True and spurious eigensolutoins for membrane and plate problems by using method of fundamental solutions

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Abstract: *In this paper, the method of fundamental solutions is utilized to solve free vibration of membrane and plate problems. Single and double-layer potential approaches are both considered for the membrane problem and 6 (C_2^4) options by adopting two potentials from the single, double, triple and quadruple potentials are chosen for the plate problem. Spurious eigenvalues appear in the method of fundamental solution for the multiply-connected domain. The occurring mechanism of the spurious eigenvalues for membrane and plate problems is studied analytically by an annular case. The degenerate kernels and circulants are utilized to derive the true and spurious eigenequations analytically in the discrete model. True eigenequation depends on the boundary condition while spurious eigenequation relies on the formulation. The remedy, Burton & Miller method, is employed to suppress the occurrence of the spurious eigenvalues. Two examples are demonstrated to check the validity of the present formulations.*

Keywords: method of fundamental solutions, eigenproblem, degenerate kernel, circulant, Burton & Miller method

1 Introduction

It is well known that the method of fundamental solutions (MFS) can deal with engineering problems when a fundamental solution is known. This method was attributed to Kupradze in 1964 [1]. The method of fundamental solutions can be applied to potential [2], Helmholtz [3], diffusion [4], biharmonic [5], Stokes [6] and elasticity problems [1]. The method of fundamental solutions can be seen as one kind of meshless method. The basic idea is to approximate the solution by a linear superposition of fundamental solution with sources located outside the domain of the problem. Moreover, it has some advantages over boundary element method, e.g., no singularity, no boundary integrals

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and mesh-free model. However, only a limited number of MFS papers have been published for problems of multiply-connected domain. Spurious eigenvalue has not been noticed in the MFS [7]. In this paper, the true and spurious eigenequations for membrane and plate eigenproblems of multiply-connected domains will be analytically and numerically studied by using the method of fundamental solutions. In the conventional MFS, only the single-layer potential approach is utilized. Based on the potential theory, two approaches (single and double-layer potential methods) are adopted for membrane problems. For plate problems, four potentials (single, double, triple and quadruple potentials) can be chosen and 6 (C_2^4) options can be considered. The spurious eigenvalue appears in the membrane and plate problems. The occurring mechanism of the true and spurious eigenequations will be studied analytically by using mathematical tools such as degenerate kernel, circulant and singular value decomposition (SVD). We will utilize the Burton & Miller method to suppress the occurrence of the spurious eigenvalues for membrane and plate problems. Two examples will be demonstrated to see the validity of the present approaches.

2 Analysis of membrane and plate eigenproblems using the method of fundamental solutions

The governing equations for membrane and plate eigenproblems are shown as follows:

$$\mathcal{L}u = \begin{cases} (\nabla^2 + k^2)u(x) = 0, \ x \in \Omega & \text{for the membrane problem,} \\ (\nabla^4 - \lambda^4)u(x) = 0, \ x \in \Omega & \text{for the plate problem,} \end{cases} \tag{1}$$

where ∇^2 is the Laplacian operator, ∇^4 is the biharmonic operator, Ω is the domain, k is the wave number which is the angular frequency over the speed of sound, λ is the frequency parameter and $u(x)$ is the field potential at x. Here, we consider the fundamental solution $U(s,x)$ as

$$U(s,x) = \begin{cases} iJ_0(kr) - Y_0(kr) & \text{for the membrane problem,} \\ \frac{1}{8\lambda^2}\{[Y_0(\lambda r) - iJ_0(\lambda r)] + \frac{2}{\pi}[K_0(\lambda r) - iI_0(\lambda r)]\} & \text{for the plate problem,} \end{cases} \tag{2}$$

where $r \equiv |s-x|$ is the distance between the source and collocation points, $i^2 = -1$, J_n denotes the first-kind Bessel function of the nth order, Y_n denotes the second-kind Bessel function of the nth order, I_n denotes the first-kind modified Bessel function of the nth order and K_n denotes the second-kind modified Bessel function of the nth order.

For the purpose of deriving the exact eigensolution, an annular domain is considered. The radii of inner and outer circles are a and b for the real boundary, and the sources are distributed on the inner (a') and outer (b') fictitious circles as shown in Figure 1. For simplicity, the membrane problem subject to the Dirichlet-Dirichlet boundary condition is considered by using the single-layer potential approach. We distribute $2N$ collocation points on each boundary. The influence matrices can be easily determined by the two-point function. By matching the boundary condition, we have

