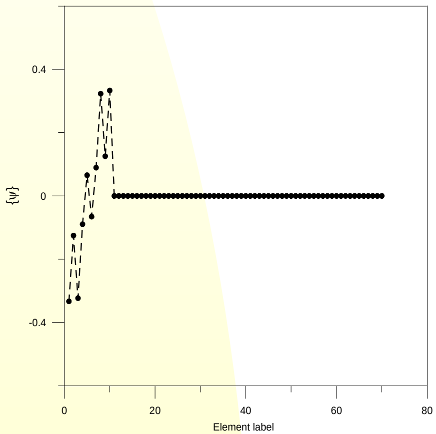
$\{\psi_1\},\ (\sigma_1 = 0)$



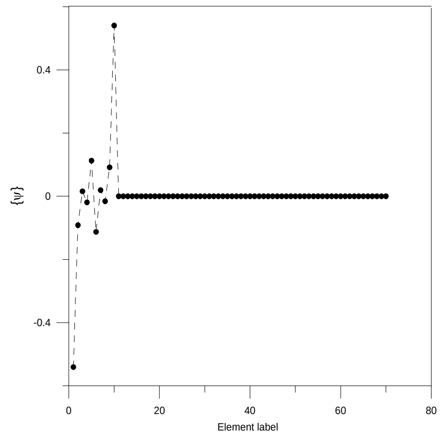
$\{\psi_2\},\ (\sigma_2 = 0)$



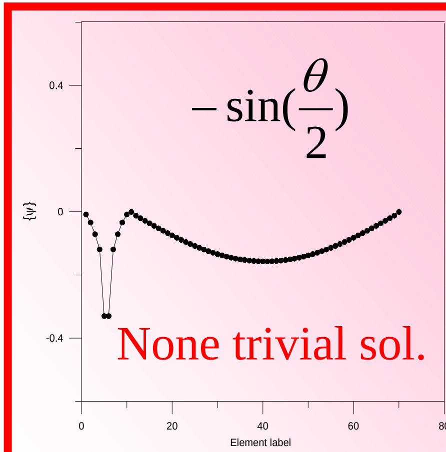
$\{\psi_3\},\ (\sigma_3 = 0)$



$\{\psi_4\},\ (\sigma_4 = 0)$



$\{\psi_5\},\ (\sigma_5 = 0)$

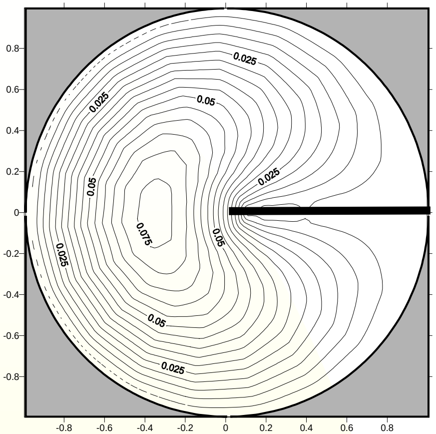


$\{\psi_6\},\ (\sigma_6 = 0)$

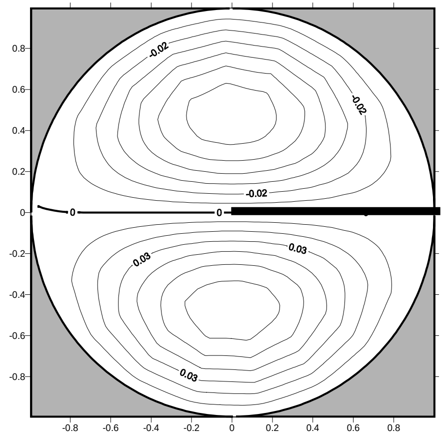
Degenerate boundary

Eigensolution

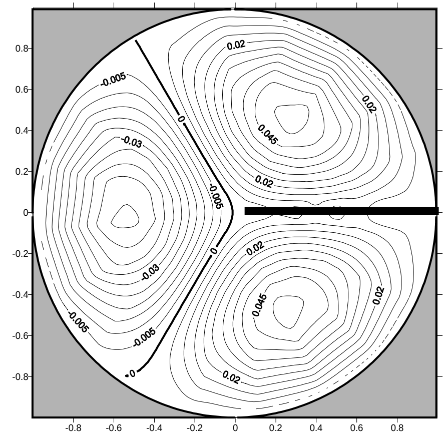
UT BEM+SVD



***k*=3.09**

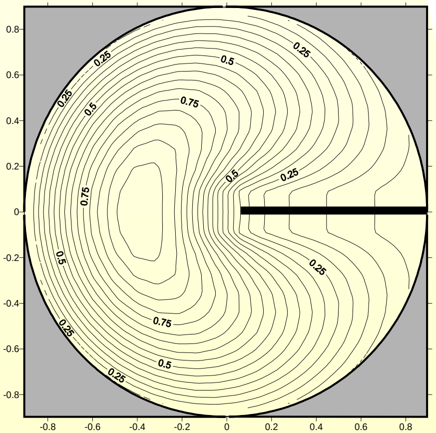


***k*=3.84**

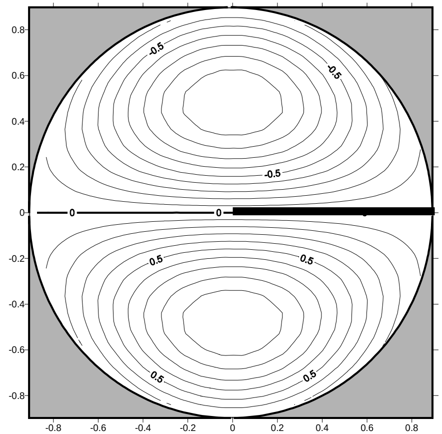


***k*=4.50**

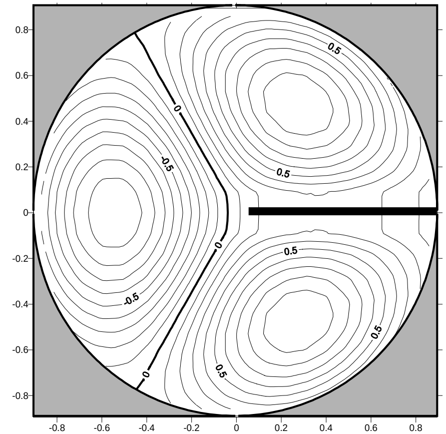
FEM (ABAQUS)



