Solutions of antiplane shear problems with a crack or a rigid line inclusion by using the indirect BIE

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Abstract. In this paper, the indirect boundary integral equations is employed to derive the analytical solutions for the anti-plane shear problem with a crack or a rigid line inclusion. The key point is expanding the closed form fundamental solution to a series form, so-called the degenerate kernel. The solution is obtained via either the single-layer or the double-layer potentials of indirect BEM in conjunction with the degenerate kernel in the elliptic coordinates. The crack and rigid line inclusion can be simulated by setting $\xi_0 = 0$ in the elliptic coordinates. Two ways can be employed to simulate the degenerate boundary problem in the analysis procedure of boundary value problem (BVP). The first way is setting $\xi_0 = 0$ after obtaining the solution of antiplane shear problem with elliptic hole/elliptic rigid inclusion. The second way is before obtaining the coefficient of the Fourier series. It is interesting to find that the single-layer and double-layer potentials can obtain the analytical solution in the first way. The unknown Fourier coefficient can be obtained in the second way if unknown coefficient term is multiplied $\cosh(\xi_0)$. The unknown Fourier coefficient cannot be obtained if it is multiplied $\sinh(\xi_0)$ ($\sinh(0) = 0$). The derivation process of can cleanly show the rank-deficiency mechanism of the degenerate boundary in the indirect BEM. Finally, the numerical results verify the validity of the present approach.

The governing equation of an elliptic hole or elliptic rigid inclusion problem under antiplane shear is shown below

$$\nabla^2 u_z = 0, \tag{1}$$

where u_z is displacement and ∇^2 is the Laplace operator. The remote antiplane shear loading (σ_{yz}^{∞} or σ_{xz}^{∞}) and the displacement (u_z^{∞}) at infinity are

$$\sigma_{yz}^{\infty} = S, |y| \to \infty, \text{ and } u_z^{\infty} = Sy/\mu, |y| \to \infty, \tag{2}$$

$$\sigma_{xz}^{\infty} = S, |x| \to \infty, \text{ and } u_z^{\infty} = Sx/\mu, |x| \to \infty, \tag{3}$$

respectively, in which μ is the shear modules.The boundary condition on the hole and rigid inclusion are shown as follows

$$t_z = \partial u_z / \partial n = 0, \text{ for the hole,} \tag{4}$$

$$u_z = 0, \text{ for the rigid inclusion.} \tag{5}$$

Based on the superposition technique, the total displacement can be decomposed into two parts. One is an free field caused by the remote antiplane shear loading. The other is an exterior boundary value problem with the corresponding boundary condition. The total displacement can be given as

$$u_z = u_z^{\infty} + u_z^{M}. \tag{6}$$

The solution representation by using the single and double-layer potentials of the indirect BEM are shown below:

$$u(\mathbf{x})=\int_B U(\mathbf{s},\mathbf{x})\alpha_s(\mathbf{s})dB,\ \mathbf{x}\in D, \tag{7}$$

$$t(\mathbf{x})=\int_B L(\mathbf{s},\mathbf{x})\alpha_s(\mathbf{s})dB,\ \mathbf{x}\in D, \tag{8}$$

$$u(\mathbf{x})=\int_B T(\mathbf{s},\mathbf{x})\alpha_d(\mathbf{s})dB,\ \mathbf{x}\in D, \tag{9}$$

$$t(\mathbf{x})=\int_B M(\mathbf{s},\mathbf{x})\alpha_d(\mathbf{s})dB,\ \mathbf{x}\in D, \tag{10}$$

where $\alpha_s(\mathbf{s})$ and $\alpha_d(\mathbf{s})$ are the unknown boundary density for the single-layer potential and double-layer potential, respectively, $\mathbf{s}$ and $\mathbf{x}$ are the source and field points, respectively, B is the boundary. The kernel function, $U(\mathbf{s},\mathbf{x})$, is the fundamental solution as follows:

$$U(\mathbf{s},\mathbf{x})=\ln r,\ r=|\mathbf{x}-\mathbf{s}|. \tag{11}$$

The other kernel functions, $T(\mathbf{s},\mathbf{x})$, $L(\mathbf{s},\mathbf{x})$ and $M(\mathbf{s},\mathbf{x})$ are defined in [1]. By employing the separable property of the kernel, $U(\mathbf{s},\mathbf{x})$, $T(\mathbf{s},\mathbf{x})$, $L(\mathbf{s},\mathbf{x})$, $M(\mathbf{s},\mathbf{x})$ can be expanded into the degenerate form by separating the source point and the field point in the elliptic coordinates [1]. Following the indirect BIE, the unknown boundary density can be expanded by using the Fourier series. The unknown coefficient of Fourier series can be obtained by using the indirect BIE in conjunction with the degenerate kernel and boundary condition. The field solution can be obtained through the obtained boundary density. The analytical solution of the elliptic hole (Eqs.(12) and (13)) under the antiplane shear can be obtained by using the single and double-layer potential approach as,