$$\{0\} = [U_{ij}^{11}]\{\phi_j^1\} + [U_{ij}^{12}]\{\phi_j^2\}, \tag{3}$$

$$\{0\} = [U_{ij}^{21}]\{\phi_j^1\} + [U_{ij}^{22}]\{\phi_j^2\}, \tag{4}$$

where the first superscript "α" in $[U_{ij}^{\alpha\beta}]$ denotes the position of collocation point (1 for B_1 and 2 for B_2), the second superscript "β" identifies the position of source point (1 for B_1' and 2 for B_2'), $\{\phi_j^1\}$ and $\{\phi_j^2\}$ are the unknown coefficients on the inner and outer boundaries, respectively. By assembling Eqs.(3) and (4) together, we have

$$[SM_{D1}]\left\{\begin{array}{c}\phi_j^1\\ \phi_j^2\end{array}\right\}=\left[\begin{array}{cc}U_{ij}^{11} & U_{ij}^{12}\\ U_{ij}^{21} & U_{ij}^{22}\end{array}\right]\left\{\begin{array}{c}\phi_j^1\\ \phi_j^2\end{array}\right\}=\left\{\begin{array}{c}0\\ 0\end{array}\right\}, \qquad (5)$$

where the subscript "$D1$" denotes the Dirichlet-Dirichlet problem by using the single-layer potential approach. For the existence of nontrivial solution, the determinant of the matrix must be zero, *i.e.*,

$$det[SM_{D1}]=0. \qquad (6)$$

By plotting the determinant versus the wave number, the curve drops at the positions of eigenvalues.
In order to check the validity of this approach, the plate problem subject to the clamped-clamped case on the outer circle B_2 ($u_2=0$ and $\theta_2=0$) and the inner circle B_1 ($u_1=0$ and $\theta_1=0$) is considered by using the U-Θ formulation. By matching the boundary condition, we have

$$\left\{\begin{array}{c}0\\ 0\end{array}\right\}=\left[\begin{array}{cc}U11 & U12\\ U21 & U22\end{array}\right]\left\{\begin{array}{c}\phi_1\\ \phi_2\end{array}\right\}+\left[\begin{array}{cc}\Theta11 & \Theta12\\ \Theta21 & \Theta22\end{array}\right]\left\{\begin{array}{c}\psi_1\\ \psi_2\end{array}\right\}, \qquad (7)$$

$$\left\{\begin{array}{c}0\\ 0\end{array}\right\}=\left[\begin{array}{cc}U11_\theta & U12_\theta\\ U21_\theta & U22_\theta\end{array}\right]\left\{\begin{array}{c}\phi_1\\ \phi_2\end{array}\right\}+\left[\begin{array}{cc}\Theta11_\theta & \Theta12_\theta\\ \Theta21_\theta & \Theta22_\theta\end{array}\right]\left\{\begin{array}{c}\psi_1\\ \psi_2\end{array}\right\}, \qquad (8)$$

where $\{\phi_1\}$, $\{\psi_1\}$, $\{\phi_2\}$ and $\{\psi_2\}$ are the generalized coefficients for B_1 and B_2 with a dimension $2N\times1$, the matrices $[Uij]$, $[\Theta ij]$, $[Uij_\theta]$ and $[\Theta ij_\theta]$ mean the influence matrices of U, Θ, U_θ and Θ_θ kernels [8] which are obtained by collocating the field and source points on B_i and B_j' with a dimension $2N\times 2N$, respectively. Similarly, the determinant of the matrix which is obtained by assembling Eqs.(7) and (8) versus the eigenvalue must be zero for the existence of nontrivial solutions. By plotting the determinant versus the frequency parameter, the curve drops at the positions of eigenvalues.

3 Mathematical tools

3.1 Degenerate kernel

The kernel function used can be typically expressed in terms of degenerate kernel as follows:

$$U(s,x)=\left\{\begin{array}{ll}U^i(s,x)=\sum_{m=0}^{\infty}\frac{i}{\lambda_m}C_m(ks)R_m(kx), & x\in\Omega^i,\\ U^e(s,x)=\sum_{m=0}^{\infty}\frac{i}{\lambda_m}C_m(kx)R_m(ks), & x\in\Omega^e,\end{array}\right. \qquad (9)$$

where Ω^i and Ω^e are the interior and exterior domains, respectively.
For the membrane case, Eq.(9) reduces to

$$U^i(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}J_m(k\rho)(iJ_m(kR)-Y_m(kR))cos(m(\theta-\phi)), \quad R>\rho, \qquad (10)$$

$$U^e(R,\theta;\rho,\phi)=\sum_{m=-\infty}^{\infty}J_m(kR)(iJ_m(k\rho)-Y_m(k\rho))cos(m(\theta-\phi)), \quad R<\rho. \qquad (11)$$