***k*=3.14**



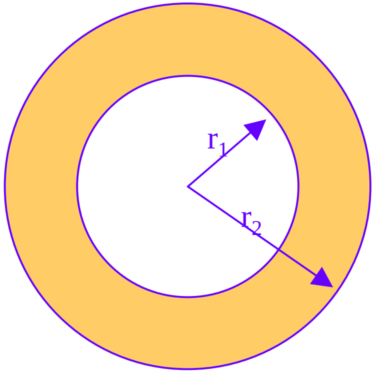
***k*=3.82**



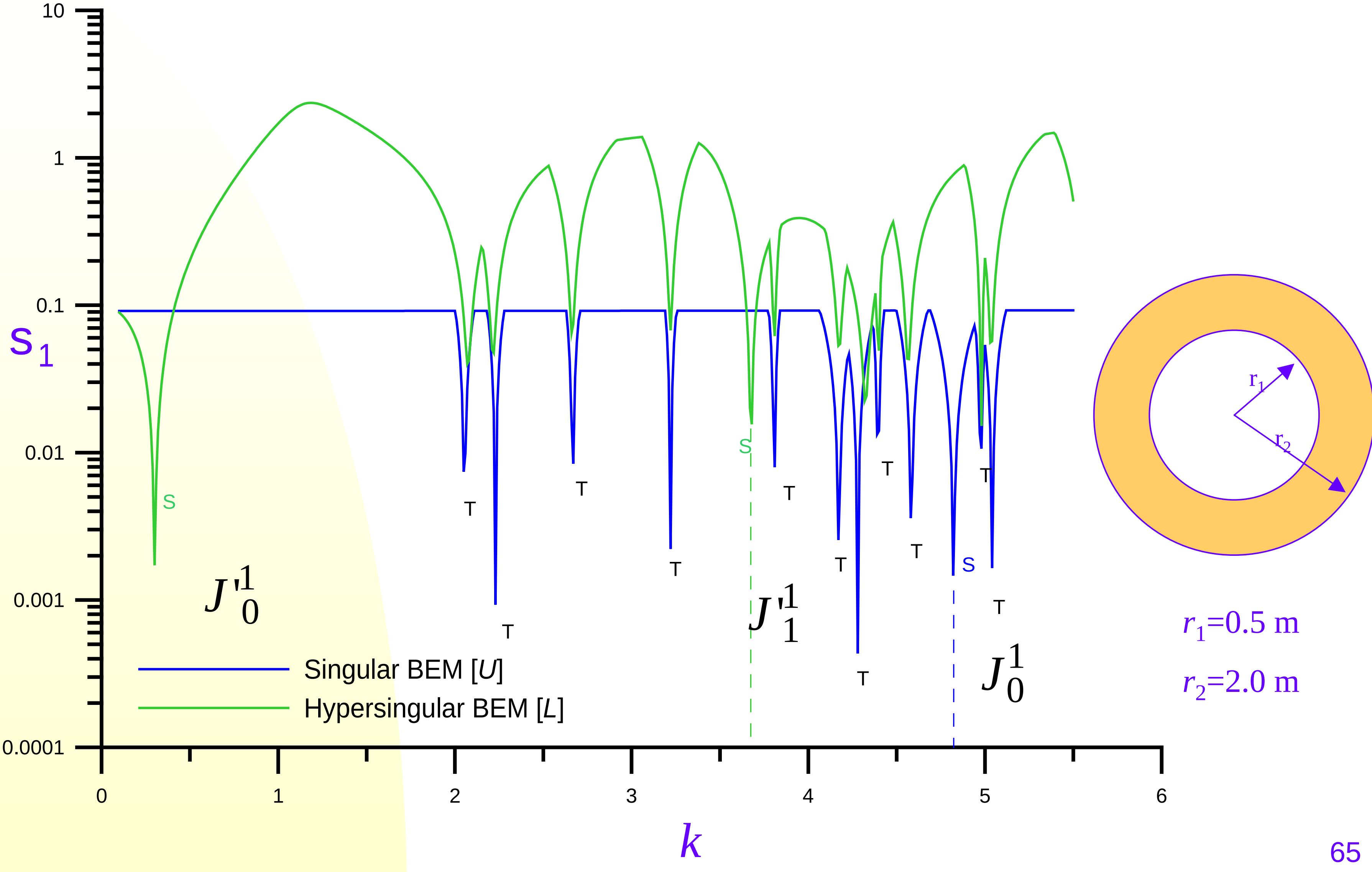
***k*=4.48**

Five pitfalls in BEM

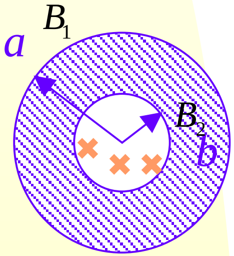
1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner



Spurious eigenvalue of membrane

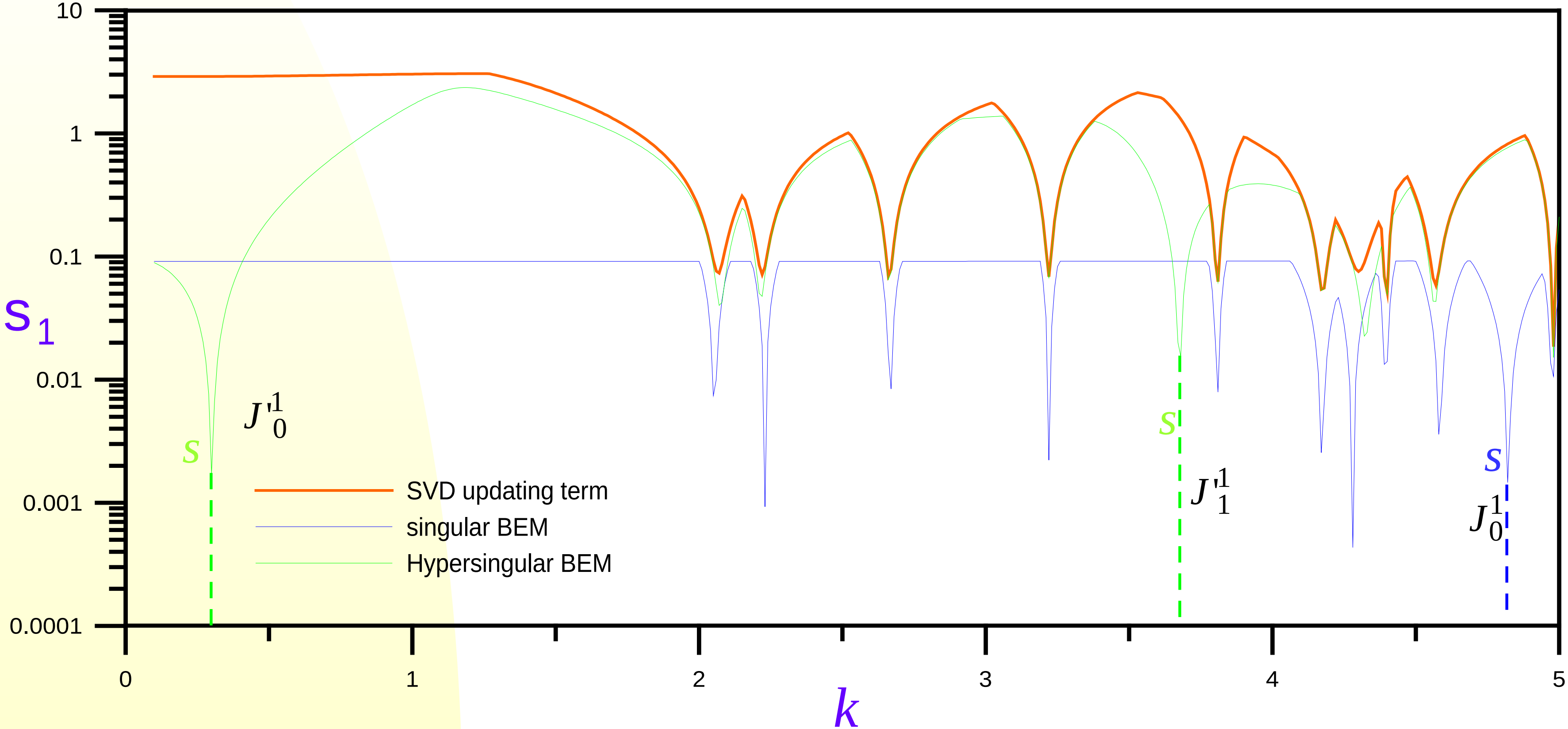


Treatments

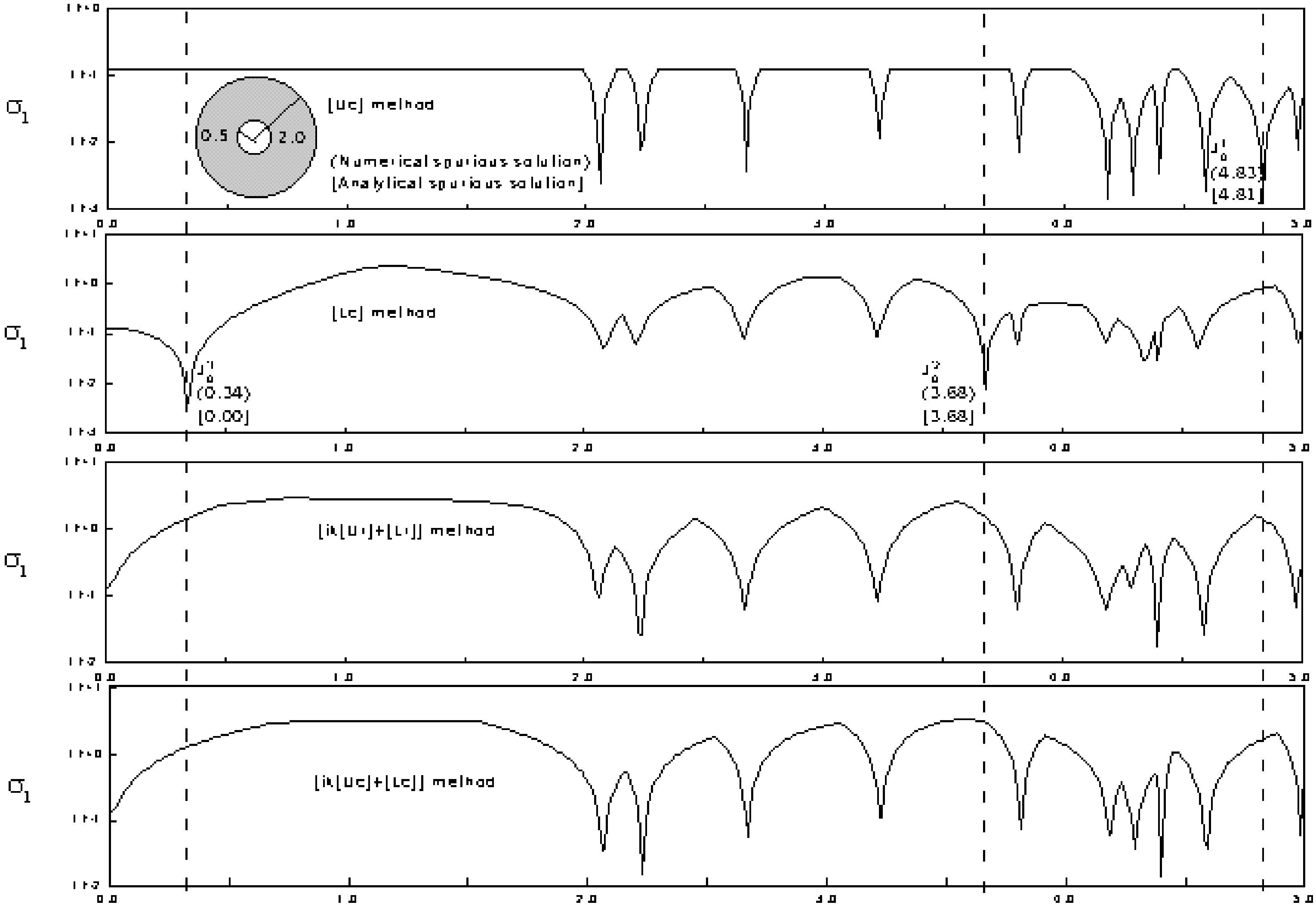
SVD updating term	$[C]=\begin{bmatrix} U \\ L \end{bmatrix}$
Burton & Miller method	$[U]+i[L]$
CHIEF method 	$[C^*]=\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{c1} & U_{c2} \end{bmatrix}_{(4N+N_c)\times 4N}$