$$u_z(\xi_x,\eta_x)=\frac{Sc}{\mu}\left(\sinh\xi_x+e^{\xi_0-\xi_x}\cosh\xi_0\right)\sin\eta_x,\ \text{under the antiplane shear}\ \sigma_{yz}^{\infty}, \tag{12}$$

$$u_z(\xi_x,\eta_x)=\frac{Sc}{\mu}\left(\cosh\xi_x+e^{\xi_0-\xi_x}\sinh\xi_0\right)\cos\eta_x,\ \text{under the antiplane shear}\ \sigma_{xz}^{\infty}, \tag{13}$$

respectively. For the solution of the elliptic rigid incluison problem can be written as

$$u_z(\xi_x,\eta_x)=\frac{Sc}{\mu}\left(\sinh\xi_x-e^{\xi_0-\xi_x}\sinh\xi_0\right)\sin\eta_x,\ \text{under the antiplane shear}\ \sigma_{yz}^{\infty}, \tag{14}$$

$$u_z(\xi_x,\eta_x)=\frac{Sc}{\mu}\left(\cosh\xi_x-e^{\xi_0-\xi_x}\cosh\xi_0\right)\cos\eta_x,\ \text{under the antiplane shear}\ \sigma_{xz}^{\infty}, \tag{15}$$

in which $2c$ is the distance of two foci in the elliptic coordinates. By comparing with those of Chen et al. [1], the results of Eqs.(12)~(15) are the same. The results of crack and rigid line inclusion are shown in Table 1. It is interesting to find that cases 2 and 4 are trivial. For the second way, the solution can be obtained by using the double-layer potential and the single-layer potential in the case 1 and 3, respectively, becomes the coefficient term includes $\cosh(\xi_0)$ instead of $\sinh(\xi_0)$. The indirect BEM can solve the crack or rigid line inclusion problem or not is our future work.

Table 1 Results of the crack and rigid line inclusion problem under the remote antiplane shear by using the different ways.

	Case 1 : Crack ($\sigma_{yz}^{\infty}=S$ and $\sigma_{xz}^{\infty}=0$)		Case 2 : Crack ($\sigma_{yz}^{\infty}=0$ and $\sigma_{xz}^{\infty}=S$)			Case 3 : Rigid line inclusion ($\sigma_{yz}^{\infty}=0$ and $\sigma_{xz}^{\infty}=S$)		Case 4 : Rigid line inclusion ($\sigma_{yz}^{\infty}=S$ and $\sigma_{xz}^{\infty}=0$)	
Geometry / BIE	Degenerate Boundary $\xi_0=0$		Degenerate Boundary $\xi_0=0$		Geometry / BIE	Degenerate Boundary $\xi_0=0$		Degenerate Boundary $\xi_0=0$	
Normal derivative of Single-layer Potential	(1)	$u_z(x)=\frac{Sc}{\mu}(\sinh\xi_x+e^{-\xi_x})\sin\eta_x$	(1)	$u_z(x)=\frac{Sc}{\mu}\cosh\xi_x\cos\eta_x$	Single-layer Potential	(1)	$u_z(\mathbf{x})=\frac{Sc}{\mu}(\cosh\xi_x-e^{-\xi_x})\cos\eta_x$	(1)	$u_z(\mathbf{x})=\frac{Sc}{\mu}\sinh\xi_x\cos\eta_x$
	(2)	$u_z(x)$ can not be found	(2)	$u_z(x)=\frac{Sc}{\mu}\cosh\xi_x\cos\eta_x$		(2)	$u_z(\mathbf{x})=\frac{Sc}{\mu}(\cosh\xi_x-e^{-\xi_x})\cos\eta_x$	(2)	$u_z(\mathbf{x})=\frac{Sc}{\mu}\sinh\xi_x\cos\eta_x$
Normal derivative of Double-layer Potential	(1)	$u_z(x)=\frac{Sc}{\mu}(\sinh\xi_x+e^{-\xi_x})\sin\eta_x$	(1)	$u_z(x)=\frac{Sc}{\mu}\cosh\xi_x\cos\eta_x$	Double-layer Potential	(1)	$u_z(\mathbf{x})=\frac{Sc}{\mu}(\cosh\xi_x-e^{-\xi_x})\cos\eta_x$	(1)	$u_z(\mathbf{x})=\frac{Sc}{\mu}\sinh\xi_x\cos\eta_x$
	(2)	$u_z(x)=\frac{Sc}{\mu}(\sinh\xi_x+e^{-\xi_x})\sin\eta_x$	(2)	$u_z(x)=\frac{Sc}{\mu}\cosh\xi_x\cos\eta_x$		(2)	$u_z(x)$ can not be found	(2)	$u_z(\mathbf{x})=\frac{Sc}{\mu}\sinh\xi_x\cos\eta_x$

Remark : (1) is first way ; (2) is second way .

References

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