The degenerate kernel of the plate problem is

$$
\begin{aligned}
U^i(R,\theta;\rho,\phi) &= \frac{1}{8\lambda^2}\sum_{\ell=-\infty}^{\infty}\{J_\ell(\lambda\rho)[Y_\ell(\lambda R)-iJ_\ell(\lambda R)] \\
&\quad +\frac{2}{\pi}(-1)^\ell I_\ell(\lambda\rho)[(-1)^\ell K_\ell(\lambda R)-iI_\ell(\lambda R)]\}cos(\ell(\theta-\phi)), R>\rho, \quad (12)\\
U^e(R,\theta;\rho,\phi) &= \frac{1}{8\lambda^2}\sum_{\ell=-\infty}^{\infty}\{J_\ell(\lambda R)[Y_\ell(\lambda\rho)-iJ_\ell(\lambda\rho)] \\
&\quad +\frac{2}{\pi}(-1)^\ell I_\ell(\lambda R)[(-1)^\ell K_\ell(\lambda\rho)-iI_\ell(\lambda\rho)]\}cos(\ell(\theta-\phi)), R<\rho, \quad (13)
\end{aligned}
$$

where $x=(\rho,\phi)$ and $s=(R,\theta)$.

3.2 Circulant

By superimposing $2N$ lumped strength along the fictitious boundary, we have the influence matrix,

$$
[U_{ij}] = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{2N-2} & a_{2N-1} \\ a_{2N-1} & a_0 & a_1 & \cdots & a_{2N-3} & a_{2N-2} \\ a_{2N-2} & a_{2N-1} & a_0 & \cdots & a_{2N-4} & a_{2N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_{2N-1} & a_0 \end{bmatrix} \quad (14)
$$

The matrix, $[U_{ij}]$, is found to be a circulant. By introducing the following bases for the ciruclants, I, $(C_{2N})^1$, $(C_{2N})^2$, $\cdots$, and $(C_{2N})^{2N-1}$, we can expand $[U]$ into

$$
[U] = a_0 I + a_1(C_{2N})^1 + a_2(C_{2N})^2 + \cdots + a_{2N-1}(C_{2N})^{2N-1}, \quad (15)
$$

where I is the unit matrix and

$$
C_{2N} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{2N\times 2N} . \quad (16)
$$

Based on the circulant theory, the eigenvalues for influence matrix, $[U]$, is found as follows:

$$
\lambda_\ell = a_0 + a_1\alpha_\ell + a_2(\alpha_\ell)^2 + \cdots + a_{2N-1}(\alpha_\ell)^{2N-1}, \ell = 0, \pm 1, \pm 2, \cdots, \pm(N-1), N, \quad (17)
$$

where λ_ℓ and α_ℓ are the eigenvalues for matrices $[U]$ and $[C_{2N}]$, respectively.

4 Numerical results and discussions

Example 1: An annular membrane with the inner radius of 0.5 meter and the outer radius of 2 meter are considered, respectively. The source points are distributed at $a' = 0.4m$ and $b' = 2.2m$. The outer and inner fictitious boundaries are both distributed 36 nodes as shown in Figure 1, respectively. Figures 2(a) and (b) show the determinant versus wave number by using the single-layer potential approach and double-layer potential approach, respectively. The drop location indicates the possible eigenvalues. As expected, the spurious eigenvalue of k=6.01 ($J_m(ka')=0$) for the single-layer potential approach and k=4.61 ($J'_m(ka')=0$) for the double-layer potential approach appear. Figure 2(c) shows the determinant versus wave number by using the Burton &

Miller method for the annular membrane where the spurious eigenvalues are suppressed. After comparing the result with the analytical solution, good agreement is made.

Example 2: An annular plate with the inner radius of 0.5 meter and the outer radius of 1 meter are considered, respectively. The source points are distributed at $a' = 0.4m$ meter and $b' = 1.2m$ meter. Forty-six nodes are uniformly distributed on the inner and outer fictitious boundaries. Figure 3(a) and (b) shows the determinant versus frequency parameter by using the $U - \Theta$ and $M - V$ formulations, respectively. The drop location indicates the possible eigenvalues. Figure 3(c) shows the determinant versus frequency parameter by using the Burton & Miller method for the annular plate. It is found that the appearance of spurious eigenvalues is suppressed. After comparing the result with the analytical solution, good agreement is made.

5 Conclusions

Mathematical analysis has shown that spurious eigenvalues occur by using degenerate kernels and circulants when the method of fundamental solutions was used to solve the eigenvalue of annular membrane and plate. The positions of spurious eigenvalues for the annular problem depend on the location of inner fictitious boundary where the sources are distributed. The spurious eigenvalues in the annular problem are found to be the true eigenvalues of the associated simply-connected problem bounded by the inner sources. Finally, we have successfully employed the Burton & Miller method to filter out the spurious eigenvalues for membrane as well as plate problems.

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