SVD updating term for true eigenvalue

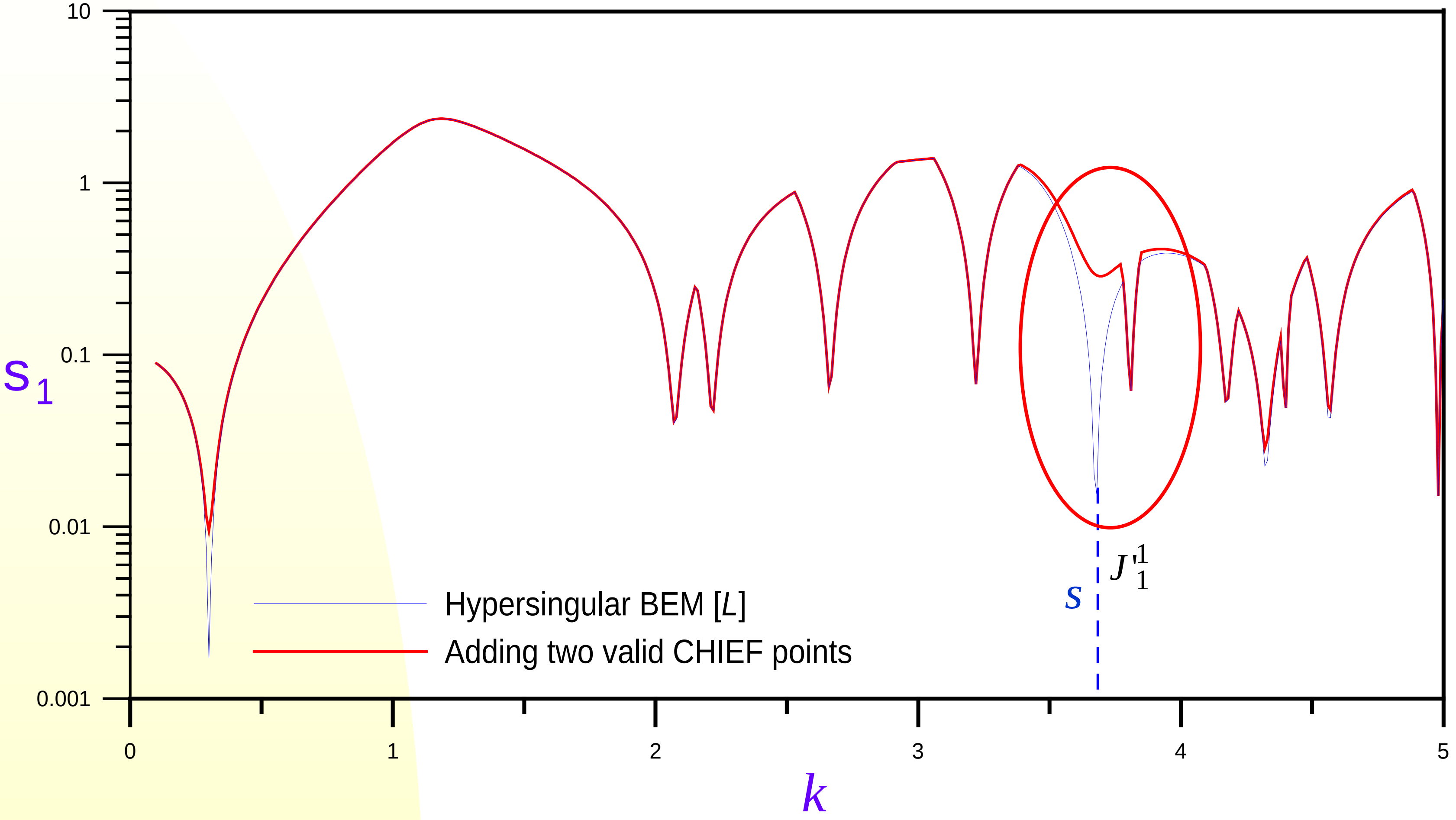
$$[U]\{t\}=\{0\}$$
$$[L]\{t\}=\{0\}$$
$$\Rightarrow \begin{bmatrix} U \\ L \end{bmatrix}\{t\}=\{0\} \Rightarrow \text{True eigenvalues}$$



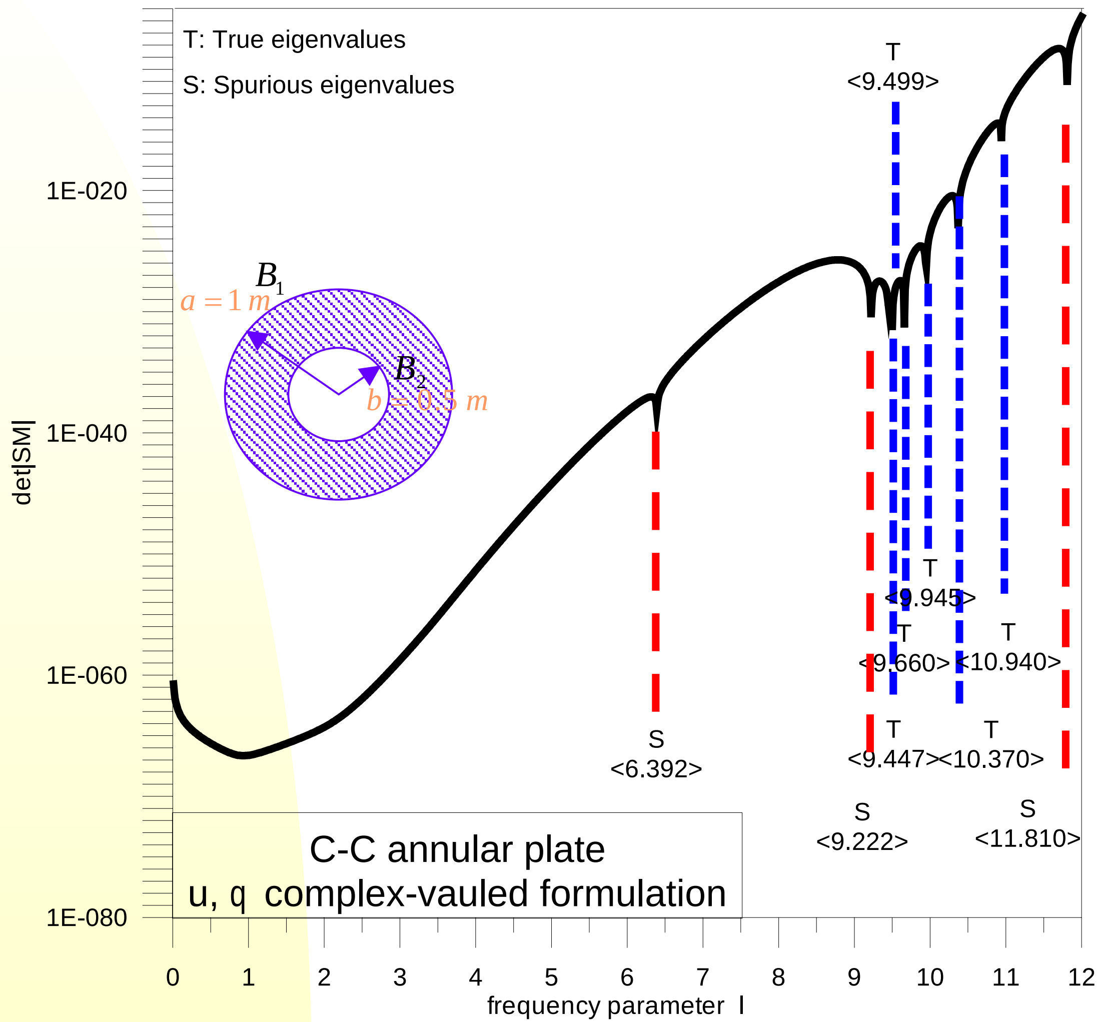
Burton & Miller method



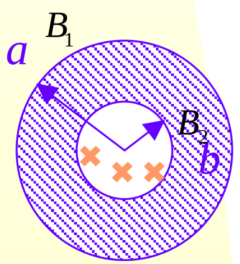
Two CHIEF points for spurious eigenvalues of multiplicity two



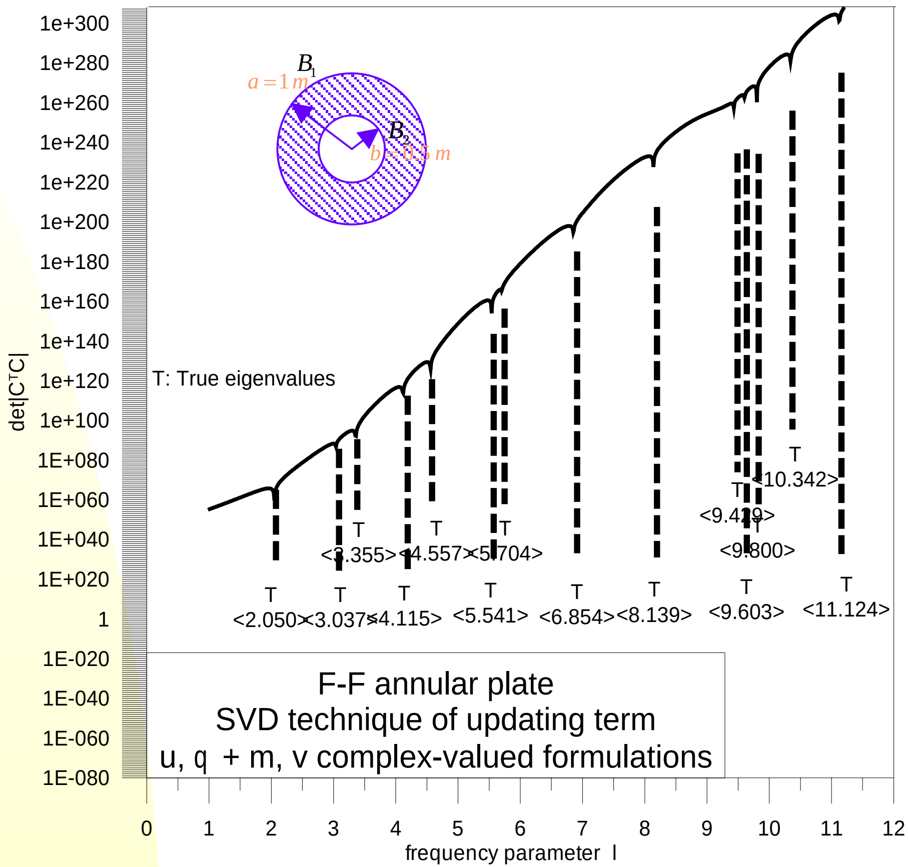
Spurious eigenvalue of plate



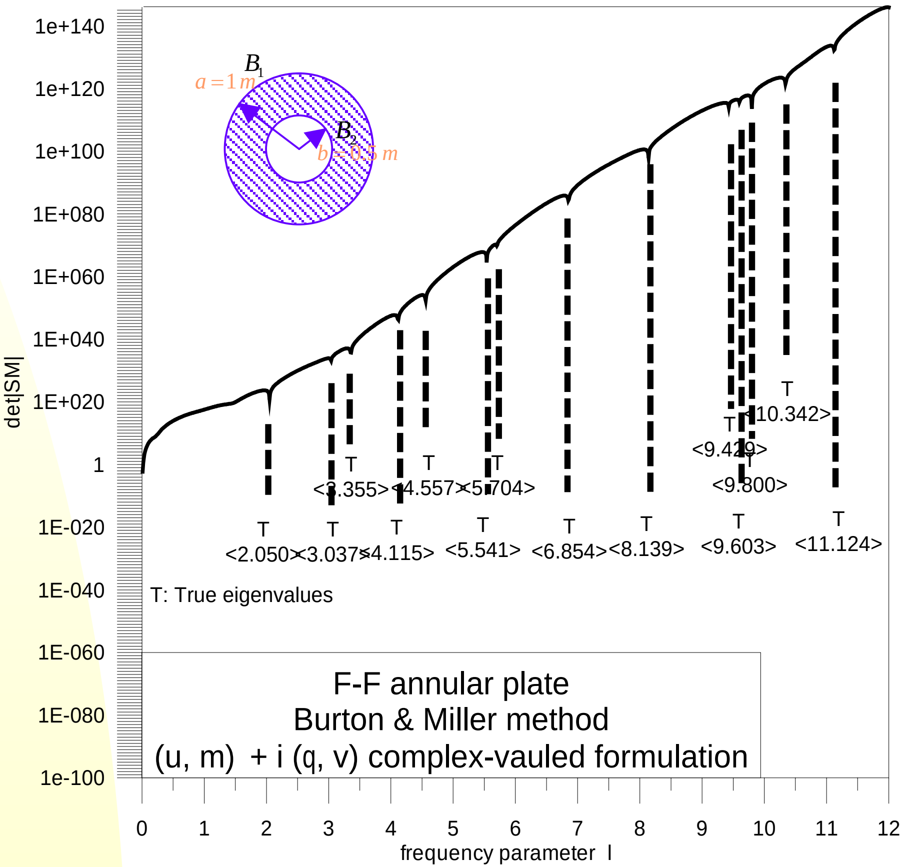
Treatments

SVD updating term	$[C]=\begin{bmatrix} SM_1^{cc} \\ SM_2^{cc} \end{bmatrix}_{16N\times 8N}$
Burton & Miller method	$[SM_1^{cc}]+i[SM_2^{cc}]$
CHIEF method 	$[C^*]=\begin{bmatrix} U11 & U12 & \Theta 11 & \Theta 12 \\ U21 & U22 & \Theta 21 & \Theta 22 \\ U11_\theta & U12_\theta & \Theta 11_\theta & \Theta 12_\theta \\ U21_\theta & U22_\theta & \Theta 21_\theta & \Theta 22_\theta \\ UC1 & UC2 & \Theta C1 & \Theta C2 \\ UC1_\theta & UC2_\theta & \Theta C1_\theta & \Theta C2_\theta \end{bmatrix}_{2(4N+N_c)\times 8N}$

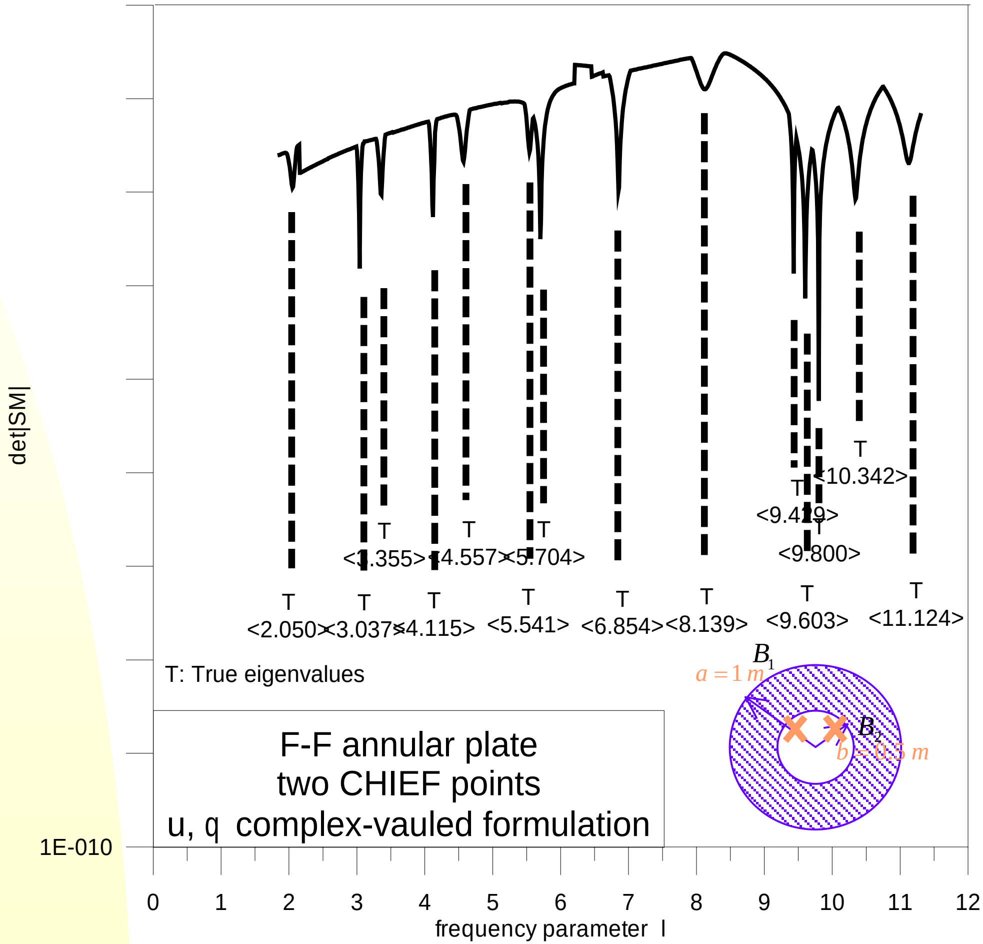
SVD updating term



The Burton & Miller concept

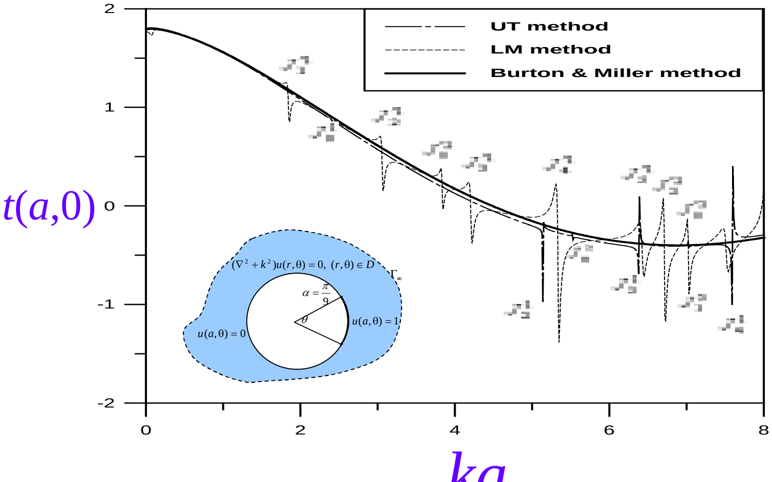


The CHIEF concept



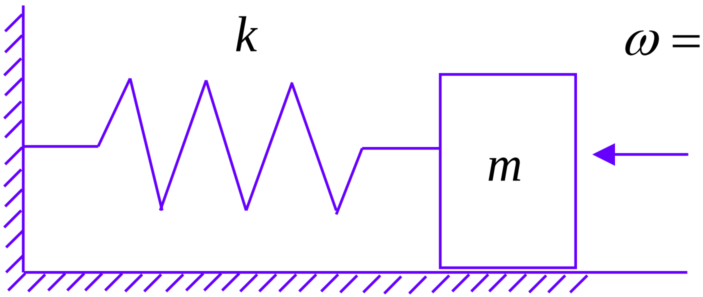
Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
2. Degenerate boundary problems
3. True and spurious eigensolution for interior eigenproblem
4. Fictitious frequency for exterior acoustics
5. Corner



On the Mechanism of Fictitious Eigenvalues in Direct and Indirect BEM

Physical resonance

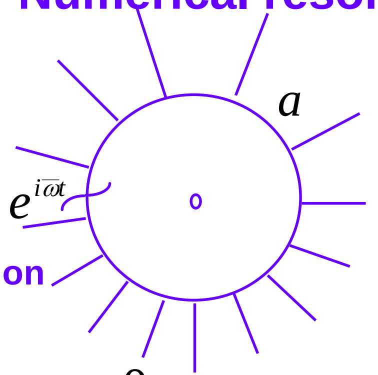


$$\omega = \sqrt{\frac{k}{m}}$$

$e^{i\overline{\omega}t}$

$$u = \frac{finite}{(\omega^2 - \overline{\omega}^2)} \to \infty, \text{ if } \overline{\omega} \to \omega$$

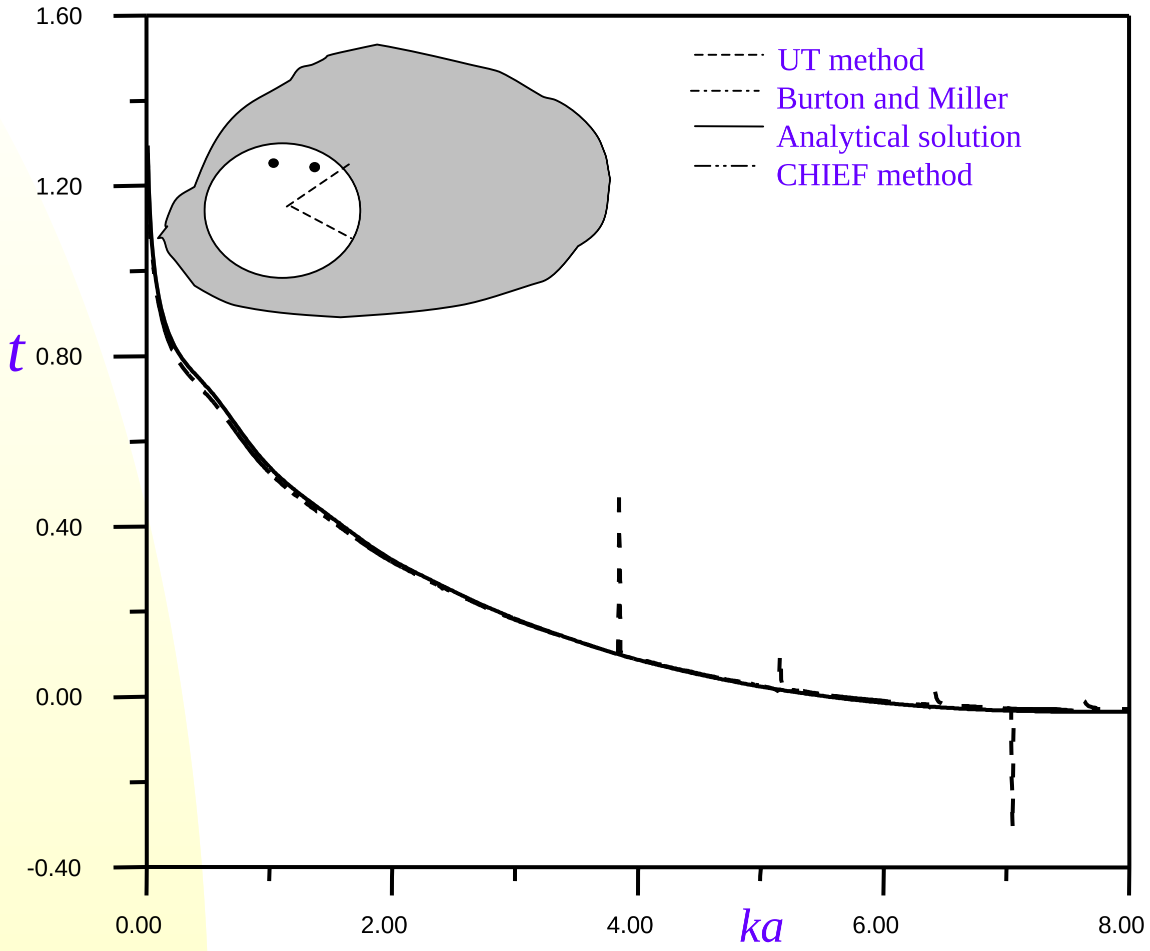
Numerical resonance



$e^{i\overline{\omega}t}$ (radiation), $e^{i\overline{\omega}t}$ (incident wave)

$$u = \lim_{\overline{\omega} \to \omega} \frac{0}{0} \to finite, \text{ if } \overline{\omega} \to \omega$$

CHIEF and Burton & Miller method



Five pitfalls in BEM

1. Degenerate scale for torsion bar problems
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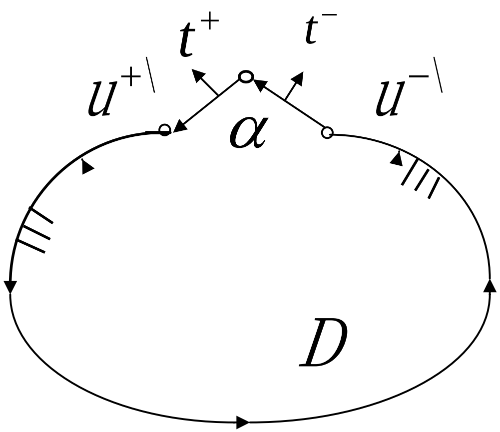
5. Corner

Theory of Dual Integral Equations for a Corner

$$\alpha u(x) = C.P.V.\int_B T(s,x)u(s)dB(s) - R.P.V.\int_B U(s,x)t(s)dB(s), \quad x \in B$$

$$\alpha t^-(x) + \sin(\alpha)t^+(x) = H.P.V.\int_B M(s,x)u(s)dB(s) - C.P.V.\int_B L(s,x)t(s)dB(s), \quad x \in B$$



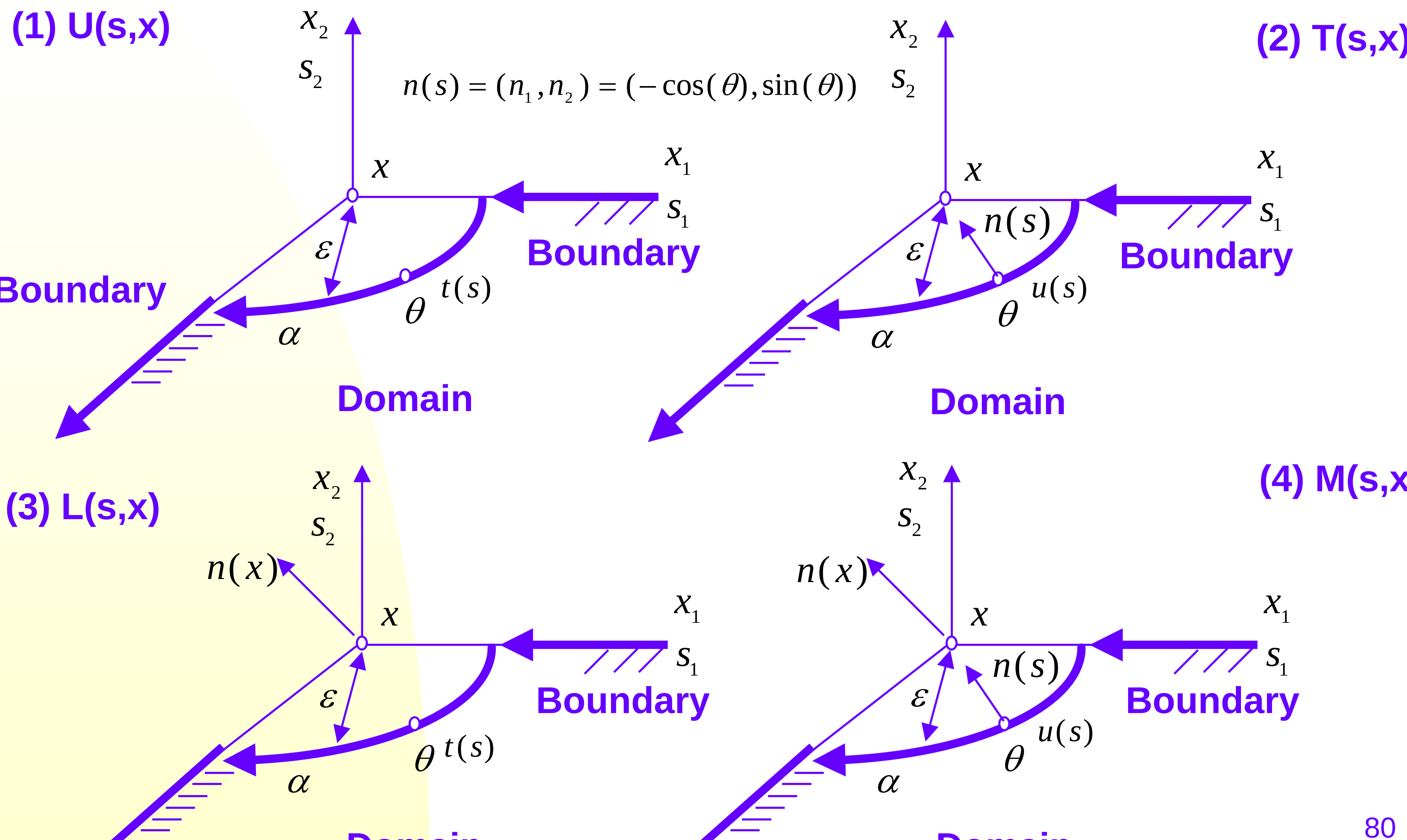
Boundary

The related symbols around the corner

$$t(s) = -\frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$$

$$u(s) = u(x) + \frac{\partial u}{\partial x}\varepsilon\cos(\theta) - \frac{\partial u}{\partial y}\varepsilon\sin(\theta)$$

$$n(s) = (n_1, n_2) = (-\cos(\theta), \sin(\theta))$$



Free terms

kernel	Laplace equation	Helmholtz equation
$U(s,x)$	$\varepsilon \ln(\varepsilon)$	$\varepsilon\left[-\frac{i\pi}{2}H_0^{(1)}(k\varepsilon)(t^+ + t^-)\right]$
$T(s,x)$	$-\alpha u(x)+\varepsilon(t^+ + t^-)$	$-\alpha u(x)+\varepsilon(t^+ + t^-)$
$L(s,x)$	$\frac{(-\sin(2\alpha)+2\alpha)}{4}t^-(x)+\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}$	$\frac{(-\sin(2\alpha)+2\alpha)}{4}t^-(x)+\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}$
$M(s,x)$	$-\frac{(-\sin(2\alpha)+2\alpha)}{4}t^-(x)$ $-\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}+B(\varepsilon)$	$-\frac{(-\sin(2\alpha)+2\alpha)}{4}t^-(x)$ $-\frac{(\cos(2\alpha)-1)}{4}u^{-\prime}+B(\varepsilon)$

Conclusions

- **The nonuniqueness in BIEM and BEM were reviewed and its treatment was addressed.**
- **The role of hypersingular BIE was examined.**
- **The numerical problems in the engineering applications using BEM were demonstrated.**
- **Several mathematical tools, SVD, degenerate kernel, …, were employed to deal with the problems.**

The End

Thanks for your kind attention

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